EXERCISES ON SPECTRA

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Contents

1. Introduction

I collect here exercises that have helped my learn about spectra. Formally, everything is happening in the land of ∞-categories, and "category" should be read as "∞-category", and similarly for terms like "(co)limit". However, I hope (and expect) that much can be gained from these exercises even without precise knowledge of this language. If you have any feedback, please feel free to send me an email.

Some of the exercises are harder or require more background knowledge, and I would not recommend them to someone just starting to learn about spectra. These are marked with a ($\star$), and I plan to give hints or links to full solutions of some of these in an appendix. Many of these are simply fun facts I've learned from reading papers or certain MathOverflow posts.

These notes are currently very much work in progress! Moreover, they are heavily biased towards my own interests, and may not cover the most "useful in practice"-material. Besides this, I have not thought too much about the precise amount of knowledge I want to assume the reader to have, though certainly a rough understanding about working with higher categorical (co)limits is expected; mostly just suspension/loops, pushouts/pullbacks, (co)fiber sequences and (co)products.

For a detailed introduction to modern stable homotopy theory (including stable ∞-categories) I can recommend Bastiaan Cnossen's lecture notes on Stable Homotopy Theory [Cno24], This includes a short modern introduction to the language of higher category theory. For an even more in-depth view of higher category theory, I can recommend Rune Haugseng's notes for his PhD course on higher categories [Hau25].

2. Arithmetic in Spectra

The basic setup and axioms we will assume are as follows:

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(1) There is a "nice" category Sp whose objects we call spectra, which formally behaves a lot like the derived category of integers $\mathcal{D}(\mathbb{Z})$. In particular, it admits all (small) limits and colimits, and stability has the following useful consequences:
   (a) A commutative square in Sp is a pushout square if and only if it is a pullback square.
   (b) The adjoint pair $\Sigma\colon \mathsf{Sp} \rightleftarrows \mathsf{Sp} :\Omega$ is one of mutually inverse equivalences. Hence one often writes $\Sigma^{-1} = \Omega$.
   (c) Sp is an additive category, so we have a zero object 0, finite coproducts and products agree via a canonical map, and $\pi_0 \operatorname{map}_{\mathsf{Sp}}(X, Y)$ is forms an abelian group, allowing us to add and subtract maps of spectra.

(2) Sp admits a closed symmetric monoidal structure, whose unit is denoted $\mathbb{S}$ and called the sphere spectrum. In particular, the tensor product $\otimes$ preserves colimits in both variables.

(3) There are functors $\pi_n\colon \mathsf{Sp} \to \mathsf{Ab}$ for each $n \in \mathbb{Z}$. These commute with infinite products and filtered colimits, and are jointly conservative, meaning that if $f\colon X \to Y$ is some map of spectra such that $\pi_n f$ is an isomorphism for each n, then f is an equivalence of spectra. Moreover, there is a natural equivalence $\pi_n(-) = \pi_0 \operatorname{map}_{\mathsf{Sp}}(\Sigma^n\mathbb{S}, -)$.

(4) A (co)fiber sequence of spectra $X \xrightarrow{f} Y \xrightarrow{g} Z$ induces a long exact sequence of homotopy groups

$$\cdots \to \pi_{n+1}Z \to \pi_n X \xrightarrow{\pi_n f} \pi_n Y \xrightarrow{\pi_n g} \pi_n Z \to \pi_{n-1}X \to \cdots$$

(5) There is a fully faithful functor $\mathsf{Ab} \hookrightarrow \mathsf{Sp}$, $A \mapsto HA$. We call HA the Eilenberg-MacLane spectrum for A (and often it is simply denoted A itself under abuse of notation). The characterizing property of an Eilenberg-MacLane spectrum is that its homotopy groups are concentrated in degree 0, i.e. $\pi_n(HA) = 0$ for $n \neq 0$, and $\pi_0(HA) \cong A$.

(6) We have $\pi_0(\mathbb{S}) = \mathbb{Z}$ and $\pi_k(\mathbb{S}) = 0$ for $k < 0$ and $\pi_k(\mathbb{S})$ is finite for $k > 0$, infinitely often non-zero, and p-torsion first occurs in degree $k = 2p - 3$.

Exercise 2.1. To familiarize yourself with these concepts, show the following properties:

(1) Also $\mathsf{Sp}^{\mathsf{op}}$ is stable and admits all (small) limits and colimits.
(2) Let $\hom = \hom_{\mathsf{Sp}}\colon \mathsf{Sp}^{\mathsf{op}} \times \mathsf{Sp} \to \mathsf{Sp}$ denote the internal hom. This is exact in both variables, i.e. both $\hom(X, -)$ and $\hom(-, X)$ preserve finite limits and colimits.
(3) There are natural equivalences $\pi_n(\Omega X) \cong \pi_{n+1}(X)$ and $\pi_n(\Sigma X) \cong \pi_{n-1}(X)$.
(4) For a spectrum X, we have $X \simeq 0$ if and only if $\pi_n X = 0$ for all n.

Exercise 2.2 (More on Cofiber Sequences)**.**

(1) Cofiber sequences can be rotated: If $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a cofiber sequence, then there is a map $Z \xrightarrow{h} \Sigma X$ such that $Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ and $Z \xrightarrow{h} \Sigma X \xrightarrow{\Sigma f} \Sigma Y$ are also cofiber sequences.[1] In fact, one can keep going in both directions

$$\cdots \to \Omega X \xrightarrow{\Omega f} \Omega Y \xrightarrow{\Omega g} \Omega Z \to X \xrightarrow{f} Y \xrightarrow{g} Z \to \Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma g} \Sigma Z \to \Sigma^2 X \to \cdots$$

so that every 2 consecutive maps form a cofiber sequence, and applying π_0 recovers the usual long exact sequence on homotopy groups

$$\cdots \to \pi_1 X \xrightarrow{\pi_1 f} \pi_1 Y \xrightarrow{\pi_1 g} \pi_1 Z \to \pi_0 X \xrightarrow{\pi_0 f} \pi_0 Y \xrightarrow{\pi_0 g} \pi_0 Z \to \pi_{-1} X \xrightarrow{\pi_{-1} f} \pi_{-1} Y \xrightarrow{\pi_{-1} g} \pi_{-1} Z \to \pi_{-2} X \to \cdots$$

(2) A map $f\colon X \to Y$ of spectra is an equivalence if and only if its cofiber vanishes, if and only if its fiber vanishes.

[1] Technically there should be $-\Sigma f$ in place of Σf here, but for purposes of exactness and cofiber sequences the signs do not matter, so we sweep them under the rug here.

(3) Suppose you have a (vertical) map of (horizontal) cofiber sequences

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\ X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' \end{array}$$

If two of the three maps a, b, c are equivalences, then also the third one is an equivalence.

(4) A commutative square of spectra

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow[k]{} & D \end{array}$$

is a pushout square if and only if $A \xrightarrow{(f,g)} B \oplus C \xrightarrow{h-k} D$ is a cofiber sequence.

(5) For a cofiber sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$, the following are equivalent:
   (a) $Z \to \mathrm{cofib}(g)$ is nullhomotopic.
   (b) g admits a section, i.e. a map $s \colon Z \to Y$ with $gs \simeq \mathrm{id}_Z$.
   (c) There is an equivalence $Y \simeq X \oplus Z$ under which f respectively g identifies with the summand inclusion respectively projection.

In this case, also the associated long exact sequence on homotopy groups consists of many split short exact sequences.

Exercise 2.3 (More on Eilenberg-MacLane Spectra)**.** Let A, B be abelian groups.

(1) The functor $H \colon \mathsf{Ab} \to \mathsf{Sp}$ preserves products, coproducts and filtered colimits. Moreover, it sends short exact sequences to cofiber sequences.

(2) Show $\pi_0 \hom(HA, \Sigma HB) = \pi_{-1} \hom(HA, HB) = \mathrm{Ext}^1_{\mathbb{Z}}(A, B)$. In particular, there can be non-trivial maps $HA \to \Sigma HB$, which are necessarily zero on all homotopy groups! (In fact, we will later see that the Steenrod squares Sq^n give rise to non-trivial maps $H\mathbf{F}_2 \to \Sigma^n H\mathbf{F}_2$ for all n.) Convince yourself that on the other hand, by fully faithfulness of $H \colon \mathsf{Ab} \to \mathsf{Sp}$, any map $\Sigma^n HA \to HB$ is zero for any abelian groups A, B and $n \geq 1$.

(3) We will later[2] see that there is a natural isomorphism $A \otimes B = \pi_0(HA) \otimes \pi_0(HB) \xrightarrow{\cong} \pi_0(HA \otimes HB)$ (where on the left the tensor product is the underived / ordinary one of abelian groups), and that $\pi_1(H\mathbb{Z} \otimes H\mathbb{Z}) = 0$. Using this, show that there is an isomorphism $\mathrm{Tor}_1^{\mathbb{Z}}(A, B) \xrightarrow{\cong} \pi_1(HA \otimes HB)$. In particular, unlike in Ab, we have $H\mathbf{F}_p \otimes H\mathbf{F}_p \not\simeq H\mathbf{F}_p$, and Eilenberg-MacLane spectra are generally not closed under tensor products.

(4) Convince yourself that for each $k \in \mathbb{Z}$ there is a natural (in spectra X, Y) map

$$\bigoplus_{m+n=k} \pi_m(X) \otimes \pi_n(Y) \to \pi_k(X \otimes Y),$$

$$[\Sigma^m \mathbb{S} \xrightarrow{f} X] \otimes [\Sigma^n \mathbb{S} \xrightarrow{g} Y] \mapsto [\Sigma^{m+n} \mathbb{S} = \Sigma^m \mathbb{S} \otimes \Sigma^n \mathbb{S} \xrightarrow{f \otimes g} X \otimes Y]$$

where we are using the natural identification $\pi_m(X) \cong \pi_0 \hom_{\mathsf{Sp}}(\Sigma^m \mathbb{S}, X)$. We may equivalently think of this as a map $\pi_*(X) \otimes \pi_*(Y) \to \pi_*(X \otimes Y)$ of graded abelian groups (where the left side uses the graded tensor product), i.e. a lax symmetric monoidal structure on $\pi_* \colon \mathsf{Sp} \to \mathsf{grAb}$. Note that by the previous point, this is generally not an isomorphism.

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[2] See Exercises 4.7 and **??**

Definition 2.4. For an abelian group A, a spectrum X, and $k \in \mathbb{Z}$, one often writes

$$H_k(X;A) \coloneqq \pi_k(X \otimes HA) \qquad \text{and} \qquad H^k(X;A) \coloneqq \pi_{-k} \hom(X, HA).$$

2.1. **Inverting Integers & Rational Spectra.** Since Sp is semi-additive, there is for each integer $n \geq 0$ and spectrum X a multiplication-by-n map

$$n = \sum_n \mathrm{id}_X = \left(X \xrightarrow{\Delta} \prod_n X \xleftarrow{\simeq} \bigoplus_n X \xrightarrow{\nabla} X \right).$$

Since all maps here are natural in X, this yields a natural transformation $n\colon \mathrm{id}_{\mathsf{Sp}} \Rightarrow \mathrm{id}_{\mathsf{Sp}}$.

Exercise 2.5. Show that $\pi_k(n)\colon \pi_k(X) \to \pi_k(X)$ is just the usual multiplication with n on the abelian group $\pi_k(X)$, for all $k \in \mathbb{Z}$. Moreover, for $m, m \in \mathbb{Z}$ we have $n \circ m \simeq (nm) = (mn) \simeq m \circ n$.

Since Sp is an additive category, we may also define $-n\colon X \to X$ as the (additive) inverse of $n\colon X \to X$. Again one checks that the effect on homotopy groups will simply be multiplication with $-n$. From now on let $n \in \mathbb{Z}$ be any integer.

Exercise 2.6. Given spectra X, Y, show that the following three endomorphisms of the internal hom spectrum $\hom(X, Y)$ are homotopic (more-or-less-equivalently, feel free to show this for the mapping space instead): n (as defined for $\hom(X, Y)$), n_* (postcomposition by $n\colon Y \to Y$) and n^* (precomposition by $n\colon X \to X$).

Let n be an integer and X a spectrum. Motivated by ordinary algebra, we define

$$X[n^{-1}] \coloneqq \mathrm{colim}(X \xrightarrow{n} X \xrightarrow{n} X \xrightarrow{n} X \to \cdots)$$

Note that we have a canonical map $X \to X[n^{-1}]$.

Exercise 2.7. Convince yourself that

(1) $X[0^{-1}] \simeq 0$.
(2) $X[n^{-1}] \simeq X$ if $n = \pm 1$.
(3) $n\colon X[n^{-1}] \to X[n^{-1}]$ is an equivalence.
(4) $\pi_k(X[n^{-1}]) \cong \pi_k(X)[n^{-1}]$, where the latter inverts n in the sense of abelian groups.

For integers $a_1, \ldots, a_{n+1}$ we define inductively

$$X[\{a_1, \ldots, a_{n+1}\}^{-1}] \coloneqq (X[\{a_1, \ldots, a_n\}^{-1}])[a_{n+1}^{-1}] \simeq X[(a_1 \cdots a_{n+1})^{-1}]$$

and if $\{a_1, a_2, \ldots\} = A \subseteq \mathbb{Z}$ is some possibly infinite set of integers, we let

$$X[A^{-1}] \coloneqq \underset{A_0 \subseteq A \text{ finite}}{\mathrm{colim}} X[A_0^{-1}] \simeq \mathrm{colim}(X \xrightarrow{a_1} X \xrightarrow{a_1 2_2} X \xrightarrow{a_1 a_2 a_3} X \to \cdots)$$

Again we have a canonical map $X \to X[A^{-1}]$.

Exercise 2.8. Let P be a set of primes.

(1) $X[(ab)^{-1}] \simeq X[\{a, b\}^{-1}] = X[a^{-1}][b^{-1}]$. For this reason, inverting n is the same as inverting its prime factors.
(2) $X[P^{-1}] \simeq \mathbb{S}[P^{-1}] \otimes X$.
(3) $\pi_k(X[P^{-1}]) \cong \pi_k(X)[P^{-1}]$ for all k.
(4) For an abelian group A, what is $(HA)[P^{-1}]$?
(5) There is an equivalence $X \simeq X[P^{-1}]$ if and only if $p\colon X \to X$ is an equivalence for all $p \in P$. In particular $\mathbb{S}[P^{-1}]$ is idempotent in the sense that $\mathbb{S}[P^{-1}] \otimes \mathbb{S}[P^{-1}] \simeq \mathbb{S}[P^{-1}]$.

Definition 2.9. In particular, one can build a functor $(-)[P^{-1}]\colon \mathsf{Sp} \to \mathsf{Sp}$ and the canonical map induces a natural transformation $\mathrm{id}_{\mathsf{Sp}} \Rightarrow (-)[P^{-1}]$.

Definition 2.10. We define the p-localization $X_{(p)}$ by inverting all primes *except* p. We say that the canonical map $X \to X_{(p)}$ exhibits $X_{(p)}$ as the p-localization of X, and say that X is p-local if it is an equivalence.

Similarly, we define the rationalization $X_{\mathbb{Q}}$ by inverting all primes, and say that X is rational if the canonical map $X \to X_{\mathbb{Q}}$ is an equivalence.

Exercise 2.11. Let X be a spectrum.

(1) Check that X is p-local (rational) if and only if there is *any* equivalence $X \simeq X_{(p)}$ ($X \simeq X_{\mathbb{Q}}$), not necessarily the canonical map.
(2) One can write $X_{\mathbb{Q}} \simeq \operatorname{colim}(X \xrightarrow{2} X \xrightarrow{3} X \xrightarrow{4} X \xrightarrow{5} X \xrightarrow{6} X \to \cdots)$.
(3) X is rational if and only if each $\pi_k(X)$ is a rational vector space.
(4) $\mathbb{S}_{\mathbb{Q}} \simeq H\mathbb{Q} \simeq (H\mathbb{Z})_{\mathbb{Q}}$, hence $X_{\mathbb{Q}} \simeq X \otimes H\mathbb{Q}$ and $H\mathbb{Q}$ is idempotent.

Exercise 2.12. For p a prime, $\mathbb{S}_{(p)} \not\simeq H\mathbb{Z}_{(p)}$.

Exercise 2.13. Let $A \in \mathsf{Ab}$. Show that there does not exist a non-zero map $HA \to \mathbb{S}$. Hint 3

Exercise 2.14. Rational spectra are as simple as it gets. Let X be rational.

(1) Show that $\hom(X, Y)$ is rational for any spectrum Y. Similarly, $\hom(Y, X)$ is rational and equivalent to $\hom(Y_{\mathbb{Q}}, X)$.
(2) Show that $\hom(H\mathbb{Q}, X) \simeq X$. In particular, there is a non-zero map $\Sigma^m H\mathbb{Q} \to \Sigma^n H\mathbb{Q}$ (i.e. an element in $\pi_0 \hom_{\mathsf{Sp}}(\Sigma^m H\mathbb{Q}, \Sigma^n H\mathbb{Q})$) if and only if $m = n$.
(3) If $([\mathbb{S} \xrightarrow{f_i} X])_i$ is a $\mathbb{Q}$-basis of $\pi_0 X$, then $\bigoplus_i \mathbb{Q} \xrightarrow{(f_i)_{\mathbb{Q}}} X$ is an isomorphism in degree 0.
(4) There is an equivalence $\bigoplus_{n \in \mathbb{Z}} \Sigma^n H\pi_n X \xrightarrow{\simeq} X$.
(5) For any spectrum Y, write out $\hom(Y, X)$ explicitly using the above.
(6) If Y is also rational, then the natural map of Exercise 2.3(4) is an isomorphism. In particular, we have a rational Künneth formula

$$H_*(X; \mathbb{Q}) \otimes_{\mathbb{Q}} H_*(Y; \mathbb{Q}) \xrightarrow{\cong} H_*(X \otimes Y; \mathbb{Q}), \quad X, Y \in \mathsf{Sp}.$$

Exercise 2.15 ($\star$)**.** Show that we also have a Künneth formula for $\mathbf{F}_p$, for any prime p:

$$H_*(X; \mathbf{F}_p) \otimes_{\mathbf{F}_p} H_*(Y; \mathbf{F}_p) \xrightarrow{\cong} H_*(X \otimes Y; \mathbf{F}_p), \quad X, Y \in \mathsf{Sp}.$$

Related Exercises. See Exercises 3.7 and 4.9.

2.2. **Modding out integers.**

Definition 2.16. For any spectrum X, we denote $X/n \coloneqq \operatorname{cofib}(X \xrightarrow{n} X)$.

Exercise 2.17. Let X be any spectrum.

(1) What is the cofiber of the zero map $0\colon X \to X$?
(2) Show $X/n \simeq \mathbb{S}/n \otimes X$.
(3) X/p is p-local. In fact, $(X_{(p)})/p \simeq X/p \simeq (X/p)_{(p)}$.
(4) Simplify $(X/p)/q$ for primes p, q (not necessarily distinct!).
(5) Given an abelian group A, describe the homotopy groups of $(HA)/n$.
(6) How do X/m, X/n and $X/(mn)$ relate?
(7) $\hom(X, Y)/n$ and $\hom(X/n, Y)$ and $\hom(X, Y/n)$ all agree up to a shift.
(8) For $X \in \mathsf{Sp}$ and a set of primes P, the following are equivalent:
    (a) $X \simeq X[P^{-1}]$.
    (b) $X/p = 0$ for all $p \in P$.
    (c) $\hom(Y/p, X) = 0$ for all Y.

Hint 3 It suffices to show this for $A = \mathbb{Z}$.

(9) $X \simeq 0$ if and only if $X/p \simeq 0$ for all primes p and $X_{\mathbb{Q}} \simeq 0$.

Exercise 2.18. $\mathbb{S}/n$ admits a unital multiplication (i.e. a map $\mu\colon \mathbb{S}/n \otimes \mathbb{S}/n \to \mathbb{S}/n$ such that $\mu \circ (\mathbb{S}/n \otimes \eta) = \mathrm{id}$ where $\eta\colon \mathbb{S} \to \mathbb{S}/n$ is the canonical map) if and only if $n\colon \mathbb{S}/n \to \mathbb{S}/n$ is nullhomotopic. In particular, $\mathbb{S}/2$ does not admit a unital multiplication.

Exercise 2.19. n^2 is always nullhomotopic on X/n. Hint 4

Exercise 2.20 ($\star$)**.** We can be a bit more specific regarding the previous exercise:

(1) Show that $\pi_2(\mathbb{S}/2) \cong \mathbb{Z}/4$. In particular $\pi_n(X/2)$ need not be 2-torsion, and $2\colon X/2 \to X/2$ need not be nullhomotopic.
(2) If n is odd, then n is nullhomotopic on X/n.

In fact, one can show that $n\colon X/n \to X/n$ is nullhomotopic if and only if $n \not\equiv 2 \mod 4$, although this is beyond the scope of a reasonable exercise.

Exercise 2.21 ($\star$)**.** Show that there exists a spectrum X so that each $\pi_n X$ is 2-torsion, but $2\colon X \to X$ is not nullhomotopic.

2.3. **Completions at primes and the Fracture Square.** The completion at a prime p is defined analogously to the completion in abelian groups:

$$X_p^\wedge \coloneqq \lim(\cdots \to X/p^3 \to X/p^2 \to X/p)$$

where the maps $X/p^{n+1} \to X/p^n$ are induced by the map of cofiber sequences

$$\begin{array}{ccccc} X & \xrightarrow{p^{n+1}} & X & \longrightarrow & X/p^{n+1} \\ {\scriptstyle p}\downarrow & & \| & & \downarrow \\ X & \xrightarrow[p^n]{} & X & \longrightarrow & X/p^n \end{array}$$

By the universal property of the limit, we obtain a canonical map $X \to X_p^\wedge$, which we say exhibits X_p as the p-completion of X. In fact, p-completion becomes a functor $(-)_p^\wedge\colon \mathsf{Sp} \to \mathsf{Sp}$, and the canonical map upgrades to a natural transformation $\mathrm{id}_{\mathsf{Sp}} \Rightarrow (-)_p^\wedge$.

Exercise 2.22. Let X be a spectrum and p a prime.

(1) X is p-complete if and only if $\hom(\mathbb{S}[p^{-1}], X) \simeq \lim(\cdots \to X \xrightarrow{p} X \xrightarrow{p} X) \simeq 0$.
(2) $X_p^\wedge$ is p-complete.
(3) $(X_{(p)})_p^\wedge \simeq X_p^\wedge \simeq (X_p^\wedge)_{(p)}$. In particular, p-complete spectra are p-local.
(4) $(X/p^n)_p^\wedge \simeq X/p^n \simeq (X_p^\wedge)/p^n$. In particular, X/p^n is p-complete.
(5) If X is p-complete and $X/p \simeq 0$, then $X \simeq 0$.
(6) If X is rational and Y is p-complete, then $\hom(X, Y) = 0$. In particular, if X is rational and p-complete, then $X = 0$.

The following is often very useful for dealing with sequential inverse limits

Exercise 2.23 (Milnor Sequence)**.** Let $X \simeq \lim(\cdots \to X_3 \to X_2 \to X_1)$ be a sequential inverse limit of spectra. Then for every $k \in \mathbb{Z}$, there is a short exact sequence

$$0 \to \lim_n{}^{(1)} \pi_{k+1} X_n \to \pi_k X \to \lim_n \pi_k X \to 0$$

Here the first term denotes the first derived limit in the sense of homological algebra. In particular, if the tower of abelian groups $(\pi_k(X_n))_n$ satisfies the Mittag–Leffler condition, then we have $\pi_k(X) \simeq \lim_n \pi_k(X_n)$.

Hint 4 Use the cofiber sequence $X/n \to \Sigma X \xrightarrow{n} \Sigma X$, which in particular composes to 0

Exercise 2.24 (*p*-completion in abelian groups vs spectra)**.** We say an abelian group A is derived p-complete if HA is a p-complete spectrum.

(1) If A is p-complete as an abelian group, then it is also derived p-complete. Moreover, if A has bounded p-power torsion, the converse also holds.
(2) Let $C_{p^\infty} := \mathbb{Z}[p^{-1}]/\mathbb{Z}$ denote the Prüfer group at the prime p. Show that the ordinary p-completion of C_{p^∞} vanishes, and compare this with the p-completion of HC_{p^∞}.
(3) Compare the ordinary and spectrum-level p-completions of $\bigoplus_{n\geq 1} \mathbb{Z}/p^n$.

Given a spectrum X, we now have canonical maps $\eta_X^p \colon X \to X_p^\wedge$ for every prime p, which induce a natural map $(\eta_X^p)_p \colon X \to \prod_p X_p^\wedge$. Moreover, every spectrum has a natural map $\eta_X^{\mathbb{Q}} \colon X \to X_{\mathbb{Q}}$ to its rationalization. We can therefore build a natural commutative square

$$\begin{array}{ccc} X & \xrightarrow{(\eta_X^p)_p} & \prod_p X_p^\wedge \\ {\scriptstyle \eta_X^{\mathbb{Q}}}\downarrow & & \downarrow{\scriptstyle \eta^{\mathbb{Q}}_{\prod_p X_p}} \\ X_{\mathbb{Q}} & \xrightarrow[((\eta_X^p)_p)_{\mathbb{Q}}]{} & (\prod_p X_p^\wedge)_{\mathbb{Q}} \end{array}$$

One can build an analogous naturality square in the category of abelian groups, and it is well-known that it is a pullback square there (convince yourself of this if you have not seen this before).

Exercise 2.25 (Arithmetic Fracture Squares)**.**

(1) Show that the above square is a pullback square in Sp.
(2) Show that one similarly has pullback squares

$$\begin{array}{ccc} X & \longrightarrow & X_p \\ \downarrow & \lrcorner & \downarrow \\ X[p^{-1}] & \longrightarrow & X_p[p^{-1}] \end{array} \qquad\qquad \begin{array}{ccc} X_{(p)} & \longrightarrow & X_p \\ \downarrow & \lrcorner & \downarrow \\ X_{\mathbb{Q}} & \longrightarrow & (X_p)_{\mathbb{Q}} \end{array}$$

(3) The functors $(-)_{\mathbb{Q}}$ and $(-)_p$ for all primes p are jointly conservative, i.e. if $X \in \mathsf{Sp}$ with $X_{\mathbb{Q}} = 0 = X_p$ for all primes p, then $X = 0$.

Related Exercises. See Exercises 3.8 and 4.8.

3. Subcategories of Spectra

Definition 3.1. Let $\mathcal{C} \subseteq \mathsf{Sp}$ be a full subcategory of spectra.

(1) We say $\mathcal{C}$ is a stable subcategory if it is closed under finite limits and colimits in $\mathcal{C}$. In other words, if $I \to \mathsf{Sp}$ is some finite diagram of spectra whose objects lie in $\mathcal{C}$, then also the limit or colimit of this diagram lies in Sp.
(2) We say $\mathcal{C}$ is a thick subcategory if it is a stable subcategory and furthermore closed under retracts, i.e. if $Y \in \mathcal{C}$ and there are maps $s \colon X \to Y$ and $r \colon Y \to X$ in Sp such that $rs \simeq \mathrm{id}_X$ (we say X is a retract of Y), then also $X \in \mathcal{C}$.
(3) We say $\mathcal{C}$ is a localizing subcategory if it is a thick subcategory and also closed under all colimits.
(4) We define the thick subcategory generated by $\mathcal{C}$ as the smallest thick subcategory of spectra containing $\mathcal{C}$, and denote it $\mathrm{Thick}(\mathcal{C})$. We analogously define the localizing subcategory $\mathrm{loc}(\mathcal{C})$ generated by $\mathcal{C}$.

The following result is also one of the most important properties of Sp:

Theorem 3.2. *Sp is generated under colimits and shifts by $\mathbb{S}$, i.e. $\mathrm{loc}(\{\mathbb{S}\}) = \mathsf{Sp}$. In particular, if $\mathcal{C} \subseteq \mathsf{Sp}$ is a localizing subcategory and $\mathbb{S} \in \mathcal{C}$, then $\mathcal{C} = \mathsf{Sp}$.*

Exercise 3.3. Let $\mathcal{C} \subseteq \mathsf{Sp}$ be a full subcategory.

(1) If $\mathcal{C}$ is closed under shifts (if $X \in \mathcal{C}$ then also $\Sigma^n X \in \mathcal{C}$ for all $n \in \mathbb{Z}$) and cofibers (if $f\colon X \to Y$ is a map of spectra and $X, Y \in \mathcal{C}$, then also $\mathrm{cofib}(f) \in \mathcal{C}$), then $\mathcal{C}$ is a stable subcategory.
(2) If $\mathcal{C}$ is a stable subcategory and $X \to Y \to Z$ is a cofiber sequence of spectra, then if two of X, Y, Z are in $\mathcal{C}$, then also the third lies in $\mathcal{C}$.
(3) If $\mathcal{C}$ is closed under direct sums (if $X_i \in \mathcal{C}$ for $i \in I$ then also $\bigoplus_{i\in I} X_i \in \mathcal{C}$) and cofibers, then $\mathcal{C}$ is closed under all colimits.
(4) If $\mathcal{C}$ is a localizing subcategory and $X \in \mathcal{C}$, then we also have $Y \otimes X \in \mathcal{C}$ for any $Y \in \mathsf{Sp}$.

The above gives rise to the extremely useful proof-principle of what I call "structural induction". Say you want to prove that some property P holds for all spectra. Then it suffices to show that the full subcategory $\mathcal{C} \subseteq \mathsf{Sp}$ of spectra satisfying the property is localizing and contains $\mathbb{S}$. A common example of this is the following:

Exercise 3.4. Let $F, G\colon \mathsf{Sp} \to \mathsf{Sp}$ be functors preserving I-indexed colimits, and suppose that $\alpha\colon F \Rightarrow G$ is a natural transformation. Then $\mathcal{C} = \{X \in \mathsf{Sp} \mid \alpha_X \text{ is an equivalence}\} \subseteq \mathsf{Sp}$ is closed under I-indexed colimits in Sp. In particular, if F, G preserve all colimits (and hence automatically also finite limits, since we are in a stable situation), and $\alpha_{\mathbb{S}}$ is an equivalence, then α is an equivalence.

Exercise 3.5. Let $\mathcal{C} \subseteq \mathsf{Sp}$ be some full subcategory. Define $\mathrm{Thick}_0(\mathcal{C}) \coloneqq 0$, and $\mathrm{Thick}_1(\mathcal{C})$ as the retract-closure of $\mathcal{C}$, and inductively $\mathrm{Thick}_{n+1}(\mathcal{C})$ as the full subcategory on those spectra that can be written as a retract of a spectrum Y which sits in a cofiber sequence $X \to Y \to Z$ with $X \in \mathrm{Thick}_1(\mathcal{C})$ and $Z \in \mathrm{Thick}_n(\mathcal{C})$. Show that:

(1) $\mathrm{Thick}(\mathcal{C}) = \bigcup_{n\geq 0} \mathrm{Thick}_n(\mathcal{C})$.
(2) Each $\mathrm{Thick}_n(\mathcal{C})$ is closed under retracts and shifts.
(3) If $X \to Y \to Z$ is a cofiber sequence with $X \in \mathrm{Thick}_m(\mathcal{C})$ and $Z \in \mathrm{Thick}_n(\mathcal{C})$, then $Y \in \mathrm{Thick}_{n+m}(\mathcal{C})$.

Exercise 3.6. Let $\mathcal{C}, \mathcal{D} \subseteq \mathsf{Sp}$ be full subcategories. Show that

$$\mathrm{Thick}(\mathcal{C}) \otimes \mathrm{Thick}(\mathcal{D}) \coloneqq \{X \otimes Y \mid X \in \mathrm{Thick}(\mathcal{C}), Y \in \mathrm{Thick}(\mathcal{D})\} \subseteq \mathrm{Thick}(\{c \otimes d \mid c \in \mathcal{C}, d \in \mathcal{D}\})$$

Exercise 3.7. Let P be a set of primes. The full subcategory $\mathsf{Sp}[P^{-1}] \subseteq \mathsf{Sp}$ of spectra for which we have $X \simeq X[P^{-1}]$ is closed under limits and colimits. In particular, the categories of p-local and rational spectra $\mathsf{Sp}_{(p)}$ and $\mathsf{Sp}_{\mathbb{Q}}$ are closed under limits and colimits.

Exercise 3.8. The full subcategory on p-complete spectra $\mathsf{Sp}_p^\wedge \subseteq \mathsf{Sp}$ is closed under limits, but not under all colimits.

Exercise 3.9. Let $F\colon \mathsf{Sp} \to \mathsf{Sp}$ be a functor which preserves colimits. Then there is a natural equivalence $F(\mathbb{S}) \otimes - \simeq F$.

Exercise 3.10 ($\star$)**.** Let $\mathcal{C} \subseteq \mathcal{D}(\mathbb{Z})$ be the smallest full subcategory which is stable, closed under limits, and contains $\mathbb{Z}$. Then $\mathcal{C} = \mathcal{D}(\mathbb{Z})$. Show that this is not the case for Sp and $\mathbb{S}$ in place of $\mathcal{D}(\mathbb{Z})$ and $\mathbb{Z}$.

4. The Postnikov t-structure

Theorem 4.1. *There is a functor $\tau_{\leq 0}\colon \mathsf{Sp} \to \mathsf{Sp}$ together with a natural transformation $\mathrm{id}_{\mathsf{Sp}} \Rightarrow \tau_{\leq 0}$ so that for every spectrum X, the spectrum $\tau_{\leq 0}X$ is coconnective and the map $X \to \tau_{\leq 0}X$ is an isomorphism on π_n for $n \leq 0$. For $n \in \mathbb{Z}$ we define*

$$\tau_{\leq n} \coloneqq \Sigma^n \circ \tau_{\leq 0} \circ \Sigma^{-n}\colon \mathsf{Sp} \to \mathsf{Sp}.$$

Then $\tau_{\leq n}X$ is n-coconnective and the map $X \to \tau_{\leq n}X$ is an isomorphism in degrees $\leq n$. Moreover:

1. *For $n \in \mathbb{Z}$, we define $\tau_{\geq n}X := \mathrm{fib}(X \to \tau_{\leq n-1}X)$. Then $\tau_{\geq n}$ is n-connective and the canonical map $\tau_{\geq n}X \to X$ is an isomorphism in degrees $\geq n$. This becomes a functor, so that we have a natural cofiber sequence*
$$\tau_{\geq n}X \to X \to \tau_{\leq n-1}X.$$
2. *Both $\tau_{\geq n}$ and $\tau_{\leq n}\colon \mathsf{Sp} \to \mathsf{Sp}$ preserve products, coproducts and filtered colimits.*
3. *For any $a, b \in \mathbb{Z}$ one has $\tau_{\leq b} \circ \tau_{\geq a} \simeq \tau_{\geq a} \circ \tau_{\leq b}$, and denotes the composite by $\tau_{[a,b]}$.*

This data is called the Postnikov t-structure on Sp.

Definition 4.2. Let X be a spectrum. We say that X is

1. n-connective if $\pi_k X = 0$ for $k < n$, i.e. $\tau_{\geq n}X \xrightarrow{\simeq} X$.
2. connective if it is 0-connective.
3. bounded below if it is n-connective for some $n \in \mathbb{Z}$.
4. n-coconnective if $\pi_k X = 0$ for $k > n$, i.e. $X \xrightarrow{\simeq} \tau_{\leq n}X$.
5. coconnective if it is 0-coconnective.
6. bounded above if it is n-coconnective for some n.
7. bounded if $\pi_n X = 0$ for $|n| \gg 0$, i.e. $X \simeq \tau_{[a,b]}X$ for some $a, b \in \mathbb{Z}$.
8. static[5] if its homotopy groups are concentrated in degree 0, in which case it lies in the image of $H\colon \mathsf{Ab} \hookrightarrow \mathsf{Sp}$. The image of this fully faithful functors is also often denoted $\mathsf{Sp}^{\heartsuit}$, the "heart" of this t-structure on spectra.

We denote by $\mathsf{Sp}_{\geq n}$ respectively $\mathsf{Sp}_{\leq n}$ the full subcategory on n-connective respectively n-coconnective spectra.

Exercise 4.3. Let X be a spectrum.

1. $\mathsf{Sp}_{\geq n}$ is closed under colimits, and $\mathsf{Sp}_{\leq n}$ is closed under limits.
2. If X is m-connective and Y is n-connective, then $X \otimes Y$ is $(m+n)$-connective.
3. If (X_n) is a collection of bounded spectra, then $\bigoplus_n \Sigma^n X_n \xrightarrow{\simeq} \prod_n \Sigma^n X_n$.
4. There are natural maps $\tau_{\geq n}X \to \tau_{\geq n-1}X$. The canonical map $\mathrm{colim}_n\, \tau_{\geq -n}X \to X$ is an equivalence. The sequence we are taking the colimit along $\tau_{\geq n}X \to \tau_{\geq n-1}X \to \tau_{\geq n-2}X \to \cdots$ is often called the Whitehead tower of X.
5. There are natural maps $\tau_{\leq n+1}X \to \tau_{\leq n}X$, and the canonical map $X \to \lim_n \tau_{\leq n}X$ is an equivalence. The sequence $\cdots \to \tau_{\leq n+1}X \to \tau_{\leq n}X \to \tau_{\leq n-1}X \to \cdots$ is often called the Postnikov tower of X.
6. We have equivalences
$$\mathrm{fib}(\tau_{\leq n}X \to \tau_{\leq n-1}X) \simeq \Sigma^n H\pi_n X \simeq \mathrm{cofib}(\tau_{\geq n+1}X \to \tau_{\geq n}X).$$
In particular, we have pullback squares of spectra
$$\begin{array}{ccccccccc}
 & & \Sigma^3 H\pi_3 X & \longrightarrow & 0 & & \Sigma H\pi_1 X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & \tau_{\leq 3}X & \longrightarrow & \tau_{\leq 2}X & \longrightarrow & \tau_{\leq 1}X & \longrightarrow & \tau_{\leq 0}X \longrightarrow \cdots \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \Sigma^2 H\pi_2 X & \longrightarrow & 0 & &
\end{array}$$

[5] often also "discrete", though nowadays it makes sense to reserve this for certain kinds of objects in condensed mathematics.

Definition 4.4. One can rotate these fiber sequences to obtain the following right pullback square

$$\begin{array}{ccccc} \Sigma^n H\pi_n X & \longrightarrow & \tau_{\leq n} X & \longrightarrow & 0 \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ 0 & \longrightarrow & \tau_{\leq n-1} X & \xrightarrow{k_n} & \Sigma^{n+1} H\pi_n X \end{array}$$

The map k_n is called the n-th k-invariant of X.

$$\pi_0 \operatorname{hom}_{\mathsf{Sp}}(\tau_{\leq n-1}X, \Sigma^{n+1}H\pi_n X) \cong \pi_{-(n+1)} \operatorname{hom}_{\mathsf{Sp}}(\tau_{\leq n-1}X, H\pi_n X) =: H^{n+1}(\tau_{\leq n-1}X; \pi_n X)$$

i.e. can be viewed as a cohomology class in degree $n+1$ of $\tau_{\leq n-1}$ with coefficients in $\pi_n X \in \mathsf{Ab}$.

Remark 4.5. Note that in the case that X is rational, it follows from Exercise 2.14 that all the k-invariants vanish, which is another way to see part (3) of that exercise.

Exercise 4.6. Let $\mathcal{C} \subseteq \mathsf{Sp}$ be a full stable subcategory with $H\mathbb{Z} \in \mathcal{C}$.

(1) $\mathcal{C}$ contains all bounded spectra.
(2) If $\mathcal{C}$ is closed under sequential (co)limits, then it contains all bounded below (above) spectra. In the case that both applies, it contains all spectra.

Exercise 4.7 (Hurewicz and Whitehead)**.** Recall the natural map $\pi_*(X) \otimes \pi_*(Y) \to \pi_*(X \otimes Y)$ from Exercise 2.3(4). By restriction, we obtain a natural map

$$\pi_0(X) \otimes \pi_0(Y) \to \pi_0(X \otimes Y),\ [\mathbb{S} \xrightarrow{f} X] \otimes [\mathbb{S} \xrightarrow{g} Y] \mapsto [\mathbb{S} \simeq \mathbb{S} \otimes \mathbb{S} \xrightarrow{f \otimes g} X \otimes Y]$$

(1) Show that the above map is an isomorphism for connective X, Y. In particular, we see that $\pi_0(HA \otimes HB) \cong A \otimes B$ for abelian groups A, B, as claimed in Exercise 2.3(3).
(2) As a special case of the above, use the canonical map $\mathbb{S} \to \tau_{\leq 0}\mathbb{S} \simeq H\mathbb{Z}$ (which corresponds to $1 \in \mathbb{Z} \cong \pi_0(H\mathbb{Z}) \cong \pi_0 \operatorname{map}_{\mathsf{Sp}}(\mathbb{S}, H\mathbb{Z})$) to show that for X connective, there is a natural isomorphism
$$\pi_0(X) \xrightarrow{\cong} \pi_0(H\mathbb{Z} \otimes X) =: H_0(X; \mathbb{Z}).$$
and a surjection $\pi_1(X) \twoheadrightarrow H_1(X; \mathbb{Z})$. More generally, if X is n-connective, the map is an isomorphism in degree n and a surjection in degree $n+1$.
(3) Suppose that X is bounded below and that $X \otimes \mathbb{Z} \simeq 0$. Then $X \simeq 0$.
(4) If $f\colon X \to Y$ is a map of bounded below spectra such that $\mathbb{Z} \otimes f$ is an equivalence, then f is an equivalence.

Exercise 4.8. Let p be a prime. Show that a spectrum X is p-complete if and only if each $H\pi_n X$ is p-complete.

Exercise 4.9. Every spectrum decomposes as the direct sum of a rational spectrum and a spectrum which does not have any (non-trivial) rational subgroups in any homotopy group.

Exercise 4.10. Let $a < b$ be integers and let $\mathsf{Sp}_{[a,b]}$ denote the full subcategory subcategory on spectra with homotopy groups concentrated in degrees $a \leq k \leq b$.

(1) Let $n = b - a$, and consider a chain of composable maps $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \to \cdots \to X_n$. If all the X_i lie in $\mathsf{Sp}_{[a,b]}$ and each f_i is zero on all homotopy groups, then the composite $X_0 \to X_n$ is nullhomotopic.
(2) Conclude that if X is a bounded spectrum and p some prime such that each $\pi_n(X)$ is p-power torsion, then p is a nilpotent endomorphism of X.

Exercise 4.11. Let X be a bounded below spectrum, say (without loss of generality) connective. Show that we can write $X = \operatorname{colim}_n X_n$ and

$$\begin{array}{ccccccccccc}
& & \bigoplus_{I_1}\mathbb{S} & \longrightarrow & 0 & & \bigoplus_{I_3}\Sigma^2\mathbb{S} & \longrightarrow & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\bigoplus_{I_0}\mathbb{S} & = & X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & \cdots \\
& & & & \uparrow & & \uparrow & & & & \\
& & & & \bigoplus_{I_2}\Sigma\mathbb{S} & \longrightarrow & 0 & & & &
\end{array}$$

for suitable sets of n-cells I_n. Show also that each $H_n(X;\mathbb{Z}) = \pi_n(X \otimes \mathbb{Z})$ is finitely generated if and only if all the I_n may be chosen finite. In this case, one says that X is of finite type. By means of a spectral sequence for computing $\pi_*(X \otimes Z)$ (see [Lur, Proposition 7.2.1.17] for $R = \mathbb{S}$) and a Serre–class argument, one can show that if X is bounded below and $\pi_*(X)$ degreewise finitely generated, then also $H_*(X;\mathbb{Z})$ is, i.e. X is of finite type.

Exercise 4.12. Let X be a spectrum of finite type. Show that the functor $X \otimes -$ preserves uniformly bounded below products and sequential limits. Concretely, this means

(1) If $Y_i, i \in I$ is a collection of spectra, and there is some n such that all of them are n-connective, then the canonical map $X \otimes \prod_i Y_i \to \prod_i X \otimes Y_i$ is an equivalence
(2) If $\cdots \to Y_3 \to Y_2 \to Y_1$ is a tower of spectra and there is some n such that all the Y_k are n-connective, then $X \otimes \lim_n Y_n \to \lim_n X \otimes Y_n$ is an equivalence.

Use this to conclude that $\mathbb{S}_p^\wedge \otimes H\mathbb{Z} \simeq H\mathbb{Z}_p^\wedge$.

Similarly, show that the functor $\hom_{\mathsf{Sp}}(X, -)$ preserves uniformly bounded above filtered colimits.

Exercise 4.13 ($\star$)**.** Find an example of spectra X, Y such that X is connective and $\tau_{\geq 0}(X \otimes Y) \not\simeq \tau_{\geq 0}X \otimes \tau_{\geq 0}Y = X \otimes \tau_{\geq 0}Y$.

Exercise 4.14 ($\star\star$)**.** Find an example of spectra X, Y such that there are equivalences $\tau_{\leq n}X \simeq \tau_{\leq n}Y$ for all n, but $X \not\simeq Y$.

5. Finite Spectra

Remark 5.1. You may freely use that every filtered diagram has a cofinal map from a filtered / directed poset, so that without loss of generality, every instance of "filtered colimit" may be replaced by "colimit over a directed poset", see Kerodon 0622 for a formal statement and proof.

Definition 5.2. Let X be a spectrum. We say that X is

(1) finite if it lies in the smallest stable subcategory generated by $\mathbb{S}$.
(2) compact if $\hom_{\mathsf{Sp}}(X, -)\colon \mathsf{Sp} \to \mathsf{Sp}$ preserves filtered colimits. This is equivalent to $\operatorname{map}_{\mathsf{Sp}}(X, -)\colon \mathsf{Sp} \to \mathsf{An}$ preserving filtered colimits, or to each $\pi_n \hom_{\mathsf{Sp}}(X, -)\colon \mathsf{Sp} \to \mathsf{Ab}$ preserving filtered colimits.
(3) dualizable if there exists a spectrum DX (the dual) and evaluation / coevaluation maps $\mathrm{ev}\colon DX \otimes X \to \mathbb{S}$ and $\mathrm{cv}\colon X \otimes DX \to \mathbb{S}$ satisfying the zig-zag/triangle/snake identities, that the composites

$$X \xrightarrow{\mathrm{cv} \otimes X} X \otimes DX \otimes X \xrightarrow{X \otimes \mathrm{ev}} X \quad \text{and} \quad DX \xrightarrow{DX \otimes \mathrm{cv}} DX \otimes X \otimes DX \xrightarrow{\mathrm{ev} \otimes DX} DX$$

are both homotopic to the identity.

Denote the corresponding full subcategories on these spectra by $\mathsf{Sp}^{\mathsf{fin}}, \mathsf{Sp}^{\omega}, \mathsf{Sp}^{\mathsf{dbl}}$.

The goal of this section is to show that $\mathsf{Sp}^{\mathsf{fin}} = \mathsf{Sp}^{\omega} = \mathsf{Sp}^{\mathsf{dbl}}$, and some other characterizations of this collection of spectra.

Exercise 5.3. Show that a finite spectrum is bounded below and has finitely generated homotopy groups.

Exercise 5.4. This exercise requires a bit more background knowledge on categories.

(1) Convince yourself that a non-empty category with colimits is filtered.
(2) Let X be any spectrum and consider the pullback of categories

$$\begin{array}{ccc} \mathsf{Sp}^{\mathsf{fin}}_{/X} & \hookrightarrow & \mathsf{Sp}_{/X} \\ \downarrow & \lrcorner & \downarrow \\ \mathsf{Sp}^{\mathsf{fin}} & \hookrightarrow & \mathsf{Sp} \end{array}$$

Informally, $\mathsf{Sp}^{\mathsf{fin}}_{/X}$ is the category with objects the maps $V \to X$ where V is finite, and morphisms commuting triangles. Because $\mathsf{Sp}^{\mathsf{fin}}, \mathsf{Sp}, \mathsf{Sp}^X$ have finite colimits and the functors $\mathsf{Sp}^{\mathsf{fin}} \subseteq \mathsf{Sp}$ and $\mathsf{Sp}_{/X} \to \mathsf{Sp}$ preserve them, we see that also $\mathsf{Sp}^{\mathsf{fin}}_{/X}$ have finite colimits, and hence is filtered.
(3) Show that the canonical map $\operatorname{colim}_{V \in \mathsf{Sp}^{\mathsf{fin}}_{/X}} V \to X$ is an equivalence. In particular, Sp is generated under filtered colimits by $\mathsf{Sp}^{\mathsf{fin}}$. Since $\mathsf{Sp}^{\mathsf{fin}}$ is itself generated as a stable category by $\mathbb{S}$, we get that $\operatorname{loc}(\mathbb{S}) = \mathsf{Sp}$, proving Theorem 3.2. Hint 6

Exercise 5.5 (More on Compactness)**.**

(1) Use the fact that each π_n preserves filtered colimits to conclude that $\mathbb{S}$ is compact.
(2) Show that $\mathsf{Sp}^{\omega} \subseteq \mathsf{Sp}$ is a thick subcategory.
(3) Suppose that that $X = \operatorname{colim}_i X_i$ is a filtered colimit, and that X is compact. Show that X is a retract of one of the X_i.
(4) Let $(X_i)_i$ be a filtered diagram of spectra, without loss of generality indexed on a directed poset I. If for each i there exists some $j \geq i$ such that $X_i \to X_j$ is nullhomotopic, then $\operatorname{colim}_i X_i \simeq 0$. If all the X_i are compact, then the converse holds.

Exercise 5.6 (More on Dualizability)**.** Let X be a dualizable spectrum with dual DX.

(1) $\mathbb{S}$ is dualizable with dual $D\mathbb{S} = \mathbb{S}$.
(2) DX is dualizable with dual $DDX \simeq X$.
(3) The coevaluation and evaluation give rise to unit and counit of an adjunction $DX \otimes - \dashv X \otimes -$. Note that we also have $X \otimes - \dashv DX \otimes -$.
(4) We have $DX \simeq \hom(X, \mathbb{S})$.
(5) Under the equivalence $DX \simeq \hom(X, \mathbb{S})$, the evaluation ev of the duality datum corresponds to the counit $\varepsilon^X_{\mathbb{S}} \colon \hom(X, \mathbb{S}) \otimes X \to \mathbb{S}$ of the adjunction $X \otimes - \dashv \hom(X, -)$ evaluated at $\mathbb{S}$.
(6) For a spectrum Y, the following are equivalent:
   (a) Y is dualizable.
   (b) There exists a map $c \colon \mathbb{S} \to Y \otimes \hom(Y, \mathbb{S})$ such that the following diagram commutes

$$\begin{array}{ccccc} & & Y \otimes \hom(Y, \mathbb{S}) \otimes Y & & \\ & \overset{c \otimes Y}{\nearrow} & & \overset{Y \otimes \varepsilon^Y_{\mathbb{S}}}{\searrow} & \\ Y & & = & & Y \end{array}$$

Hint 6 Use that π_n preserves filtered colimits, argue for surjectivity and injectivity separately.

(c) For every spectrum Z, the canonical map $\varphi_Z\colon \hom(Y,\mathbb{S})\otimes Z \to \hom(Y,Z)$ is an equivalence. This map is adjoint to $\hom(Y,\mathbb{S})\otimes Y\otimes Z \xrightarrow{\varepsilon_{\mathbb{S}}^Y\otimes Z} Z$.
(d) $\varphi_Y\colon \hom(Y,\mathbb{S})\otimes Y \to \hom(Y,Y)$ is an equivalence.
(e) $Y\otimes -$ preserves limits, i.e. is a right adjoint (Sp is presentable and the adjoint functor theorem applies.)[7]

(7) Conclude from (c) that $\mathsf{Sp}^{\mathsf{dbl}}\subseteq \mathsf{Sp}$ is a thick subcategory.

Exercise 5.7. Here we show that $\mathsf{Sp}^{\mathsf{fin}} = \mathsf{Sp}^{\mathsf{dbl}} = \mathsf{Sp}^{\omega}$.

(1) Conclude from Exercises 5.4, 5.5 and 5.6 that $\mathsf{Sp}^{\mathsf{fin}} \subseteq \mathrm{Thick}(\mathbb{S}) = \mathsf{Sp}^{\mathsf{dbl}} = \mathsf{Sp}^{\omega}$ (note that $\mathrm{Thick}(\mathbb{S})$ is just the retract-closure of $\mathsf{Sp}^{\mathsf{fin}}$).
(2) Show that the following full subcategory of spectra is thick:

$$\mathcal{C} = \{X\in \mathsf{Sp} \mid X \text{ bounded below and } H_*(X;\mathbb{Z}) \text{ finitely generated}\}$$

where finitely generated does not mean degreewise, but $\bigoplus_{n\in\mathbb{Z}} H_n(X;\mathbb{Z})$ should be finitely generated, so in particular only be non-zero in finitely many degrees.
(3) Show that $\mathcal{C} = \mathsf{Sp}^{\mathsf{fin}}$. Conclude that $\mathsf{Sp}^{\mathsf{fin}}$ is already thick, hence agrees with Sp^{ω}. Hint 8
(4) Finite spectra are closed under tensor products.

Exercise 5.8. Let X be finite and Y have finitely generated homotopy groups. Then also $\hom(X,Y)$ has finitely generated homotopy groups $\hom(X,Y)$.

Exercise 5.9. Show that the set of isomorphism classes of finite spectra is countable (and the group of maps between any two of these is also countable, in fact finitely generated).

Exercise 5.10 (Invertible Spectra)**.** Call a spectrum X invertible if there exists a spectrum X^{-1} with $X\otimes X^{-1}\simeq \mathbb{S}$. Show that if X is invertible, there is some $n\in\mathbb{Z}$ and an equivalence $X\simeq \Sigma^n\mathbb{S}$. Feel free to use Exercises 2.14(6) and 2.15.

Exercise 5.11 ($\star$)**.** Show that if a finite spectrum is bounded above, then it is already zero.

Exercise 5.12 ($\star$)**.** An object X in a category $\mathcal{C}$ is called projective if $\mathrm{map}_{\mathcal{C}}(X,-)$ preserves geometric realizations. Show that the only projective object in Sp is 0.[9]

Exercise 5.13 ($\star$)**.** Suppose we have a map of spectra $\eta\colon \mathbb{S}\to R$ such that $R\otimes(\mathbb{S}\to R)\colon R\to R\otimes R$ is an equivalence (one says that R is an idempotent ring spectrum). Show that if R is also dualizable, then $\mathbb{S}\to R$ is a split epimorphism, i.e. R is a retract of $\mathbb{S}$.

Exercise 5.14 ($\star$)**.** Let κ be an infinite regular cardinal. There is an associated notion of κ-filtered category / κ-filtered poset, where instead of asking for cones over finite subdiagrams, we ask for the existence of cones over κ-small subdiagrams. Here κ-small always means cardinality *less* than κ. In the case $\kappa=\omega$ we recover our previous notion of filtered diagrams. As before, one defines κ-compact objects as those for which $\mathrm{map}(X,-)$ preserves κ-filtered colimits. For an in-depth treatment of these notions, I recommend [Hau25, Section 9]. Here, we will only consider them specifically in spectra. Let $\mathcal{C}\subseteq\mathsf{Sp}$ be the smallest stable stable subcategory that contains $\mathbb{S}$ and is closed under κ-small colimits, and let Sp^{κ} be the full subcategory on κ-compact spectra.

(1) Show that $\mathcal{C}\subseteq \mathsf{Sp}^{\kappa}$.

[7] Warning: The equivalence of this point with the remaining ones is a special property of Sp, and is generally false for other stable presentably symmetric monoidal categories $\mathcal{C}$. The crucial point is that we need the adjunction to be $\mathcal{C}$-linear. For Sp this is automatic since Sp is idempotent in $\mathsf{Pr}^L_{\mathsf{st}}$).

Hint 8 For the inclusion $\supseteq$, argue by induction over the lowest non-trivial homology group and use Exercise 3.3(2).

[9] This is in fact true in any stable category.

(2) Show that $\mathcal{C}$ is small, i.e. has only a set of isomorphism classes of objects. Hint 10
(3) Convince yourself that Exercise 5.5(3,4) still hold true for "compact" replaced by "κ-compact".
(4) Convince yourself that the same argument as in Exercise 5.4 shows that $\mathcal{C}$ generates Sp under κ-filtered colimits.
(5) Note that $\mathcal{C}$ is closed under retracts (for $\kappa = \omega$ we showed this in Exercise 5.7, and for uncountable κ this follows since $\mathcal{C}$ is closed under sequential colimits) Conclude that $\mathcal{C} = \mathsf{Sp}^\kappa$.
(6) For uncountable κ, a spectrum X is κ-compact if and only if $|\pi_* X| < \kappa$. To this end, proceed in the following steps:[11]
   (a) $\mathcal{D} \coloneqq \{X \in \mathsf{Sp} \mid |\pi_* X| < \kappa\}$ is thick and closed under κ-small coproducts.
   (b) Convince yourself that $\mathcal{D}$ is therefore closed under κ-small colimits, and conclude $\mathsf{Sp}^\kappa \subseteq \mathcal{D}$.
   (c) Given $X \in \mathcal{D}$, show that $F \coloneqq \bigoplus_{\alpha \in \pi_* X} \Sigma^{|\alpha|}\mathbb{S}$ lies in Sp^κ and the map $X \to \operatorname{cofib}(F \to X)$ is zero on all homotopy groups.
   (d) Use the previous point to inductively construct a sequence $X_0 \to X_1 \to \cdots$ so that $\operatorname{colim}_n X_n \simeq X$ and each X_n lies in Sp^κ. Conclude that $X \in \mathsf{Sp}^\kappa$, and hence $\mathsf{Sp}^\kappa = \mathcal{D}$.

6. Bousfield Localizations of Spectra

Exercise 6.1. We show that $\langle I_{\mathbb{Q}/\mathbb{Z}} \rangle$ contains many spectra.
(1) Show that $\mathbb{Z} \otimes I_{\mathbb{Q}/\mathbb{Z}} = 0$.
(2) Conclude that $I_{\mathbb{Q}/\mathbb{Z}}$ kills all bounded above spectra, and hence $I_{\mathbb{Q}/\mathbb{Z}} \otimes I_{\mathbb{Q}/\mathbb{Z}} = 0$. In particular, this gives an example of a non-trivial spectrum E which is not E-local!
(3) $I_{\mathbb{Q}/\mathbb{Z}}$ kills every homotopy MU-module, hence every homotopy-module over every complex-orientable ring spectrum.
(4) $I_{\mathbb{Q}/\mathbb{Z}}$ does not kill any finite spectrum.
(5) Find a non-finite spectrum that is not killed by $I_{\mathbb{Q}/\mathbb{Z}}$.

7. Rings and Modules in Spectra

Exercise 7.1. Every $\mathbb{Z}$-module splits.

Exercise 7.2 (Connective Idempotent Spectra)**.** Call a ring spectrum idempotent if the canonical map $R \xrightarrow{\eta \otimes R} R \otimes R$ is an equivalence. Show that if R is connective and idempotent, then there is an equivalence $R \simeq HA$ where $\mathbb{Z} \leq A \leq \mathbb{Q}$.

8. Phantom Maps

Phantom maps are one crucial distinction of Sp and more algebraic categories such as $\mathcal{D}(\mathbb{Z})$.

Definition 8.1. A map $\varphi\colon X \to Y$ of spectra is called phantom if for every map $F \to X$ from a finite spectrum F, the composite $F \to X \to Y$ is nullhomotopic.

Exercise 8.2. Let $\varphi\colon X \to Y$ be a map of spectra.
(1) phantom maps form a 2-sided ideal, i.e. if φ is phantom then also $f\varphi g$ is phantom for suitably composable maps f, g.
(2) The following are equivalent:
   (a) φ is phantom.

Hint 10 Start with $\mathsf{Sp}^{\mathsf{fin}}$ and inductively add κ-small colimits. Show that each step stays small, and this process terminates at stage κ.

[11] I learned this proof strategy from Denis Nardin.

   (b) $\pi_*(\varphi \otimes V)$ is null for every spectrum V.
   (c) $\pi_*(\varphi \otimes V)$ is null for every finite spectrum V.
(3) For any collection of spectra $(X_i)_{i \in I}$, consider the map inc: $\bigoplus_i X_i \to \prod_i X_i$. Then $\mathrm{fib}(\mathrm{inc}) \to \bigoplus_i X_i$ is phantom.
(4) For any spectrum X, we have a cofiber sequence $\bigoplus_{F \in \mathsf{Sp}^{\mathrm{fin}}_{/X}} F \to X \xrightarrow{\varphi} \widetilde{X}$, where the first map is the obvious one, and φ is a phantom map. Show that every phantom map out of X factors through φ.
(5) Conclude that there are no non-zero phantom maps out of X if and only if X is a retract of a direct sum of finite spectra.

Note that since phantom maps are zero on homotopy groups, it follows that a countably infinite composition (sequential colimit) of them is 0. Moreover, Exercise 4.10 shows that on the subcategory of spectra bounded in a fixed finite range, phantom maps are nilpotent. As it turns out, something much stronger is true: phantom maps form a square-zero ideal!

Theorem 8.3 ([CS98, Corollary 4.7])**.** *The composite of two phantom maps is zero.*

Exercise 8.4. Use the above theorem and Exercise 8.2(4) to show that every spectrum is a retract of a spectrum X sitting in a cofiber sequence of the form $\bigoplus_i F_i \to \bigoplus_j F'_j \to X$, where all the F_i and F'_j are finite spectra. In particular, we have $\mathrm{Thick}(\mathcal{C}) = \mathsf{Sp}$ where $\mathcal{C}$ is the full subcategory of spectra on direct sums of finite spectra.

Exercise 8.5. We investigate how phantom maps relate to Brown-Comenetz duality. Let Z be a spectrum such that each $\pi_n Z$ is finitely generated.

(1) For any spectrum X, the map $\mathrm{fib}(X \to I^2_{\mathbb{Q}/\mathbb{Z}} X) \to X$ is phantom.
(2) A map $X \to I_{\mathbb{Q}/\mathbb{Z}} Y$ is phantom if and only if it is zero.
(3) If $\varphi\colon X \to Y$ is phantom, then $I_{\mathbb{Q}/\mathbb{Z}} \varphi$ is zero.
(4) A map $X \to Y$ is phantom if and only if $X \to Y \to I^2_{\mathbb{Q}/\mathbb{Z}} Y$ is zero. Conclude that there exists no non-zero phantom map to Y if and only if $Y \to I^2_{\mathbb{Q}/\mathbb{Z}} Y$ is a split monomorphism.
(5) $\mathrm{fib}(Z \to I^2_{\mathbb{Q}/\mathbb{Z}} Z)$ is rational.
(6) The subgroup of phantom maps $\mathrm{Phantom}(X, Z) \leq \pi_0 \hom(X, Z)$ is divisible.
(7) There exists a non-zero phantom map to Z if and only if $Z_{\mathbb{Q}} \neq 0$.
(8) $\varphi\colon X \to Z$ is phantom if and only if $[\varphi] \in \pi_0 \hom(X, Z)$ is divisible Hint 12
(9) For any X, the following are equivalent:
   (a) $X \otimes I_{\mathbb{Q}/\mathbb{Z}} Z = 0$.
   (b) $\mathrm{Phantom}(V \otimes X, Z) = \pi_0 \hom(V \otimes X, Z)$ for all V.
   (c) $\mathrm{Phantom}(V \otimes X, Z) = \pi_0 \hom(V \otimes X, Z)$ for all finite V.
   (d) $\hom(X, Z)$ is rational.
(10) All maps from a bounded above spectrum to a finite spectrum are phantom.

Exercise 8.6. By the previous exercise, we see that $\hom(\mathbb{Z}, \mathbb{S})$ is rational, and consists only of phantom maps. Show that it is given by $\Sigma^{-1} H\,\mathrm{Ext}(\mathbb{Q}, \mathbb{Z})$. Similarly, in combination with Exercise 6.1, we see that $\hom(X, \mathbb{S})$ is rational for many non-finite spectra, which then allows for an easy calculation of said hom-spectrum.

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