Parametrized Higher Algebra and Global Picard Spectra

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Born 7th November 1999 in Essen, Germany

26th February 2024

Master's Thesis Mathematics

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Abstract

This thesis consists of two parts; we first develop some foundations for parametrized higher algebra with respect to so-called orbital subcategories P of arbitrary indexing ∞-categories T, purely from the perspective of categorical Mackey functors. The second part answers a question of Schwede on the existence of global Picard spectra associated to his ultracommutative ring spectra; given an ultracommutative ring spectrum R, we show there exists a global spectrum $\mathsf{pic}_{\mathsf{gl}}(R)$ with $\mathsf{pic}_{\mathsf{gl}}(R)^G \simeq \mathsf{pic}(\mathsf{Mod}_{\mathrm{res}_G R}(\mathsf{Sp}^G))$ for all finite groups G.

More specifically, using the framework of algebraic patterns of [BHS22], we define P-symmetric monoidal T-∞-categories and their commutative algebras in an analogous way to the normed categories of [BH17], and compare our definitions to those of [NS22] and to the P-commutative T-monoids of [CLL23a]. Moreover, we construct parametrized symmetric monoidal module categories using the techniques of [LNP22]. We investigate a generalization of the classical "Borelification" construction, which in the G-equivariant case has already been done in [Hil24], where it enhances a symmetric monoidal ∞-category with G-action to a G-symmetric monoidal one.

In the second part we then focus on constructing the equivariantly symmetric monoidal structures on the global categories equivariant and global spectra, and use the results from the first part to define global Picard spectra.

Contents

1 Introduction

Global Homotopy Theory

Global homotopy theory studies objects which admit simultaneous and suitably compatible actions by all compact Lie groups or some family of subgroups thereof. For example, we have a G-equivariant K-theory spectrum $\mathrm{KU}_G \in \mathsf{Sp}^G$ for every compact Lie group G, and these assemble into one coherent object in Schwede's ∞-category of global spectra $\mathsf{Sp}^{\mathsf{gl}}_{\mathsf{Lie}}$, see [Sch18, Chapter 6]. The intuition here, originally conjectured by Schwede and later proven by Linskens, Nardin and Pol in [LNP22], is that a global spectrum $X \in \mathsf{Sp}^{\mathsf{gl}}_{\mathsf{Lie}}$ should consist of the following data:

- An underlying genuine G-spectrum $X_G \in \mathsf{Sp}^G$ for every compact Lie group G,
- comparison maps $f_\alpha : \alpha^* X_K \to X_G$ in Sp^G for all continuous group homomorphisms $\alpha : G \to K$,
- a homotopy between f_{c_g} induced by the conjugation isomorphism and the map $\ell_g : c_g^* X_G \to X_G$ given by left multiplication with g,
- higher coherences for the homotopies.

This data is required to satisfy the following conditions:

- the maps f_α are functorial, i.e. $f_{\beta\circ\alpha} \simeq f_\beta \circ \beta^*(f_\alpha)$ whenever α and β are composable. In particular, $f_{\mathrm{id}} = \mathrm{id}$,
- for every injection $i : H \hookrightarrow G$, the map f_i is an equivalence.

Formally, one considers the global indexing ∞-category of compact Lie groups $\mathsf{Glo}_{\mathsf{Lie}}$ which essentially consists of compact Lie groups and continuous group homomorphisms up to conjugation. One can construct a functor $\mathsf{Glo}_{\mathsf{Lie}}^{\mathsf{op}} \to \mathsf{CMon}(\mathsf{Cat}), G \mapsto \mathsf{Sp}^G$ encoding the symmetric monoidal structure on genuine G-spectra with its functoriality in restriction of the group action. Then [LNP22, Theorem 11.10] provides a symmetric monoidal equivalence $\mathsf{Sp}^{\mathsf{gl}}_{\mathsf{Lie}} \simeq \mathrm{laxlim}^{\dagger}{}_{G\in\mathsf{Glo}_{\mathsf{Lie}}^{\mathsf{op}}} \mathsf{Sp}^G$ between Schwede's ∞-category of global spectra and the partially lax limit over this functor, with marked edges the injective group homomorphisms. In the present text, we will only be concerned with finite groups, which Schwede calls Fin-global homotopy theory. So for us, a global spectrum is actually a Fin-global spectrum in his sense, and we denote the corresponding category by $\mathsf{Sp}^{\mathsf{gl}}$.

Succinctly, the goal of this thesis is to generalize the classical notion of a Picard spectrum to the context of equivariant and global homotopy theory. Let us first recall this classical case.

Picard Spectra

The Picard group of a (symmetric) monoidal category is an interesting invariant capturing information about objects which are invertible with respect to the monoidal structure. Classically, one considers line bundles with their tensor product on some geometric object. For example, on a paracompact Hausdorff space X, the Picard group of real respectively complex line bundles is isomorphic to the cohomology group $H^1(X;\mathbb{Z}/2)$ respectively $H^2(X;\mathbb{Z})$. In the context of higher category theory, a symmetric monoidal ∞-category $\mathcal{C}$ has an underlying $\mathbb{E}_\infty$-monoid $\mathcal{C}^{\simeq}$ in spaces, and taking units we obtain an $\mathbb{E}_\infty$-group $(\mathcal{C}^{\simeq})^\times$ which is often called the Picard space of $\mathcal{C}$. By May's recognition principle, we may equivalently consider this as a connective spectrum, the so-called Picard spectrum $\mathsf{pic}(\mathcal{C})$ of $\mathcal{C}$. Most of the interesting information lies in the Picard group $\pi_0 \,\mathsf{pic}(\mathcal{C})$, but we prefer to work with the whole Picard space or spectrum for better categorical properties such as descent. One of the first interesting examples one encounters is that of the ∞-category of spectra Sp; one checks that the only invertible spectra are shifts of the sphere, and the Picard group is thus isomorphic to $\mathbb{Z}$. More generally, one often considers $\mathbb{E}_\infty$-rings $R \in \mathrm{CAlg}(\mathsf{Sp})$, and the Picard groups/spectra of their symmetric monoidal module categories $\mathsf{pic}(R) \coloneqq \mathsf{pic}(\mathsf{Mod}_R(\mathsf{Sp}))$ to learn about $\otimes_R$-invertible R-modules. This can be made into a functor $\mathsf{pic} : \mathrm{CAlg}(\mathsf{Sp}) \to \mathsf{Sp}, R \mapsto \mathsf{pic}(R)$, which is what we are interested in generalizing to the setting of equivariant and global homotopy theory. In the equivariant case, the idea is that given a strictly commutative orthogonal G-ring spectrum R, we should be able to construct a G-spectrum $\mathsf{pic}_G(R)$ so that $\pi_0^H(\mathsf{pic}_G(R)) \cong \pi_0 \,\mathsf{pic}(\mathsf{Mod}_{\mathrm{res}^G_H R}(\mathsf{Sp}^H))$ for $H \leq G$.

Ultracommutative Ring Spectra

In trying to adapt the construction of pic to the world of equivariant homotopy theory, one is led to the following questions: What sort of algebraic structure is needed on a G-space to deloop it to a *genuine* G-spectrum, and with which algebraic structure do we need to endow a genuine G-spectrum so that we can build a G-Picard spectrum of its module category in a manner analogous to the classical Picard-spectrum construction above? The former is answered by the field of equivariant infinite loop space theory, where Costenoble and Waner [CW91] developed the notion of a G-$\mathbb{E}_\infty$-operad as the correct equivariant generalization of $\mathbb{E}_\infty$-operads. Their algebras, the G-E_∞-monoids, are endowed with (coherently commutative, associative and unital) equivariant multiplications $\mathrm{Coind}_H^G \mathrm{res}_H^G X \to X$, where one thinks of the domain as an indexed product $\prod_{G/H} X$ endowed with a mixed G-action that both permutes the factors via the action on G/H and acts diagonally on each factor. Strictly commutative topological G-spaces automatically admit such equivariant multiplications, but homotopy-coherently this is extra data, which has been formalized into the notion of a G-$\mathbb{E}_\infty$-operad. For example, this extra structure allows one to define transfers on homotopy groups to obtain a Mackey-Functor $G/H \mapsto \pi_0(X^H)$ which is to be expected since this structure is already present on π_0 of any genuine G-spectrum. And indeed, it was shown that such grouplike G-$\mathbb{E}_\infty$-monoids in spaces deloop to genuine G-spectra.

The theory of G-$\mathbb{E}_\infty$-operads applies just as well to G-spectra. Indeed, the effective use of the multiplicative norms by Hill-Hopkins-Ravenel in their landmark solution of the Kervaire Invariant One Problem [HHR16] inspired Blumberg-Hill [BH15] to initiate a detailed study of the different kinds of equivariant levels of commutativity between "naive $\mathbb{E}_\infty$" and "G-$\mathbb{E}_\infty$" in both G-spaces and G-spectra, giving rise to their N_∞-operads. As in G-spaces, a G-$\mathbb{E}_\infty$-ring spectrum is endowed with equivariant multiplications $N_H^G \operatorname{res}_H^G R \to R$ where $N_H^G : \mathsf{Sp}^H \to \mathsf{Sp}^G$ is the multiplicative norm, which can be thought of an indexed tensor product $\bigotimes_{G/H}$ analogously, and these allow one to define multiplicative transfers on the homotopy groups, which ultimately endow $\underline{\pi}_0(R)$ with the structure of a Tambara Functor, c.f. [Bru07, Section 7.2].

The corresponding notion in global homotopy theory has been introduced by Schwede in [Sch18, Chapter 5] under the name of ultracommutative global ring spectra. As in the equivariant case, such a spectrum admits significantly more structure than a (coherently) commutative global ring spectrum, i.e. an $\mathbb{E}_\infty$-algebra in some category of global spectra. Schwede simply defined these as strictly commutative ring spectra in his model category of global spectra, but there is also an analogous notion of global-$\mathbb{E}_\infty$-operad developed by Barrero [Bar23b]. Some evidence for the existence of global Picard spectra has been given in [Sch18, Remark 5.1.18], but Schwede was unable to construct them in his model.

Parametrized Higher Algebra

Ideally one would like a purely ∞-categorical framework to reason about ultracommutative equivariant and global ring spectra. We claim that the recent methods of parametrized higher category theory and higher algebra are suitable for this.

The field of parametrized higher category theory as originally developed by Barwick, Dotto, Glasman, Nardin and Shah [BDG$^+$16a, BDG$^+$16b, Nar16] was inspired by the perspective on equivariant stable homotopy theory developed by Hill-Hopkins-Ravenel [HHR16], which centers around the study of indexed (co)products and indexed symmetric monoidal structures (incorporating the norms). In G-equivariant category theory, one works with so-called G-∞-categories, which are functors $\mathsf{Orb}_G^{\mathsf{op}} \to \mathsf{Cat}_\infty$, and tries to lift all usual categorical and higher algebraic notions into this setting. For example, by now we have established notions of G-adjunctions, G-localizations, G-(co)limits, G-presentability, G-stability, etc... Equivalently, this can be viewed as doing higher category theory internal to the ∞-topos of G-spaces $\mathsf{Spc}^G \simeq \mathrm{PSh}(\mathsf{Orb}_G^{\mathsf{op}})$ as opposed to spaces Spc. This has been investigated in a series of papers by Martini and Wolf, see e.g. [MW24].

In the case of global homotopy theory, one uses the global indexing category Glo, the $(2,1)$-category of finite groups, group homomorphisms and conjugations, or equivalently the full subcategory of Spc on the finite connected 1-groupoids. For example, in [CLL23b] Cnossen, Lenz and Linskens construct the global category of equivariant spectra $\underline{\mathsf{Sp}} : \mathsf{Glo}^{\mathsf{op}} \to \mathsf{Cat}_\infty, G \mapsto \mathsf{Sp}^G$ with restriction functoriality, and prove that it admits a universal property analogous to the ∞-category of spectra Sp being the free stable presentable ∞-category on one generator. In his PhD-thesis [Len21], Lenz has introduced the category of G-global spectra $\mathsf{Sp}^{G\text{-gl}}$, as a common generalization of both G-equivariant spectra Sp^G and global spectra $\mathsf{Sp}^{\mathsf{gl}}$. Specifically, Sp^G is both a left- and right Bousfield localization of $\mathsf{Sp}^{G\text{-gl}}$, and the categories of G-global spectra are again contravariantly functorial in the group via restriction of the action, giving rise to the global category of global spectra $\underline{\mathsf{Sp}}^{\mathsf{Orb}}_{\mathsf{Glo}} : \mathsf{Glo}^{\mathsf{op}} \to \mathsf{Cat}_\infty, G \mapsto \mathsf{Sp}^{G\text{-gl}}$ constructed in [CLL23a]. As shown there, it also admits a universal property analogous to that of Sp.

An extensive study of parametrized higher algebra has already been carried out by Nardin and Shah in [NS22], however their theory is built entirely internal to parametrized higher category theory and thus requires a significant amount of familiarity with it. Instead, we will follow the ideas of Bachmann-Hoyois [BH17] and develop some basic building blocks of parametrized higher algebra viewing symmetric monoidal categories as categorical Mackey functors.

For example, a G-symmetric monoidal structure on a G-category $\mathsf{Orb}_G^{\mathsf{op}} \to \mathsf{Cat}_\infty$ is an extension to a categorical G-Mackey functor $\mathrm{Span}(G) \xrightarrow{\times} \mathsf{Cat}_\infty$, analogously to how non-parametrized symmetric monoidal ∞-categories can be viewed as finite-product preserving functors $\mathrm{Span}(\mathbb{F}) \xrightarrow{\times} \mathsf{Cat}_\infty$. It has been shown in [NS22, Theorem 2.3.9] that this agrees with their notion of G-symmetric monoidal ∞-category, and more recently Barkan, Haugseng and Steinebrunner have shown in [BHS22] that the G-∞-operads developed by Nardin-Shah can equivalently be viewed as certain fibrations over $\mathrm{Span}(G)$ satisfying conditions analogous to operads over the base-category of finite pointed sets $\mathbb{F}_*$. This is the viewpoint we will take in this document, and we will recall the important constructions and theorems of [BHS22] in the main text.

1.1 Sketch of the main constructions and results

There is a model structure on strictly commutative symmetric ring spectra modeling the homotopy theory of ultracommutative ring spectra. We denote its underlying ∞-category by UCom. Analogously, we have an ∞-category UCom_G underlying strictly commutative symmetric G-ring spectra. The main goal of this thesis is the following theorem

Theorem A (Constructions 5.13, 5.14 and 5.16)**.** There exist functors

$$\mathsf{pic}_{\mathsf{gl}}, \mathsf{pic}_{\mathsf{eqv}} : \mathsf{UCom} \to \mathsf{Sp}^{\mathsf{gl}}$$

which send an ultracommutative ring spectrum $R \in \mathsf{UCom}$ to global spectra that satisfy $\mathsf{pic}_{\mathsf{gl}}(R)^G \simeq \mathsf{pic}(\mathsf{Mod}_{\mathrm{res}_G R}(\mathsf{Sp}^{G\text{-gl}}))$ and $\mathsf{pic}_{\mathsf{eqv}}(R)^G \simeq \mathsf{pic}(\mathsf{Mod}_{\mathrm{res}_G R}(\mathsf{Sp}^G))$ for every finite group G. Analogously, for a fixed finite group G there exists a functor $\mathsf{pic}_G : \mathsf{UCom}_G \to \mathsf{Sp}^G$ with $\mathsf{pic}_G(R)^H \simeq \mathsf{pic}(\mathsf{Mod}_{\mathrm{res}_H R}(\mathsf{Sp}^H))$.

All three functors are constructed in the same way. Let us sketch how pic_G is constructed, which highlights some special cases of the main results of this thesis. First, we need to construct a G-symmetric monoidal G-category of genuine G-spectra $\underline{\mathsf{Sp}}_G^{\otimes} \in \mathsf{Mack}_G(\widehat{\mathsf{Cat}}(\mathsf{sift}))$. This latter category is the category of G-Mackey functors valued in the category $\widehat{\mathsf{Cat}}(\mathsf{sift})$ of large categories admitting sifted colimits and functors preserving these. To do this, we consider the symmetric monoidal model category of symmetric spectra $\mathsf{Sp}^\Sigma \in \mathsf{Mack}(\mathsf{Cat}) = \mathsf{Fun}^\times(\mathrm{Span}(\mathbb{F}), \mathsf{Cat}) \simeq \mathsf{CMon}(\mathsf{Cat})$. In Section 3.2 we construct the "Borelification" functor $\mathrm{Bor}_G : \mathsf{Mack}(\mathsf{Cat})^{BG} \to \mathsf{Mack}_G(\mathsf{Cat})$ where $\mathrm{Bor}_G(\mathrm{infl}_G \mathsf{Sp}^\Sigma)(G/H)$ encodes the pointwise symmetric monoidal structures on $\mathsf{Fun}(BH, \mathsf{Sp}^\Sigma)$ together with symmetric monoidality of restrictions and indexed tensor products. We can then invert the H-stable equivalences in level H to obtain the desired functor $\underline{\mathsf{Sp}}^\otimes_G$, compare Definition 4.18 and Construction 4.14. To compare strictly commutative G-ring spectra to our notion of G-commutative algebras in $\underline{\mathsf{Sp}}^\otimes_G$, we have the following theorem:

Theorem B (Theorem 3.15)**.** Let G be a finite group and $s : \mathrm{Span}(\mathbb{F}) \to \mathrm{Span}(G)$ be induced by $* \mapsto G/G$. Restriction along s induces an equivalence natural in $\mathcal{C} \in \mathsf{Mack}(\mathsf{Cat})^{BG}$:

$$\mathrm{ev}_{G/G} = s^* : \mathrm{CAlg}_G(\mathrm{Bor}_G(\mathcal{C})) \xrightarrow{\simeq} \mathrm{CAlg}(\mathcal{C}^{hG}).$$

Using this theorem, we can then construct a comparison functor $\Phi_G : \mathsf{UCom}_G \to \mathrm{CAlg}_G(\underline{\mathsf{Sp}}^\otimes_G)$ which we conjecture to be an equivalence, see Construction 4.23. In the global case we instead use version of the above theorem for arbitrary "Borel inclusions", see Proposition 3.7.

We want to define the spectral Mackey functor $\mathsf{pic}_G(R) \in \mathsf{Mack}_G(\mathsf{Sp})$, and working backwards the last step is given by applying $\mathsf{pic}_* : \mathsf{Mack}_G(\mathsf{Cat}) \simeq \mathsf{Mack}_G(\mathsf{CMon}(\mathsf{Cat})) \to \mathsf{Mack}_G(\mathsf{Sp})$ to a certain parametrized module category $\underline{\mathsf{Mod}}_R(\underline{\mathsf{Sp}}^\otimes_G)$. To construct the latter, we prove a much more general version of the following theorem:

Theorem C (Theorem 2.37)**.** Let $\mathcal{C} \in \mathsf{Mack}_G(\mathsf{Cat}(\{\Delta^{\mathsf{op}}\}))$ be a G-symmetric monoidal G-category compatible with geometric realizations. Then there exists a functor

$$\underline{\mathsf{Mod}}_{(-)}(\mathcal{C}) : \mathrm{CAlg}_G(\mathcal{C}) \to \mathsf{Mack}_G(\mathsf{Cat})$$

with $\underline{\mathsf{Mod}}_R(\mathcal{C})(G/H) = \mathsf{Mod}_{R(G/H)}(\mathcal{C}(G/H))$. A morphism $f : R \to S$ is sent to $S \otimes_R - : \underline{\mathsf{Mod}}_R(\mathcal{C}) \to \underline{\mathsf{Mod}}_S(\mathcal{C})$ which at $G/H \in \mathsf{Orb}_G^{\mathsf{op}}$ is given by the symmetric monoidal left adjoint $S(G/H) \otimes_{R(G/H)} - :$ $\mathsf{Mod}_{R(G/H)}(\mathcal{C}(G/H)) \to \mathsf{Mod}_{S(G/H)}(\mathcal{C}(G/H))$.

Finally, one can define pic_G by considering the composite

$$\mathsf{UCom}_G \xrightarrow{\Phi_G} \mathrm{CAlg}_G(\underline{\mathsf{Sp}}^\otimes_G) \xrightarrow{\underline{\mathsf{Mod}}_{(-)}(\underline{\mathsf{Sp}}^\otimes_G)} \mathsf{Mack}_G(\mathsf{Cat}) \xrightarrow{\mathsf{pic}_*} \mathsf{Mack}_G(\mathsf{Sp}) \simeq \mathsf{Sp}^G$$

In the global case, this last equivalence uses a spectral Mackey functor description $\mathsf{Sp}^{\mathsf{gl}} \simeq \mathsf{Fun}^\times(\mathrm{Span}_{\mathsf{all},\mathsf{Orb}}(\mathbb{F}_{\mathsf{Glo}}), \mathsf{Sp})$ of global spectra. This is also a consequence of the following comparison theorem, c.f. Remark 2.57

Theorem D (Theorem 2.54 and Corollary 2.56)**.** Let $P \subset T$ be atomic orbital. There is an equivalence of T-categories natural in $\mathcal{E} \in \mathsf{Cat}^\times$

$$\mathsf{Fun}^\times(\mathrm{Span}_{\mathsf{all},P}((\mathbb{F}_T)_{/\bullet}), \mathcal{E}) = \underline{\mathsf{Mack}}^P_T(\mathcal{E}) \simeq \underline{\mathsf{CMon}}^P_T(\underline{\mathcal{E}}_T).$$

Here the right hand side is the T-category of P-commutative T-monoids from [CLL23a] and the functoriality on the left side is induced from the postcomposition functoriality of the slice. As a consequence one also obtains a parametrized spectral Mackey functor description

$$\underline{\mathsf{Mack}}_T^P(\mathsf{Sp}) \simeq \underline{\mathsf{Sp}}_T^P$$

of the T-category of P-genuine T-spectra $\underline{\mathsf{Sp}}_T^P$ from [CLL23a].

1.2 Outline

Although the original goal of this thesis is the construction of equivariant and global Picard spectra as detailed in the Introduction above, the main content of this thesis are the methods and constructions in parametrized higher algebra developed to do this. The actual construction of these Picard spectrum functors will follow by piecing together our general results. Moreover, although we are ultimately interested in the equivariant and global versions of most results, will follow [CLL23a] and work in the generality of higher category theory parametrized by an arbitrary (small) base ∞-category T and some orbital subcategory $P \subset T$ determining the "level of commutativity".

We begin in Section 2 by introducing the framework we will be working in for most of this thesis; algebraic patterns and the main theorems of [BHS22]. We go on to define our model for parametrized symmetric monoidal categories as categorical Mackey functors. For these we can define categories of parametrized commutative algebras, and (under certain hypotheses) parametrized symmetric monoidal module categories, see Section 2.2 and Section 2.3. In Section 2.4 we compare our Mackey functors to the P-commutative T-monoids of [CLL23a], which also yields a Mackey functor description of their T-category of P-genuine T-spectra. We also compare our definitions of parametrized symmetric monoidal categories and commutative algebras therein to those of [NS22].

Section 3 will focus on a parametrized symmetric monoidal generalization of "Borelification" – the classical construction which embeds Borel-G-equivariant objects[1] into genuinely G-equivariant objects. In the G-equivariant case this has already been considered in [Hil24]. Here one constructs a G-symmetric monoidal ∞-category $\mathrm{Bor}_G(\mathcal{C})$ from a symmetric monoidal ∞-category with G-action, and the main result identifies G-commutative algebras in $\mathrm{Bor}_G(\mathcal{C})$ with ordinary commutative algebras in $\mathcal{C}^{hG}$. We reproduce this with an added description of the functor inducing this equivalence, and also generalize this to arbitrary "Borel inclusions" $(T, P) \subseteq (S, Q)$" instead of $(BG, BG) \subseteq (\mathsf{Orb}_G, \mathsf{Orb}_G)$. This is crucial to construct a comparison functor between strictly commutative objects in the 1-categorical world and parametrized commutative objects in our setting.

In Section 4 we begin in Section 4.1 by recording how to obtain a parametrized symmetric monoidal ∞-category from a 1-categorical version by Dwyer-Kan localization. After recalling the model categories of G-equivariant and G-global spectra in Section 4.2 we use the above to construct the equivariantly

[1] Also known as "naively G-equivariant objects", "objects with G-action" or "local systems on BG".

symmetric monoidal global categories of equivariant spectra $\underline{\mathsf{Sp}}^{\otimes}$ respectively global spectra $\underline{\mathsf{Sp}}^{\otimes}_{\mathsf{Glo}}$. They encode the ∞-category of genuine G-spectra Sp^G respectively G-global spectra $\mathsf{Sp}^{G\text{-}\mathsf{gl}}$ for every finite group G together with symmetric monoidal structures on the categories, restriction functors, and multiplicative norms.

Finally, Section 5 begins with some motivation and background on the classical notion of Picard groups and Picard spectra. Using the above results, we then finally construct two global Picard spectrum functors $\mathsf{pic}_{\mathsf{gl}}, \mathsf{pic}_{\mathsf{eqv}} : \mathsf{UCom} \to \mathsf{Sp}^{\mathsf{gl}}$, where $\mathsf{pic}_{\mathsf{gl}}(R)^G \simeq \mathsf{pic}(\mathsf{Mod}_{\mathrm{res}_G R}(\mathsf{Sp}^{G\text{-}\mathsf{gl}}))$ and $\mathsf{pic}_{\mathsf{eqv}}(R)^G \simeq \mathsf{pic}(\mathsf{Mod}_{\mathrm{res}_G R}(\mathsf{Sp}^G))$. All of this also works in the G-equivariant case, giving for every finite group G a functor $\mathsf{pic}_G : \mathsf{UCom}_G \to \mathsf{Sp}^G$ with $\mathsf{pic}_G(R)^H \simeq \mathsf{pic}(\mathsf{Mod}_{\mathrm{res}^G_H R}(\mathsf{Sp}^H))$ for each $H \leq G$.

1.3 Conventions and common Notation

From now on, by "category" we mean $(\infty, 1)$-category as developed by Lurie in [Lur09, Lur17]. Unless otherwise specified, all categorical notions are to be understood in this sense. For example, "unique" means unique up to a contractible space of choices. Moreover, we write Cat for what is usually denoted Cat_∞. Let us also remind the reader that we have included an index of notation at the end of this thesis. For a category $\mathcal{C}$, we often denote the mapping space of morphisms from c to d by $\mathcal{C}(c, d) \coloneqq \mathrm{map}_{\mathcal{C}}(c, d)$. A wide subcategory $\mathcal{C}_0 \subset \mathcal{C}$ is one that contains all equivalences (in particular all objects). We use the homotopy-invariant definitions of (co)cartesian morphisms and fibrations as recalled in [HHLN23a, Section 2.1]. Concretely, for a functor $p : \mathcal{E} \to \mathcal{C}$, a morphism $f : x \to y$ in $\mathcal{E}$ is p-cocartesian if the following square is cartesian:

$$\begin{tikzcd}
\mathcal{E}(y,z) \arrow[r, "f^*"] \arrow[d, "p"] & \mathcal{E}(x,z) \arrow[d, "p"'] \\
\mathcal{C}(py,pz) \arrow[r, "(pf)^*"'] & \mathcal{C}(px,pz)
\end{tikzcd}$$

We then say that p is a cocartesian fibration if for every map $g : c \to d$ in $\mathcal{C}$ and $x \in \mathcal{E}$ with $px = c$, there exists a p-cocartesian morphism $f : x \to y$ with $pf \simeq g$. We follow the diction of Kerodon 02MZ regarding cofinality; we call a functor $F : \mathcal{C} \to \mathcal{D}$ right cofinal if for every colimit cone $K^{\triangleright} \to \mathcal{C}$ also $K^{\triangleright} \to \mathcal{C} \xrightarrow{F} \mathcal{D}$ is a colimit cone. Dually, we say F is left cofinal if the corresponding condition on limit cones holds. Finally, we will sometimes need to use some basic facts on $(\infty, 2)$-categories. We briefly recall the relevant statements in Remark D.5.

1.4 Acknowledgements

I would like to thank Bastiaan Cnossen, Kaif Hilman, Sil Linskens and Maxime Ramzi for their patience in answering what feels like uncountably many of my questions, and helping me not get lost in the jungle of higher category theory. I am thankful to Tobias Lenz for some helpful exchanges regarding the contents of this thesis. I would also like to thank my advisor Stefan Schwede for introducing me to (stable, equivariant, global) homotopy theory, for countless insightful conversations during my stay in Bonn, and for suggesting his question from [Sch18, Remark 5.1.18] as a master thesis project and giving me complete freedom in how to realize it. Finally, I am grateful to my family for their continuous support, and to my friends for the many good times we have had – whether mathematical or not; Alessandro, Alex, Anton, Daniël, David, Fabio, Heiko, Joe, Maria and Yordan.

2 Parametrized Higher Algebra

This section forms the foundation of this thesis and introduces the main objects of study; P-symmetric monoidal T-categories for orbital $P \subset T$. The basis for our approach to parametrized higher algebra will be the framework of algebraic patterns, their Segal objects and fibrous patterns, as developed in [CH21] and more recently [BHS22]. We start in Section 2.1 by recalling the relevant background on algebraic patterns and specializing to the class of algebraic patterns of the form $\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T;T)$ for $P \subset T$ orbital, see Example 2.15. In Section 2.2 we investigate the resulting notion of P-symmetric monoidal T-category, defined as categorical Mackey functors on the above span categories, and P-commutative T-algebras therein. We continue in Section 2.3 with defining module categories for these algebras under some mild hypotheses. In Section 2.4 we conclude by comparing our $\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)$ Mackey functors to the P-commutative T-monoids of [CLL23a], and our specialized definitions of T-symmetric monoidal T-categories and algebras therein to those of [NS22].

2.1 Algebraic Patterns, Segal Objects and Fibrous Patterns

The main idea of the framework of algebraic patterns is to generalize Lurie's theory of (symmetric) operads by replacing the base category $\mathbb{F}_*$ of finite pointed sets with categories satisfying similar properties, such as admitting an inert-active orthogonal factorization system. After recalling the basic definitions and results on algebraic patterns and related objects, we will give a brief reminder on span categories to then focus our attention on a specific class of algebraic patterns arising from them. For example, if G is a finite group, we can consider the category $\mathrm{Span}(G)$ of spans of finite G-sets. This category has a long history in representation theory and equivariant homotopy theory, giving rise to G-Mackey functors and also genuine G-spectra by the Guillou-May Theorem [GM22]. It was shown in [BHS22] that their framework of algebraic patterns, specialized to the algebraic pattern arising from $\mathrm{Span}(G)$, precisely recovers the theory of G-operads and G-symmetric monoidal G-categories developed in [NS22]. We will consider algebraic patterns arising from span patterns of the form $\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)$, where $P \subset T$ is orbital. The main example to keep in mind will be the global indexing category Glo and its maximal atomic orbital subcategory $\mathsf{Orb} \subset \mathsf{Glo}$, as defined in Example A.8.

Definition 2.1 ([CH21, Definitions 2.1, 4.1])**.** An algebraic pattern is a category $\mathcal{O}$ equipped with:

1. a factorization system $(\mathcal{O}^{\mathsf{int}}, \mathcal{O}^{\mathsf{act}})$ of *inert* and *active* morphisms,

2. a full subcategory $\mathcal{O}^{\mathsf{el}} \subseteq \mathcal{O}^{\mathsf{int}}$ of *elementary objects*.

A morphism of algebraic patterns is a functor which preserves the subcategories of inert morphism, active morphisms, and elementary objects.

Following [CH21, Definition 5.4], one defines the category of algebraic patterns $\mathsf{AlgPatt}$ as a full subcategory of $\mathrm{Ar}(\mathsf{Cat}) \times_{t,\mathsf{Cat},\mathrm{ev}_0} \mathsf{Fun}(\Lambda^2_2, \mathsf{Cat})$ on the objects $\mathcal{C}' \to \mathcal{C}_L \to \mathcal{C} \leftarrow \mathcal{C}_R$ where $\mathcal{C}' \to \mathcal{C}_L$ is a full subcategory inclusion, and $\mathcal{C}_L, \mathcal{C}_R \to \mathcal{C}$ are essentially surjective subcategory inclusions, so that $(\mathcal{C}^L, \mathcal{C}^R)$ forms an orthogonal factorization system on $\mathcal{C}$, in the sense of [Lur09, Section 5.8.2], specifically, this means that the restriction $\mathsf{Fun}_{L,R}(\Delta^2, \mathcal{C}) \to \mathsf{Fun}(\Lambda^2_2, \mathcal{C})$ is an equivalence, where the former category is defined as the pullback

$$
\begin{tikzcd}
\mathsf{Fun}_{L,R}(\Delta^2, \mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Fun}(\Delta^2, \mathcal{C}) \arrow[d] \\
\mathsf{Fun}(\Delta^1, \mathcal{C}_L) \times \mathsf{Fun}(\Delta^1, \mathcal{C}_R) \arrow[r] & \mathsf{Fun}(\Delta^{0,1}, \mathcal{C}) \times \mathsf{Fun}(\Delta^{1,2}, \mathcal{C})
\end{tikzcd}
$$

Lemma 2.2 ([CH21, 5.5])**.** The full subcategory $\mathsf{AlgPatt} \subseteq \mathrm{Ar}(\mathsf{Cat}) \times_{t,\mathsf{Cat},\mathrm{ev}_0} \mathsf{Fun}(\Lambda^2_2, \mathsf{Cat})$ is closed under limits and filtered colimits.

Definition 2.3 ([CH21, Definitions 2.7, 4.2])**.** For an algebraic pattern $\mathcal{O}$, let $\mathcal{O}^{\mathsf{el}}_{X/} \coloneqq \mathcal{O}^{\mathsf{el}} \times_{\mathcal{O}^{\mathsf{int}}} \mathcal{O}^{\mathsf{int}}_{X/}$.

1. A category $\mathcal{C}$ is $\mathcal{O}$-complete if it admits all limits of shape $\mathcal{O}^{\mathsf{el}}_{X/}$ for $X \in \mathcal{O}$.

2. A Segal $\mathcal{O}$-object in $\mathcal{C}$ is a functor $F : \mathcal{O} \to \mathcal{C}$ such that for every $X \in \mathcal{C}$ the canonical map

$$F(X) \to \lim_{E \in \mathcal{O}^{\mathsf{el}}_{X/}} F(E) \tag{1}$$

is an equivalence. This is equivalent to $F|_{\mathcal{O}^{\mathsf{int}}}$ being right Kan extended from $F|_{\mathcal{O}^{\mathsf{el}}}$. We denote by $\mathrm{Seg}(\mathcal{O}, \mathcal{C}) \subseteq \mathsf{Fun}(\mathcal{O}, \mathcal{C})$ the full subcategory on the Segal $\mathcal{O}$-objects.

3. A morphism of algebraic patterns $f : \mathcal{O} \to \mathcal{P}$ is called a Segal morphism if for every $\mathcal{O}$-complete category $\mathcal{C}$, the functor $f^* : \mathsf{Fun}(\mathcal{P}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{O}, \mathcal{C})$ restricts to $f^* : \mathrm{Seg}(\mathcal{P}, \mathcal{C}) \to \mathrm{Seg}(\mathcal{O}, \mathcal{C})$. We say f is a strong Segal morphism if the induced functors $\mathcal{O}^{\mathsf{el}}_{X/} \to \mathcal{P}^{\mathsf{el}}_{f(X)/}$ are left cofinal for each $X \in \mathcal{O}$.

As mentioned in the introduction, the standard example of an algebraic pattern is that of finite pointed sets $\mathbb{F}_*$, equipped with its usual inert-active factorization system and sole elementary object 1_+. Given an object $n_+ \in \mathbb{F}_*$, we note that $(\mathbb{F}_*)^{\mathsf{el}}_{n_+/} = \{1_+\} \times_{\mathbb{F}^{\mathsf{int}}_*} (\mathbb{F}^{\mathsf{int}}_*)_{n_+/}$ is discrete on the usual Segal morphisms $\rho^i : n_+ \to 1_+$ sending everything but i to the basepoint. Thus the general Segal condition Eq. (1) reduces to the usual Segal condition in this special case. In particular a category $\mathcal{C}$ is $\mathbb{F}_*$-complete if and only if it admits finite products, and a Segal object for the pattern $\mathbb{F}_*$ is the definition of a commutative monoid in $\mathcal{C}$:

$$\mathsf{CMon}(\mathcal{C}) \coloneqq \mathrm{Seg}(\mathbb{F}_*, \mathcal{C}) \subseteq \mathsf{Fun}(\mathbb{F}_*, \mathcal{C})$$

compare [Lur17, 2.4.2]. In particular, taking $\mathcal{C} = \mathsf{Cat}$, we see that Cat-valued Segal objects for the algebraic pattern form the category of symmetric monoidal categories and strong symmetric monoidal functors. In general, symmetric monoidal categories are special cases of operads, and so there ought to also be an analogue for operads over any algebraic pattern, giving back the usual notion in the case of $\mathbb{F}_*$. Indeed, weak Segal fibrations have been investigated for this purpose in [CH21], and more recently [BHS22] have used the notion of fibrous patterns we recall below. The former are more similar to Lurie's definition of operads, however under the additional technical assumption that the base algebraic pattern is *sound*, both of these notions agree, see [BHS22, 4.1.7]. The formal definition of sound patterns is given in [BHS22, 3.1.2], but we will not need to know the details; all algebraic patterns we care about in this text will even be soundly extendable in the sense of [BHS22, 3.3.16], such as $\mathbb{F}_*$ or the span patterns we will consider below, see Lemma 2.16.

Definition 2.4 ([BHS22, Definition 4.1.2])**.** Let $\mathcal{O}$ be an algebraic pattern. A fibrous $\mathcal{O}$-pattern is a functor $\pi : \mathcal{P} \to \mathcal{O}$ such that:

1. $\mathcal{P}$ has all π-cocartesian lifts of inert morphisms in $\mathcal{O}$.
2. For all $O \in \mathcal{O}$, the following square is cartesian:

$$\begin{tikzcd} \mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\mathsf{act}}_{/O} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \lim_{E \in \mathcal{O}^{\mathsf{el}}_{O/}} \mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\mathsf{act}}_{/E} \arrow[d] \\ \mathcal{O}^{\mathsf{act}}_{/O} \arrow[r] & \lim_{E \in \mathcal{O}^{\mathsf{el}}_{O/}} \mathcal{O}^{\mathsf{act}}_{/E} \end{tikzcd}$$

The horizontal functors are induced by the functoriality of $\mathcal{O}^{\mathsf{act}}_{/\bullet} : \mathcal{O} \to \mathsf{Cat}$ we describe below.

For any orthogonal factorization system $(\mathcal{C}^L, \mathcal{C}^R)$ on some category $\mathcal{C}$, we can make the slices $\mathcal{C}^R_{/\bullet}$ covariantly functorial in $\mathcal{C}$ using the uniqueness of the factorization. Intuitively, for $f : c \to d$ in $\mathcal{C}$, the induced $\mathcal{C}^R_{/c} \to \mathcal{C}^R_{/d}$ takes $(e \xrightarrow{R} c) \in \mathcal{C}^R_{/c}$, factors the composite $e \xrightarrow{R} c \xrightarrow{f} d$ uniquely into $e \xrightarrow{L} x \xrightarrow{R} d$, and finally projects to $(x \xrightarrow{R} d) \in \mathcal{C}^R_{/d}$. Formally, one checks that the target projection $t : \mathrm{Ar}_R(\mathcal{C}) \subseteq \mathrm{Ar}(\mathcal{C}) \to \mathcal{C}$ is still a cocartesian fibration, where $\mathrm{Ar}_R(\mathcal{C})$ is full on the arrows in $\mathcal{C}^R$, compare [BHS22, Prop. 2.2.2] or [LNP22, Prop. 6.7]. In particular, for an algebraic pattern $\mathcal{O}$ we obtain $\mathcal{O}^{\mathsf{act}}_{/\bullet} : \mathcal{O} \to \mathsf{Cat}$ as cocartesian straightening of $t : \mathrm{Ar}_{\mathsf{act}}(\mathcal{O}) \to \mathcal{O}$.

Remark 2.5 ([BHS22, 4.1.11])**.** Let $\pi : \mathcal{P} \to \mathcal{O}$ be a fibrous $\mathcal{O}$-pattern. A morphism in $\mathcal{P}$ is inert (active) if it is a π-cocartesian lift of an inert (active) morphism in $\mathcal{O}$. By [Lur17, 2.1.2.5], this yields a factorization system on $\mathcal{P}$, which we complete into an algebraic pattern by taking the elementary objects to be all those lying over elementary objects in $\mathcal{O}$.

Definition 2.6 ([BHS22, 4.1.12])**.** A morphism of fibrous $\mathcal{O}$-patterns is a commutative triangle

$$\begin{tikzcd} \mathcal{P} \arrow[rr, "f"] \arrow[dr, "\pi"'] & & \mathcal{P}' \arrow[dl, "\pi'"] \\ & \mathcal{O} & \end{tikzcd}$$

where π, π' are fibrous $\mathcal{O}$-patterns and f is a morphism of algebraic patterns. Equivalently, it suffices to assume that f preserves inert morphisms. One defines $\mathsf{Fbrs}(\mathcal{O})$ as the full subcategory of $\mathsf{AlgPatt}_{/\mathcal{O}}$ on the fibrous $\mathcal{O}$-patterns, or equivalently as a full subcategory of $\mathsf{Cat}_{/\mathcal{O}}^{\mathsf{int-cc}}$. It inherits the structure of an $(\infty, 2)$-category from the latter, with mapping categories $\mathsf{Fun}_{\mathsf{Fbrs}(\mathcal{O})}(\mathcal{P}, \mathcal{P}') \coloneqq \mathsf{Fun}_{/\mathcal{O}}^{\mathsf{int-cc}}(\mathcal{P}, \mathcal{P}')$, see [BHS22, 5.3.1, 5.3.12].

Example 2.7 ([BHS22, 4.1.9])**.** Fibrous $\mathbb{F}_*$-patterns are precisely operads as defined in [Lur17], and $\mathsf{Fbrs}(\mathbb{F}_*)$ agrees with Lurie's $(\infty, 2)$-category Op of operads.

We are mostly interested in algebraic patterns coming from span categories. For a detailed treatment, we refer the reader to [HHLN23b, Section 2][2] but let us give a brief overview. The basic layout of a span category is that a morphism from X to Y is given by a span $X \xleftarrow{f} T \xrightarrow{g} Y$ in some underlying category, and composition is given by pullback. Formally, an adequate triple is a category $\mathcal{X}$ equipped with wide subcategories of backwards and forwards morphisms $\mathcal{X}^b, \mathcal{X}^f \subset \mathcal{X}$, such that the basechange of a backwards morphism along a forwards morphism exists and is again a backwards morphism, and vice-versa. Given such an adequate triple $(\mathcal{X}, \mathcal{X}^b, \mathcal{X}^f)$, one can construct a span category $\mathrm{Span}_{b,f}(\mathcal{X}) \coloneqq \mathrm{Span}(\mathcal{X}, \mathcal{X}^b, \mathcal{X}^f)$ which has the same objects as $\mathcal{X}$, and morphism given by spans $x \xleftarrow{b} z \xrightarrow{f} y$ with labels indicating backwards and forwards morphisms. Composition is given by pullback

$$\begin{tikzcd} & & \bullet \arrow[dl, "b"'] \arrow[dr, "f"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] \arrow[ddll, bend right, "b"'] \arrow[ddrr, bend left, "f"] & & \\ & \bullet \arrow[dl, "b"'] \arrow[dr, "f"] & & \bullet \arrow[dl, "b"'] \arrow[dr, "f"] & \\ \bullet & & \bullet & & \bullet \end{tikzcd}$$

A more precise description of this is given in Lemma B.1. Note that every morphism then factors uniquely as the composite of a backwards morphism $x \xleftarrow{b} y = y$ followed by a forwards morphism $y = y \xrightarrow{f} z$. In fact, these backwards and forwards morphisms $(\mathcal{X}^b)^{\mathsf{op}}, \mathcal{X}^f \subset \mathrm{Span}_{b,f}(\mathcal{X})$ form the left and right classes of an orthogonal factorization system on $\mathrm{Span}_{b,f}(\mathcal{X})$ in the sense of [Lur09, Section 5.2.8], see [HHLN23b, Proposition 4.9]. Importantly, span categories are self-dual up to swapping backwards and forwards morphisms:

$$\mathrm{Span}_{b,f}(\mathcal{X})^{\mathsf{op}} \simeq \mathrm{Span}_{f,b}(\mathcal{X})$$

[2] Warning: Their terminology uses "ingressive" respectively "egressive" for what we call forwards respectively backwards maps. Moreover, they write adequate triples in the order $(\mathcal{X}, \mathcal{X}^f, \mathcal{X}^b)$ instead of our $(\mathcal{X}, \mathcal{X}^b, \mathcal{X}^f)$.

and allowing only equivalences for one class yields $\mathcal{X} \simeq \mathrm{Span}_{\simeq,\mathsf{all}}(\mathcal{X})$ and $\mathcal{X}^{\mathsf{op}} \simeq \mathrm{Span}_{\mathsf{all},\simeq}(\mathcal{X})$. Note that any category $\mathcal{C}$ which admits pullbacks yields an adequate triple $(\mathcal{C}, \mathcal{C}, \mathcal{C})$, whose associated span category we denote by $\mathrm{Span}(\mathcal{C})$. Let us also remark that even if $\mathcal{X}$ is a 1-category, $\mathrm{Span}_{b,f}(\mathcal{X})$ will generally be a $(2,1)$-category, as e.g. in the case of $\mathrm{Span}(\mathbb{F})$, because pullbacks and hence composition is only unique up to isomorphism. The reason we are interested in span categories is their utility in encoding algebraic data. Specifically, there is a close connection between span categories and Lawvere theories, originally investigated in [Cra10].

Example 2.8. By Lemma B.2 the category $\mathrm{Span}(\mathbb{F})$ is semiadditive, and the inclusion $\mathbb{F} \to \mathrm{Span}(\mathbb{F})$ creates coproducts, so that the biproduct of $X, Y \in \mathbb{F}$ in $\mathrm{Span}(\mathbb{F})$ has underlying object $X \sqcup Y$ in $\mathbb{F}$. Moreover, the projection from $X \sqcup Y$ to X in $\mathrm{Span}(\mathbb{F})$ is given by the backwards summand inclusion $\mathrm{pr}_X = (X \sqcup Y \leftarrow X = X)$. In fact, one can show that $\mathrm{Span}(\mathbb{F})$ is the free semiadditive category on one generator. For any category $\mathcal{C}$ admitting finite products, restriction along $i : \mathbb{F}_* \simeq \mathrm{Span}_{\mathsf{inj},\mathsf{all}}(\mathbb{F}) \to \mathrm{Span}(\mathbb{F})$[3] induces an equivalence

$$i^* : \mathsf{Mack}(\mathcal{C}) \coloneqq \mathsf{Fun}^{\times}(\mathrm{Span}(\mathbb{F}), \mathcal{C}) \xrightarrow{\simeq} \mathsf{CMon}(\mathcal{C})$$

with inverse given by right Kan extension, see e.g. [BH17, C.1] for a short proof. The way such a functor $\Phi : \mathrm{Span}(\mathbb{F}) \xrightarrow{\times} \mathcal{C}$ encodes a commutative monoids structure is as follows.

1. $\Phi(\varnothing) = * \in \mathcal{C}$ is terminal, and $C \coloneqq \Phi(*) \in \mathcal{C}$ is the underlying object.
2. Most morphisms encode auxiliary data: forwards summand inclusions (injections) give rise to unit maps $\Phi(\varnothing \to *) : * \to C$, backwards summand inclusions (injections) give rise to projections $\Phi(X \sqcup Y \hookleftarrow X) : C^X \times C^Y \to C^X$, and backwards fold maps (surjections) are sent to diagonals $\Phi(X \leftarrow X \sqcup X) : C^X \to C^X \times C^X$.
3. Multiplications are encoded by forwards fold maps (surjections) $\nabla : * \sqcup * \to *$:
$$C \times C = \Phi(*) \times \Phi(*) \xleftarrow[\simeq]{(\mathrm{pr}_1, \mathrm{pr}_2)} \Phi(* \sqcup *) \xrightarrow{\nabla} \Phi(*) = C.$$
Generally for $f : X \to Y$ a surjection of finite sets, $\Phi(f) : \prod_{x \in X} C \to \prod_{y \in Y} C$ multiplies everything in each fiber over y together. If C had elements, we would write this as sending $(c_x)_{x \in X} \mapsto (\prod_{f(x)=y} c_x)_{y \in Y}$.
4. Functoriality of Φ allows C to inherit unitality, associativity, commutativity and the higher coherence data all from the corresponding data of the canonical commutative monoid structure on $* \in \mathbb{F}$ in the cocartesian monoidal structure on finite sets.

[3] Here the first equivalence is induced by sending a map $f : X_+ \to Y_+$ to the span $X \hookleftarrow f^{-1}(Y) \to Y$, see Lemma A.10 for a more general statement.

One can recover the above from a more general result comparing the algebraic patterns $\mathbb{F}_*$ and $\mathrm{Span}(\mathbb{F})$, see Corollary 2.14. Because of the following very classical example, product-preserving functors $\mathrm{Span}_{b,f}(\mathcal{X}) \xrightarrow{\times} \mathcal{C}$ are generally called $\mathcal{C}$-valued Mackey functors.

Example 2.9. Let G be a finite group. Then $\mathsf{Fun}^{\times}(\mathrm{Span}(G), \mathsf{Ab})$ is equivalent to the well-known category of G-Mackey functors from representation theory. These also play an important role in G-equivariant stable homotopy theory, as this is the structure on $\underline{\pi}_0(X)$ for any genuine G-spectrum X. Even better, it was shown in [GM22] that the category of genuine G-spectra may be modeled by spectral Mackey functors. For the (∞-categorical) equivalence $\mathsf{Sp}^G \simeq \mathsf{Mack}_G(\mathsf{Sp}) \coloneqq \mathsf{Fun}^{\times}(\mathrm{Span}(G), \mathsf{Sp})$, we refer the reader to [CMNN22, Appendix A]. Let $X \in \mathsf{Sp}^G$ be a genuine G-spectrum. The associated spectral Mackey functor looks as follows:

1. We have $\Phi(G/H) = X^H$ the genuine H-fixed points for $H \leq G$, and restriction along $\mathrm{Span}(\mathbb{F}) \to \mathrm{Span}(G), * \mapsto G/H$ encodes the canonical additive group structure on X^H that any object in an additive category possesses.

2. For $K \leq H \leq G$, the backwards morphism $G/H \leftarrow G/K$ is sent by Φ to the "restriction"
$$X^H \xrightarrow{(\eta_X)^H} (\mathrm{Coind}_K^H \mathrm{res}_K^H X)^H \simeq X^K$$
whereas the forwards morphism $G/K \to G/H$ is sent to the "transfer"
$$X^K \simeq (\mathrm{Coind}_K^H \mathrm{res}_K^H X)^H \simeq (\mathrm{Ind}_K^H \mathrm{res}_K^H X)^H \xrightarrow{(\varepsilon_X)^H} X^H,$$
where we employ the Wirthmüller isomorphism $\mathrm{Coind}_K^H \simeq \mathrm{Ind}_K^H$. One can check $\pi_0 \circ \Phi \cong \underline{\pi}_0(X)$ is the usual Mackey functor structure. For example, the double-coset formula is then a consequence of the definition of composition in $\mathrm{Span}(G)$, and the orbit-decomposition of a product $G/H \times G/K$ in the category of finite G-sets $\mathbb{F}_G$.

While such a Mackey functor description of genuine G-spectra only exists for finite groups, a big upside is that it yields a purely categorical definition of the category of genuine G-spectra, without reference to any model categories. For example, restriction and (co)induction can be defined by precomposing with certain functors of span categories, and the Wirthmüller isomorphism is then a formal consequence of the self-duality of $\mathrm{Span}(G)$.[4]

Coming back to general span categories, a morphism of adequate triples is a functor which preserves backwards and forwards maps as well as their basechanges along each other.

[4]Maxime Ramzi wrote an expository account of the basic notions in equivariant stable homotopy theory based on the spectral Mackey functor model of genuine G-spectra, which can be found on his homepage https://sites.google.com/view/maxime-ramzi-en.

Definition/Lemma 2.10 ([HHLN23b, 2.1, 2.4, 2.5])**.** Adequate triples and their morphisms assemble into a subcategory $\mathsf{AdTrip} \subset \mathsf{Fun}(\Lambda^2_2, \mathsf{Cat})$. It admits limits and filtered colimits, which are computed pointwise in Cat, i.e. separately for $\mathcal{X}, \mathcal{X}^b$ and $\mathcal{X}^f$. Moreover, AdTrip is cartesian closed, giving AdTrip the structure of an $(\infty, 2)$-category with mapping categories $\mathsf{Fun}_{\mathsf{AdTrip}} \subseteq \mathrm{Nat}_{\Lambda^2_2}$ full on the functors of adequate triples.

Given any category $\mathcal{C}$, let $\mathrm{Tw}^r(\mathcal{C})$ denote the associated twisted arrow category with the convention that $(s,t) : \mathrm{Tw}^r(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\mathsf{op}}$ is a right fibration (classified by the mapping space functor). This can be endowed with the structure of an adequate triple, with forwards morphisms those inverted by $t : \mathrm{Tw}^r(\mathcal{C}) \to \mathcal{C}^{\mathsf{op}}$, and backwards morphisms those inverted by $s : \mathrm{Tw}^r(\mathcal{C}) \to \mathcal{C}$.

Theorem 2.11 ([HHLN23b, Theorem 2.18])**.** We have an adjunction $\mathrm{Tw}^r : \mathsf{Cat} \rightleftarrows \mathsf{AdTrip} : \mathrm{Span}$.

Moreover, one easily checks that Tw^r preserves compact objects and hence Span preserves filtered colimits. An augmented adequate triple $(\mathcal{X}, \mathcal{X}^b, \mathcal{X}^f; \mathcal{X}_0)$ is an adequate triple together with a full subcategory $\mathcal{X}_0 \subseteq \mathcal{X}$, which we think of as a choice of objects. This yields a further category of augmented adequate triples $\mathsf{AdTrip}_{\mathsf{aug}} \subseteq \mathsf{AdTrip} \times_{\mathsf{Cat}} \mathrm{Ar}(\mathsf{Cat})$ full on those objects $(\mathcal{X}, \mathcal{X}^b, \mathcal{X}^f, \mathcal{X}_0 \to \mathcal{X})$ where $\mathcal{X}_0 \to \mathcal{X}$ is fully faithful. Again this admits all limits and filtered colimits, computed pointwise in Cat.

Example 2.12 ([BHS22, Example 3.2.7])**.** An augmented adequate triple $(\mathcal{X}, \mathcal{X}^b, \mathcal{X}^f; \mathcal{X}_0)$ induces the algebraic pattern $\mathrm{Span}_{b,f}(\mathcal{X}; \mathcal{X}_0)$ equipped with elementary objects $\mathcal{X}_0$ and the inert-active factorization system of backwards and forwards maps (cf. [HHLN23b, Proposition 4.9]). It follows immediately from the definitions and Lemma 2.2 that this construction upgrades to a functor

$$\mathrm{Span} : \mathsf{AdTrip}_{\mathsf{aug}} \to \mathsf{AlgPatt}$$

which again preserves limits and filtered colimits. The Segal conditions for a functor $F : \mathrm{Span}_{b,f}(\mathcal{X}; \mathcal{X}_0) \to \mathcal{C}$ demands that for every $x \in \mathcal{X}$ the canonical map

$$F(x) \to \lim_{(e \to x) \in (\mathcal{X}^b_{0/x})^{\mathsf{op}}} F(e) \tag{2}$$

is an equivalence. This is because $\mathrm{Span}_{b,f}(\mathcal{X})^{\mathsf{int}} = \mathrm{Span}_{b,\simeq}(\mathcal{X}) \simeq (\mathcal{X}^b)^{\mathsf{op}}$, so $\mathrm{Span}_{b,f}(\mathcal{X})^{\mathsf{el}}_{x/} \simeq (\mathcal{X}^b_{0/x})^{\mathsf{op}}$, where $\mathcal{X}^b_{0/x} \coloneqq \mathcal{X}^b_0 \times_{\mathcal{X}^b} \mathcal{X}^b_{/x}$ and $\mathcal{X}^b_0 \subseteq \mathcal{X}^b$ is the full subcategory on the objects in $\mathcal{X}_0$.

The following theorem will be used repeatedly throughout this text.

Theorem 2.13 ([BHS22, 3.1.16, 5.1.1, 5.1.12, 5.3.17])**.** Consider augmented adequate triples $(\mathcal{X}, \mathcal{X}^b, \mathcal{X}^f; \mathcal{X}_0)$ and $(\mathcal{Y}, \mathcal{Y}^b, \mathcal{Y}^f; \mathcal{Y}_0)$ and a morphism of augmented adequate triples $F : \mathcal{X} \to \mathcal{Y}$.

1. If the induced $\mathcal{X}^b_0 \times^b_{\mathcal{X}} \mathcal{X}^b_{/x} \to \mathcal{Y}^b_0 \times_{\mathcal{Y}^b} \mathcal{Y}_{/Fx}$ is right cofinal for all $x \in \mathcal{X}$, then F is a strong Segal morphism (Definition 2.3).

2. Suppose F is a strong Segal morphism and that the induced functors $F : \mathcal{X}_0^b \to \mathcal{Y}_0^b$ as well as $F : (\mathcal{X}^f_{/x})^{\simeq} \to (\mathcal{Y}^f_{/Fx})^{\simeq}$ are equivalences. For every complete category $\mathcal{C}$, restriction along F induces an equivalence

$$F^* : \mathrm{Seg}(\mathrm{Span}_{b,f}(\mathcal{Y};\mathcal{Y}_0),\mathcal{C}) \xrightarrow{\simeq} \mathrm{Seg}(\mathrm{Span}_{b,f}(\mathcal{X};\mathcal{X}_0),\mathcal{C})$$

with inverse given by right Kan extension F_*. Note that if $\mathcal{C} = \mathsf{Cat}$ this is an equivalence of $(\infty,2)$-categories by Lemma D.6.

3. If F is as in (2) and additionally $\mathrm{Span}_{b,f}(\mathcal{Y};\mathcal{Y}_0)$ is soundly extendable, then pullback along F induces an equivalence of $(\infty,2)$-categories

$$F^* : \mathsf{Fbrs}(\mathrm{Span}_{b,f}(\mathcal{Y};\mathcal{Y}_0)) \xrightarrow{\simeq} \mathsf{Fbrs}(\mathrm{Span}_{b,f}(\mathcal{X},\mathcal{X}_0)).$$

Corollary 2.14 ([BHS22, 5.1.13])**.** Let $i : \mathbb{F}_* \simeq \mathrm{Span}_{\mathsf{inj},\mathsf{all}}(\mathbb{F}) \to \mathrm{Span}(\mathbb{F})$ be the inclusion.[5]

1. Then i satisfies all the assumptions of Theorem 2.13(3). In particular, restriction along i induce an equivalences of $(\infty,2)$-categories

$$i^* : \mathsf{Mack}(\mathsf{Cat}) = \mathrm{Seg}(\mathrm{Span}(\mathbb{F}),\mathsf{Cat}) \xrightarrow{\simeq} \mathrm{Seg}(\mathbb{F}_*,\mathsf{Cat}) = \mathsf{CMon}(\mathsf{Cat}).$$

and pullback along i induces an equivalence of $(\infty,2)$-categories

$$i^* : \mathsf{Fbrs}(\mathrm{Span}(\mathbb{F})) \xrightarrow{\simeq} \mathsf{Fbrs}(\mathbb{F}_*) = \mathsf{Op}.$$

2. In particular, given $\mathcal{O} \in \mathsf{Fbrs}(\mathrm{Span}(\mathbb{F}))$, we obtain a natural equivalence

$$i^* : \mathrm{CAlg}_{\mathbb{F}}(\mathcal{O}) \coloneqq \mathsf{Fun}^{\mathbb{F}^{\mathsf{op}}-\mathsf{cc}}_{/\,\mathrm{Span}(\mathbb{F})}(\mathrm{Span}(\mathbb{F}),\mathcal{O}) \xrightarrow{\simeq} \mathsf{Fun}^{\mathsf{int}-\mathsf{cc}}_{/\mathbb{F}_*}(\mathbb{F}_*, i^*\mathcal{O}) \eqqcolon \mathrm{CAlg}(i^*\mathcal{O}).$$

We now specialize to the span patterns which will play the main role in the rest of this document. We will assume familiarity with the contents of Appendix A, specifically with orbital categories (Definition A.5) and the notation for free finite coproduct completions and related objects (Notation A.3).

Example 2.15. Let $P \subset T$ be orbital. Then $(\mathbb{F}_T, \mathbb{F}_T, \mathbb{F}_T^P)$ is an adequate triple, which we augment with the objects T. This gives the algebraic pattern $\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T;T)$. In the notation of Example 2.12, we have $\mathcal{X}^b_{0/X} = T \times_{\mathbb{F}_T} (\mathbb{F}_T)_{/X} \eqqcolon T_{/X} \simeq \coprod_{i=1}^n T_{/X_i}$ for all finite T-sets $X = \coprod_{i=1}^n X_i$. Here the last equivalence comes from Lemma A.4. In particular, we see that the discrete subcategory on the summand inclusions $\{X_i \to X\}$ is right cofinal in $T_{/X}$, so that the Segal condition (2) reduces to

$$\prod_{i=1}^n F(\rho_i) : F(X) \xrightarrow{\simeq} \prod_{i=1}^n F(X_i) \tag{3}$$

[5] Here we are using Lemma A.10 for the first equivalence.

where $\rho_i = (X \leftarrow X_i = X_i)$ are the backwards morphisms associated to the summand inclusions. The ρ_i are precisely the projections witnessing X as a biproduct of the X_i in the semiadditive category $\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)$ (c.f. Lemma B.2), so the above shows that Segal objects for the algebraic pattern $\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T;T)$ are precisely Mackey-functors:

$$\mathrm{Seg}(\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T;T),\mathcal{C}) = \mathsf{Fun}^\times(\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T),\mathcal{C}) =: \mathsf{Mack}_T^P(\mathcal{C}).$$

Note that for $P = T^\simeq$, we have $\mathrm{Span}_{\mathsf{all},\simeq}(\mathbb{F}_T) = \mathbb{F}_T^{\mathsf{op}}$, and $\mathsf{Fun}^\times(\mathbb{F}_T^{\mathsf{op}},\mathcal{C}) \simeq \mathsf{Fun}(T^{\mathsf{op}},\mathcal{C})$.

As mentioned previously, the reader will not need to know the precise definition of soundly extendable pattern from [BHS22, 3.3.16]. It is a technical condition which is a hypothesis for some important theorems of [BHS22], and we show now that it is satisfied for the above class of span patterns.

Lemma 2.16. Let $P \subset T$ be orbital.

1. The algebraic pattern $\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T;T)$ is soundly extendable.
2. Let $f : T \to S$ be a functor sending an orbital $P \subset T$ into an orbital subcategory $Q \subset S$, and suppose the induced $F : \mathbb{F}_T \to \mathbb{F}_S$ sends pullbacks along morphisms in $\mathbb{F}_T^P$ to pullbacks along morphisms in $\mathbb{F}_S^Q$. Then $\mathrm{Span}(F) : \mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T;T) \to \mathrm{Span}_{\mathsf{all},Q}(\mathbb{F}_S;S)$ is a strong Segal morphism and preserves finite products.

Proof. For the first point, the pattern is sound by [BHS22, Corollary 3.3.24]. The remaining condition is shown analogously to [BHS22, Example 3.3.26]. We have to prove that for every $X \in \mathbb{F}_T$, the functor

$$(\mathbb{F}_T^P)_{/X} \to \lim_{E \in (T_{/X})^{\mathsf{op}}} (\mathbb{F}_T^P)_{/E}$$

is an equivalence. But using right cofinality of the summand inclusions of $X = \coprod_{i=1}^n X_i$ in $T_{/X}$ as in the above example, this reduces to the statement that pulling back along the summand inclusions induces a decomposition

$$\prod_{i=1}^n (X_i \to X)^* : (\mathbb{F}_T^P)_{/X} \xrightarrow{\simeq} \prod_{i=1}^n (\mathbb{F}_T^P)_{/X_i}.$$

This functor is well-defined because by orbitality P is stable under basechange, and it is an equivalence by Lemma A.4 since $\mathbb{F}_T^P = \mathbb{F}_P$.

For (2), note that preservation of finite products follows immediately from Lemma B.2. Let $X \in \mathbb{F}_T$ with coproduct-decomposition $X = \coprod_{i=1}^n X_i$. We need to show that the induced $T_{/X} \to S_{/FX}$ is right cofinal. Since $F = f^{\sqcup}$ preserves coproducts by definition, we obtain a commutative diagram

$$\begin{tikzcd}
T_{/\coprod_{i=1}^n X_i} \arrow[r, "F_{/X}"] & S_{/F\coprod_{i=1}^n X_i} \arrow[r, "\simeq"] & S_{/\coprod_{i=1}^n fX_i} \\
\coprod_{i=1}^n T_{/X_i} \arrow[u, "\simeq"] \arrow[rr, "\coprod_{i=1}^n f_{/X_i}"'] & & \coprod_{i=1}^n S_{/fX_i} \arrow[u, "\simeq"']
\end{tikzcd}$$

where we use the equivalences from Lemma A.4. Clearly $\coprod_{i=1}^n f_{/X_i}$ is right cofinal, so we are done. $\square$

For $X \in \mathbb{F}_T$, note that by Lemma A.9 we can replace $P \subset T$ with $\pi_X^{-1}(P) \subset T_{/X}$. in Example 2.15. The equivalence $\mathbb{F}_{T_{/X}} \simeq (\mathbb{F}_T)_{/X}$ from Lemma A.9 identifies $\mathbb{F}_{T_{/X}}^{\pi_X^{-1}(P)}$ with $\pi_X^{-1}(\mathbb{F}_T^P) \subset (\mathbb{F}_T)_{/X}$, hence

$$((\mathbb{F}_T)_{/X})_{\mathsf{all},P} \coloneqq ((\mathbb{F}_T)_{/X}, (\mathbb{F}_T)_{/X}, \pi_X^{-1}(\mathbb{F}_T^P); T_{/X})$$

is also an augmented adequate triple.

Lemma 2.17. Let $P \subset T$ be orbital. The postcomposition-functoriality of the slices $(\mathbb{F}_T)_{/\bullet}$ induces a functor

$$\mathrm{Span}_{\mathsf{all},P}((\mathbb{F}_T)_{/\bullet}; T_{/\bullet}) : \mathbb{F}_T \to \mathsf{AlgPatt} \times_{\mathsf{Cat}} \mathsf{Cat}^{\oplus}.$$

Moreover, the projections $\pi_X : (\mathbb{F}_T)_{/X} \to \mathbb{F}_T$ assemble into a natural transformation

$$\mathrm{Span}(\pi_\bullet) : \mathrm{Span}_{\mathsf{all},P}((\mathbb{F}_T)_{/\bullet}; T_{/\bullet}) \Rightarrow \mathrm{const}\, \mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T; T)$$

which exhibits $\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)$ as the colimit of $\mathrm{Span}_{\mathsf{all},P}((\mathbb{F}_T)_{/\bullet})$ in $\mathsf{Cat}^{\oplus}, \mathsf{Cat}^{\times}$ and Cat.

Proof. Since colimits and pullbacks in $(\mathbb{F}_T)_{/X}$ are computed in $\mathbb{F}_T$, it follows that π_X and $(\mathbb{F}_T)_{/f} : (\mathbb{F}_T)_{/X} \to (\mathbb{F}_T)_{/Y}$ induce morphisms of augmented adequate triples

$$(((\mathbb{F}_T)_{/X})_{\mathsf{all},P}; T_{/X}) \to ((\mathbb{F}_T)_{\mathsf{all},P}; T) \qquad \text{and} \qquad (((\mathbb{F}_T)_{/X})_{\mathsf{all},P}; T_{/X}) \to (((\mathbb{F}_T)_{/Y})_{\mathsf{all},P}; T_{/Y}).$$

Hence the functor and natural transformation

$$((\mathbb{F}_T)_{/\bullet})_{\mathsf{all},P} : \mathbb{F}_T \to \mathsf{AdTrip}_{\mathsf{aug}} \quad \text{and} \quad \pi_\bullet : (((\mathbb{F}_T)_{/\bullet})_{\mathsf{all},P}; T_{/\bullet}) \Rightarrow \mathrm{const}((\mathbb{F}_T)_{\mathsf{all},P}; T)$$

exist. Postcomposing with $\mathrm{Span} : \mathsf{AdTrip}_{\mathsf{aug}} \to \mathsf{AlgPatt}$ from Example 2.12 yields the desired functor and natural transformation. Forgetting to the underlying categories, it lands in $\mathsf{Cat}^{\oplus}$ by Lemma B.2. For the remaining statement, we will first show that we have the colimit in Cat. Taking cocartesian unstraightenings, we have the following commutative diagram

$$\begin{tikzcd}
& \mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T) & \\
\int \mathrm{Span}_{\mathsf{all},P}((\mathbb{F}_T)_{/\bullet}) \arrow[ur, "\gamma"] \arrow[rr, "\int \mathrm{Span}(\pi_\bullet)"] \arrow[dr] & & \mathbb{F}_T \times \mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T) \arrow[ul, "\mathrm{pr}"'] \arrow[dl] \\
& \mathbb{F}_T &
\end{tikzcd}$$

and the claim is equivalent to showing that γ is a localization at cocartesian arrows, see Kerodon 02V0. We will use [HHLN23b, Theorem 3.9] to compute the cocartesian unstraightening. The cartesian unstraightening of $(\mathbb{F}_T)_{/\bullet}$ respectively $\mathbb{F}_{T_\bullet}^{\pi_\bullet^{-1}(P)}$ is given by the target projections $t : \mathrm{Tw}^r(\mathbb{F}_T) \to \mathbb{F}_T^{\mathsf{op}}$, respectively $t : \mathrm{Tw}^r_{sP}(\mathbb{F}_T) \to \mathbb{F}_T^{\mathsf{op}}$, where $\mathrm{Tw}^r_{sP}(\mathbb{F}_T) \subset \mathrm{Tw}^r_{sP}(\mathbb{F}_T)$ denotes the subcategory on those morphisms (squares) whose top morphism lands in $\mathbb{F}_T^P$. In both cases the cartesian morphisms are those inverted by the source projection. Then the cocartesian unstraightening is given by

$$\mathrm{Span}(t) : \mathrm{Span}(\mathrm{Tw}^r(\mathbb{F}_T), \mathrm{Tw}^r(\mathbb{F}_T), \mathrm{Tw}^r_{sP,\mathsf{tdeg}}(\mathbb{F}_T)) \to \mathrm{Span}(\mathbb{F}_T^{\mathsf{op}}, \mathbb{F}_T^{\mathsf{op}}, \mathbb{F}_T^{\simeq}) \simeq \mathbb{F}_T$$

where $\mathrm{Tw}^r_{sP,\mathsf{tdeg}}(\mathbb{F}_T) \subset \mathrm{Tw}^r_{sP}(\mathbb{F}_T)$ is the subcategory on those morphisms (squares) whose image under the target projection is an equivalence. Moreover, the theorem also tells us that the $\mathrm{Span}(t)$-cocartesian morphisms are precisely those backwards morphisms which are t-cartesian. Thus a morphism is $\mathrm{Span}(t)$-cocartesian if and only if it gets inverted by

$$\mathrm{Span}(s) : \mathrm{Span}(\mathrm{Tw}^r(\mathbb{F}_T), \mathrm{Tw}^r(\mathbb{F}_T), \mathrm{Tw}^r_{sP,\mathsf{tdeg}}(\mathbb{F}_T)) \to \mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T).$$

Using the above explicit model for $\int \mathrm{Span}_{\mathsf{all},P}((\mathbb{F}_T)_{/\bullet})$, we moreover see that γ is precisely given by this $\mathrm{Span}(s)$. Now $s : \mathrm{Tw}^r_{sP,\mathsf{tdeg}}(\mathbb{F}_T) \to \mathbb{F}_T^P$ is a right fibration since the fiber over X is the groupoid $((F_T)_{X/})^{\simeq}$. Since $s : \mathrm{Tw}^r(\mathbb{F}_T) \to \mathbb{F}_T$ is a localization, it then follows from Lemma B.4 that $\mathrm{Span}(s)$ is also a localization (at the morphisms it inverts). This proves the claim for Cat.

Now $\mathsf{Cat}^{\oplus}$ is a right Bousfield localization of $\mathsf{Cat}^{\times}$ (see [HW, II.19]), hence colimits in $\mathsf{Cat}^{\oplus}$ are computed in $\mathsf{Cat}^{\times}$ and it remains to see that that we also have the colimit there. So let $\mathcal{E} \in \mathsf{Cat}^{\times}$ and consider the commutative diagram

$$\begin{tikzcd}
\mathsf{Fun}(\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T), \mathcal{E}) \arrow[r, "\simeq"] & \lim_{X \in \mathbb{F}_T^{\mathsf{op}}} \mathsf{Fun}(\mathrm{Span}_{\mathsf{all},P}((\mathbb{F}_T)_{/X}), \mathcal{E}) \\
\mathsf{Fun}^{\times}(\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T), \mathcal{E}) \arrow[r] \arrow[u, hook] & \lim_{X \in \mathbb{F}_T^{\mathsf{op}}} \mathsf{Fun}^{\times}(\mathrm{Span}_{\mathsf{all},P}((\mathbb{F}_T)_{/X}, \mathcal{E}) \arrow[u, hook]
\end{tikzcd}$$

where the horizontal maps are induced by $\mathrm{Span}(\pi_\bullet)$. Since fully faithful functors are closed under limits in $\mathrm{Ar}(\mathsf{Cat})$, it follows that the right vertical and hence also the bottom horizontal morphism are fully faithful, and it remains to see essential surjectivity. Given an object in the lower right limit, we can write it as $(\mathrm{Span}(\pi_X)^*\Phi)_{X\in\mathbb{F}_T^{\mathsf{op}}}$ for some $\Phi : \mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T) \to \mathcal{E}$ by essential surjectivity of the top horizontal. But since each $\mathrm{Span}(\pi_X)^*\Phi$ preserves finite products, it easily follows that also Φ does; given $\mathrm{pr}_X = (X \sqcup Y \xleftarrow{i_X} X = X)$ with i_X the summand inclusion, then also $\mathrm{id}_{X\sqcup Y} = i_X \sqcup i_Y$ in $(\mathbb{F}_T)_{/X\sqcup Y}$, and $\pi_{X\sqcup Y}(\mathrm{pr}_{i_X}) = \mathrm{pr}_X$. This proves that also the lower horizontal in the above diagram is an equivalence, as desired. ☐

The above allows us to define a T-category of Mackey functors. Recall that a T-category is simply a functor $T^{\mathsf{op}} \to \mathsf{Cat}$, or equivalently $\mathbb{F}_T^{\mathsf{op}} \xrightarrow{\times} \mathsf{Cat}$ by limit extending. The category of T-categories is thus defined as $\mathsf{Cat}_T \coloneqq \mathsf{Fun}(T^{\mathsf{op}}, \mathsf{Cat}) \simeq \mathsf{Fun}^{\times}(\mathbb{F}_T^{\mathsf{op}}, \mathsf{Cat})$.

Definition 2.18. Let $P \subset T$ be orbital and $\mathcal{E}$ be a category admitting finite products. We define the T-category

$$\underline{\mathsf{Mack}}_T^P(\mathcal{E}) \coloneqq \mathsf{Fun}^{\times}(\mathrm{Span}_{\mathsf{all},P}((\mathbb{F}_T)_{/\bullet}), \mathcal{E}) : \mathbb{F}_T^{\mathsf{op}} \xrightarrow{\times} \mathsf{Cat}.$$

By the above lemma, the underlying category $\Gamma\underline{\mathsf{Mack}}_T^P(\mathcal{E}) \coloneqq \lim_{\mathbb{F}_T^{\mathsf{op}}} \underline{\mathsf{Mack}}_T^P(\mathcal{E})$ is given by $\mathsf{Mack}_T^P(\mathcal{E})$. This construction induces a limit-preserving functor $\underline{\mathsf{Mack}}_T^P : \mathsf{Cat}^{\times} \to \mathsf{Cat}_T$.

Remark 2.19. Let us record a few facts about $\underline{\mathsf{Mack}}_T^P$.

1. For $X \in \mathbb{F}_T$ the forgetful π_X induces a natural equivalence $(\mathbb{F}_{T_{/X}})_{/\bullet} \simeq ((\mathbb{F}_T)_{/X})_{/\bullet} \simeq (\mathbb{F}_T)_{/\pi_X(\bullet)}$ compatible with the wide subcategories on morphisms (forgetting to morphisms) in $\mathbb{F}_T^P$. This in turn induces an equivalence of $T_{/X}$-categories natural in $\mathcal{E} \in \mathsf{Cat}^{\times}$:

$$\pi_X^* \underline{\mathsf{Mack}}_T^P(\mathcal{E}) \simeq \underline{\mathsf{Mack}}_{T_{/X}}^{\pi_X^{-1}(P)}(\mathcal{E}).$$

2. Since $\mathrm{Span}_{\mathsf{all},P}((\mathbb{F}_T)_{/\bullet})$ factors through semiadditive categories by Lemma B.2, the forgetful transformation $U : \mathsf{CMon}(-) \to (-)$ induce a natural equivalences of functors $\mathsf{Cat}^{\times} \to \mathsf{Cat}_T$

$$\underline{\mathsf{Mack}}_T^P \circ \mathsf{CMon} \simeq \underline{\mathsf{Mack}}_T^P.$$

2.2 P-symmetric monoidal T-categories

In this subsection we define P-symmetric monoidal T-categories as categorical $\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)$ Mackey functors. We use the theory of fibrous $\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T;T)$-patterns to define P-commutative T-algebras, and give some examples. We mention the results of [BHS22] on envelopes of fibrous patterns and determine the P-symmetric monoidal structure on the envelope $\mathcal{A}_T^P$ of the terminal $\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)$-pattern, which contains the "free P-commutative T-algebra" A_T^P, see Lemma 2.29(2).

Definition 2.20. Let $P \subset T$ be orbital.

1. The T-category of P-symmetric monoidal T-categories is given by $\underline{\mathsf{Mack}}_T^P(\mathsf{Cat})$. The category of P-symmetric monoidal T-categories $\mathsf{Mack}_T^P(\mathsf{Cat})$ upgrades to an $(\infty, 2)$-category with mapping categories of P-symmetric monoidal T-functors (cf. Remark D.5):

$$\mathsf{Fun}_T^{P-\otimes}(\mathcal{C}, \mathcal{D}) = \mathrm{Nat}_{\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)}(\mathcal{C}, \mathcal{D}).$$

2. For ease of notation we let $\mathsf{Fbrs}_T^P \coloneqq \mathsf{Fbrs}(\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T; T))$. The P-commutative T-operad is the terminal object $\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T) \in \mathsf{Fbrs}_T^P$. In particular, given $\mathcal{O} \in \mathsf{Fbrs}_T^P$, we define

$$\mathrm{CAlg}_T^P(\mathcal{O}) \coloneqq \mathsf{Fun}_{\mathsf{Fbrs}_T^P}(\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T), \mathcal{O}) = \mathsf{Fun}^{\mathbb{F}_T^{\mathsf{op}}-\mathsf{cc}}_{/\,\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)}(\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T), \mathcal{O}).$$

   If $\mathcal{C} \in \mathsf{Mack}_T^P(\mathsf{Cat})$ we let $\mathrm{CAlg}_T^P(\mathcal{C}) \coloneqq \mathrm{CAlg}_T^P(\smallint\mathcal{C})$, where $\smallint\mathcal{C} \in \mathsf{Fbrs}_T^P$ by Theorem 2.28 below.

Remark 2.21. Orb-symmetric monoidal Glo-categories will also be called equivariantly symmetric monoidal global categories.

Remark 2.22. In view of Corollary 2.14, this agrees with the usual theory of symmetric monoidal categories and operads of Lurie in the non-parametrized case $P = T = *$. In most of this text, we prefer to work with $\mathrm{Span}(\mathbb{F})$ over $\mathbb{F}_*$.

Definition 2.23. For every $X \in \mathbb{F}_T$, the unique coproduct-preserving functor $\mathbb{F} \to \mathbb{F}_T^P, * \mapsto X$ induces a finite-product preserving strong Segal morphism $i_X : \mathrm{Span}(\mathbb{F}) \to \mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)$ by Lemma 2.16. Using Theorem 2.13, this induces a forgetful functor natural in $\mathcal{C} \in \mathsf{Mack}_T^P(\mathsf{Cat})$:

$$i_X^* : \mathrm{CAlg}_T^P(\mathcal{C}) \to \mathrm{CAlg}(\mathcal{C}(X)).$$

Consider the contravariant slice functoriality $\mathrm{PSh}(T)_{/-}$ given by pullback. Restricting along $\mathbb{F}_T^{\mathsf{op}} \to \mathrm{PSh}(T)^{\mathsf{op}}$, this defines the T-category of T-spaces $\underline{\mathsf{Spc}}_T$.

Definition 2.24. Let $P \subset T$ be orbital. The pointwise full subcategories $\underline{\mathbb{F}}_T^P(X) \subseteq \underline{\mathsf{Spc}}_T(X)$ on morphisms $Y \to X$ in $\mathbb{F}_T^P$ assemble via orbitality of P and Lemma D.3 into the full T-subcategory of finite P-sets $\underline{\mathbb{F}}_T^P \subseteq \underline{\mathsf{Spc}}_T$. Equivalently, $\underline{\mathbb{F}}_T^P \simeq \mathrm{Un}^{\mathsf{ct}}(t : \mathrm{Ar}_P(\mathbb{F}_T) \to \mathbb{F}_T) \in \mathsf{Fun}^\times(\mathbb{F}_T^{\mathsf{op}}, \mathsf{Cat})$ where $\mathrm{Ar}_P(\mathbb{F}_T) \subseteq \mathrm{Ar}(\mathbb{F}_T)$ is full on morphisms in $\mathbb{F}_T^P$.

Example 2.25. Let $P \subset T$ be orbital, and let $\mathcal{C}$ be a T-category admitting finite P-coproducts. Adapting [NS22, 2.4.1], one can define a P-cocartesian P-symmetric monoidal structure on $\mathcal{C}$ via Barwick's unfurling construction. Indeed, the necessary prerequisites are precisely that $\mathcal{C}$ admits finite P-coproducts; one needs left-adjoints $p_! : \mathcal{C}(X) \to \mathcal{X}(Y)$ to p^* for all $p : X \to Y$ in $\mathbb{F}_T^P$ which further satisfy a certain Beck-Chevalley condition. This is spelled out in detail in [HHLN23b, 3.4]. The result of this construction is then a Mackey functor[6]

$$\mathcal{C}^{P-\sqcup} : \mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T) \xrightarrow{\times} \mathsf{Cat}, X \mapsto \mathcal{C}(X)$$

whose underlying T-category agrees with $\mathcal{C}$, where restriction along $\mathrm{Span}(\mathbb{F}) \to \mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T), * \mapsto X$ encodes the ordinary cocartesian monoidal structure on $\mathcal{C}(X)$, and where the norms are given by $p_\otimes = p_!$ for p in $\mathbb{F}_T^P$. Of course, the case of P-cartesian monoidal structures on T-categories admitting finite P-products is entirely analogous.

For an explicit example, consider $\underline{\mathbb{F}}_T^P$, which admits finite P-coproducts where the left adjoints are restricted from the postcomposition $p_! : (\mathbb{F}_T)_{/X} \to (\mathbb{F}_T)_{/Y}$. In fact, it was shown in [CLL23a, 4.2.17] that $\underline{\mathbb{F}}_T^P$ is even the free T-category admitting finite P-coproducts on one generator. By [HHLN23b, Example 3.4] the cocartesian fibration encoding Barwick's unfurling construction is given by

$$\mathrm{Span}(t) : \mathrm{Span}_{\mathsf{ct},tP}(\mathrm{Ar}_P(\mathbb{F}_T)) \to \mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T).$$

Specifically, morphisms in $\mathrm{Span}_{\mathsf{ct},tP}(\mathrm{Ar}_P(\mathbb{F}_T))$ are given by commutative diagrams (in $\mathbb{F}_T$)

$$\begin{tikzcd} Y \arrow[d, "P"'] & Y'' \arrow[l] \arrow[r] \arrow[d, "P"] \arrow[dl, phantom, "\llcorner"] & Y' \arrow[d, "P"'] \\ X & X'' \arrow[l] \arrow[r, "P"'] & X' \end{tikzcd}$$

where the left square is cartesian and morphisms labeled P lie in $\mathbb{F}_T^P$. Such a morphism is $\mathrm{Span}(t)$-cocartesian if and only if the top right horizontal morphism is an equivalence. One P-commutative T-algebra in $\underline{\mathbb{F}}_T^{P-\sqcup}$ which is easy to construct is the identity section $\mathrm{id}_{(-)} : \mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T) \to \mathrm{Span}_{\mathsf{ct},tP}(\mathrm{Ar}_P(\mathbb{F}_T))$. We will see below in Lemma 2.29 that in the case that P is atomic orbital, this is in some sense the "free P-commutative T-algebra".

Of course one would rather have a recognition criterion of P-cocartesian symmetric monoidal structures, analogous to the non-parametrized case in [Lur17, 2.4.0.1], as well as the statement that all objects are canonically P-commutative T-algebras analogously to [Lur17, 2.4.3.10]. However, to the author's knowledge no such results are known currently.

Example 2.26. Let G be a finite group. In Definition 4.18 we will define the G-symmetric monoidal category of genuine G-spectra $\underline{\mathsf{Sp}}_G^\otimes \in \mathsf{Mack}_G(\widehat{\mathsf{Cat}})$. Each restriction along $\mathrm{Span}(\mathbb{F}) \to \mathrm{Span}(G), * \mapsto G/H$ encodes the unique closed symmetric monoidal structure on Sp^H with $\mathbb{S}_H = \mathrm{infl}_H \mathbb{S}$ as unit. For every inclusion $K \leq H$, the corresponding backwards morphism $G/H \leftarrow G/K$ is sent to the restriction $\mathrm{res}^H_K : \mathsf{Sp}^H \to \mathsf{Sp}^K$, and $G/K \to G/H$ is sent to the Hill-Hopkins-Ravenel Norm $N^H_K : \mathsf{Sp}^K \to \mathsf{Sp}^H$. For a good overview of these functors in the model of orthogonal spectra we refer the reader to [Sch23, Section 10].

[6] By Lemma B.2(1), this extension preserves finite products because $\mathcal{C} \in \mathsf{Fun}^\times(\mathbb{F}_T^{\mathsf{op}}, \mathsf{Cat})$ does.

Now suppose that $R \in \mathrm{CAlg}_G(\underline{\mathsf{Sp}}_G^{\otimes})$. Using the notation of Definition 2.23 we obtain for every $H \leq G$ an underlying commutative H-ring spectrum $R_H := i_{G/H}^* R \in \mathrm{CAlg}(\mathsf{Sp}^H)$, and by functoriality we have $R_H \simeq \mathrm{res}_H^G R_G$. So what is the extra structure of $R \in \mathrm{CAlg}_G(\underline{\mathsf{Sp}}_G^{\otimes})$ as opposed to $R_G \in \mathrm{CAlg}(\mathsf{Sp}^G)$? This is precisely where the norms $N_H^G : \mathsf{Sp}^H \to \mathsf{Sp}^G$ come in: By definition R is a section of $\int \underline{\mathsf{Sp}}_G^{\otimes} \to \mathrm{Span}(G)$ which is cocartesian over backwards morphisms. For every inclusion $K \leq H \leq G$ we thus obtain a morphism $R(G/K) \to R(G/H)$ which we can factor uniquely as a cocartesian $R(G/K) \to N_K^H R(G/K)$ followed by an "equivariant multiplication" morphism of H-spectra

$$\mu_K^H : \bigotimes_{H/K} R(G/K) := N_K^H R(G/K) \to R(G/H).$$

In fact, using functoriality of R and symmetric monoidality of the norm this upgrades to a morphism $\mu_K^H : N_K^H R_K \to R_H$ in $\mathrm{CAlg}(\mathsf{Sp}^H)$.

Using these equivariant multiplications, one can define power operations and norm maps on the equivariant homotopy groups of R_G. For example, given an element $x \in \pi_0^H(R_G)$, represented by a map of spectra $f : \mathbb{S}_H \to \mathbb{R}_H$ (not of ring spectra), then we can send this to the map of spectra $\mathbb{S}_G = N_H^G \mathbb{S}_H \to N_H^G R_H \xrightarrow{\mu_H^G} R_G$, which represents an element of $\pi_0^G(R_G)$. Ultimately, this defines a function $\mathrm{norm}_H^G : \pi_0^H(R_G) \to \pi_0^G(R_G)$ also called multiplicative transfer, in analogy with the additive transfers $\mathrm{tr}_H^G : \pi_0^H(X) \to \pi_0^G(X)$ part of the Mackey functor structure on $\underline{\pi}_0(X)$ for any $X \in \mathsf{Sp}^G$. While μ_H^G is a map of spectra and hence additive, the map norm_H^G is generally not. Its properties and relations with the Mackey functor structure are elaborated on in [Sch23, Proposition 11.9]. As shown in [Bru07, Section 7.2], these extra operations ultimately endow $\underline{\pi}_0(R_G)$ with the structure of a Tambara Functor. In the global setting of ultracommutative ring spectra, the analogous results were obtained in [Sch18, Section 5.1]. Admitting such equivariant multiplications is still a fairly restrictive condition on a commutative G-ring spectrum, and provides a lot of extra structure one can leverage. For example, this was famously used by Hill-Hopkins-Ravenel in [HHR16] to solve the Kervaire invariant one problem.

For an explicit example, the unit $\mathbb{S}_G \in \mathsf{Sp}^G$ upgrades to a G-commutative algebra $\mathbb{S}_G \in \mathrm{CAlg}_G(\underline{\mathsf{Sp}}_G^{\otimes})$[7] and on π_0 we get the Burnside ring Tambara functor. Concretely, let $A(G)$ denote the Burnside ring, i.e. the group completion of the monoid of isomorphism classes of finite G-sets under direct sum, i.e. $K_0(\mathbb{F}_G, \oplus, \varnothing)$, with induced ring structure coming from the product of finite G-sets. The underlying abelian group is free on the cosets G/H, i.e. of rank the number of conjugacy classes of subgroups $H \leq G$. The rings $A(H)$ for $H \leq G$ assemble into a Mackey functor $\underline{A}$ by defining restrictions to simply restrict the action, and additive transfers by induction $\mathrm{Ind}_H^G : A(H) \to A(G)$. By [Sch23, Theorem 6.16, 11.11] there is an isomorphism of Mackey functors $\underline{A} \cong \underline{\pi}_0(\mathbb{S}_G)$ which is pointwise an isomorphism of rings, and norm_H^G corresponds to coinduction $\mathrm{Coind}_H^G : A(H) \to A(G)$.

[7]Formally, we can use the comparison functor $\Phi_G : \mathsf{UCom}_G \to \mathrm{CAlg}_G(\underline{\mathsf{Sp}}_G^{\otimes})$ we constructed in Construction 4.23 and note that it sends the strictly commutative sphere G-symmetric ring spectrum to the desired $\mathbb{S}_G \in \mathrm{CAlg}_G(\underline{\mathsf{Sp}}_G^{\otimes})$.

Remark 2.27. It is expected that such G-commutative algebras in $\underline{\mathsf{Sp}}_G^{\otimes}$ agree with the classical notion of G-$\mathbb{E}_\infty$-algebras in say orthogonal G-spectra, as mentioned in the introduction (and also conjectured in the introduction of [NS22] and claimed without proof below [BH17, 9.14]). White showed in [Whi14] that there is a Quillen equivalence between the model category of strictly commutative orthogonal G-ring spectra, also called ultracommutative G-ring spectra, and the model category of G-$\mathbb{E}_\infty$-algebras in orthogonal G-spectra. In Construction 4.23 we build a comparison functor $\Phi_G : \mathsf{UCom}_G \to \mathrm{CAlg}_G(\underline{\mathsf{Sp}}_G^{\otimes})$ which we expect to realize this conjectured equivalence.

We specialize one of the main results of [BHS22] to our class of examples.

Theorem 2.28. Let $P \subset T$ be orbital.

1. There is an adjunction of $(\infty, 2)$-categories

$$\mathrm{Env}_T^P : \mathsf{Fbrs}_T^P \rightleftarrows \mathsf{Mack}_T^P(\mathsf{Cat}) : \textstyle\int$$

where $\mathrm{Env}_T^P(\mathcal{P}) = \mathcal{P} \times_{\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)} \mathcal{A}_T^P$ where $\mathcal{A}_T^P = \mathrm{Env}_T^P(\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T))$ is the P-symmetric monoidal category corresponding to the P-commutative T-operad $\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)$. In particular, we obtain a composite equivalence natural in $\mathcal{C} \in \mathsf{Mack}_T^P(\mathsf{Cat})$:

$$\mathsf{Fun}_T^{P-\otimes}(\mathcal{A}_T^P, \mathcal{C}) \xrightarrow{\int} \mathsf{Fun}_{\mathsf{Fbrs}_T^P}(\textstyle\int \mathcal{A}_T^P, \int \mathcal{C}) \xrightarrow{\eta^*_{\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)}} \mathrm{CAlg}_T^P(\mathcal{C}). \tag{4}$$

2. $\mathcal{A}_T^P$ has cocartesian unstraightening given by the target projection

$$t : \mathrm{Ar}_{\mathsf{act}}(\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)) \to \mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)$$

where $\mathrm{Ar}_{\mathsf{act}}$ denotes the full subcategory on the active morphisms, i.e. forwards morphisms. Moreover, the unit transformation of the above adjunction, evaluated at $\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)$, is given by the identity section

$$\eta_{\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)} = \mathrm{id}_{(-)} : \mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T) \to \mathrm{Ar}_{\mathsf{act}}(\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)) \simeq \textstyle\int \mathcal{A}_T^P.$$

Proof. The existence of the adjunction and description of Env_T^P is part of [BHS22, Theorem C]. This is upgraded to an adjunction of $(\infty, 2)$-categories in [BHS22, Corollary 5.3.13]. The description of $\mathrm{Un}^{\mathsf{cc}}(\mathcal{A}_T^P)$ follows from [BHS22, Corollary 2.1.5]. $\square$

In the following lemma we determine what the P-symmetric monoidal structure on $\mathcal{A}_T^P$ looks like exactly. Let us first fix some notation. Recall the adjunction $\mathrm{Tw}^r : \mathsf{Cat} \rightleftarrows \mathsf{AdTrip} : \mathrm{Span}$ from Theorem 2.11. The adequate triple $\mathrm{Tw}^r\,\Delta^1$ has underlying category the span $\Lambda_1^2 = (0 \leftarrow 1 \to 2)$ with backwards morphism $0 \leftarrow 1$ and forwards $1 \to 2$. Let $\gamma = s_2 : \Lambda_1^2 \to \Delta^1$ send $0 \leftarrow 1$ to id_0 and $1 \to 2$ to $0 \to 1$. Note that this is the Dwyer-Kan localization of $\mathrm{Tw}^r(\Delta^1) = \Lambda_1^2$ at the backwards morphism. Indeed, recall from [Hin, 1.1.2] that we can compute this localization as the pushout given by the whole rectangle

$$\begin{tikzcd}
* \arrow[r, "0"] \arrow[d, "0"'] & \Delta^1 \arrow[r] \arrow[d] & {|\Delta^1| = *} \arrow[d] \\
\Delta^1 \arrow[r] & \mathrm{Tw}^r(\Delta^1) \arrow[r, "\gamma"'] \arrow[ul, phantom, "\ulcorner", very near start] & \Delta^1 \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Lemma 2.29. Let $P \subset T$ be orbital. We have a commutative diagram of cocartesian fibrations

$$\begin{tikzcd}
\int \mathcal{A}_T^P \arrow[rrr] \arrow[d, equal] & & & \int \underline{\mathbb{F}}_T^{P-\sqcup} \arrow[d, equal] \\
\mathrm{Ar}_{\mathsf{act}}(\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)) \arrow[r, "\simeq"] \arrow[dr, "t"'] & \mathrm{Span}(\mathrm{Ar}_P(\mathbb{F}_T), \mathrm{Ar}_P(\mathbb{F}_T)^{\mathsf{ct}}, \mathrm{Ar}(\mathbb{F}_T^P)) \arrow[rr] \arrow[d, dashed, "\mathrm{Span}(t)"] & & \mathrm{Span}_{\mathsf{ct},\mathrm{tP}}(\mathrm{Ar}_P(\mathbb{F}_T)) \arrow[dll, "\mathrm{Span}(t)"] \\
& \mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T) & &
\end{tikzcd}$$

Here $\underline{\mathbb{F}}_T^{P-\sqcup}$ and its explicit cocartesian fibration encode the P-cocartesian monoidal structure on $\underline{\mathbb{F}}_T^P$, as presented in Example 2.25. The equivalence is induced by the adjunction $\mathrm{Tw}^r : \mathsf{Cat} \rightleftarrows \mathsf{AdTrip} : \mathrm{Span}$ and restriction along γ. Moreover:

1. A morphism from $(X \to Y)$ to $(X' \to Y')$ in $\int \mathcal{A}_T^P$ can be identified with a commutative diagram (in $\mathbb{F}_T$)

$$\begin{tikzcd}
Y \arrow[d, "P"'] & Y'' \arrow[l] \arrow[r, "P"] \arrow[d, "P"] \arrow[dl, phantom, "\llcorner", very near start] & Y' \arrow[d, "P"] \\
X & X'' \arrow[l] \arrow[r, "P"'] & X'
\end{tikzcd}$$

   where the left square is cartesian and all morphisms labeled by P lie in $\mathbb{F}_T^P$. Such a morphism is $\mathrm{Span}(t)$-cocartesian if and only if the top right horizontal morphism is an equivalence. Note that $\int \mathcal{A}_T^P \subset \int \underline{\mathbb{F}}_T^{P-\sqcup}$ thus have the same $\mathrm{Span}(t)$-cocartesian morphisms.

2. The identity section $\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T) \to \int A_T^P$ gives a P-commutative T-algebra $A_T^P \in \mathrm{CAlg}(\mathcal{A}_T^P)$. This is the free in the sense that for every $\mathcal{C} \in \mathsf{Mack}_T^P(\mathsf{Cat})$ and $R \in \mathrm{CAlg}_T^P(\mathcal{C})$ there is a unique P-symmetric monoidal T-functor $F : \mathcal{A}_T^P \to \mathcal{C}$ so that the induced $\mathrm{CAlg}_T^P(F)$ sends A_T^P to R.

3. The resulting P-symmetric monoidal T-functor $\mathcal{A}_T^P \to \underline{\mathbb{F}}_T^{P-\sqcup}$ is fiberwise the inclusion $(\mathbb{F}_T^P)_{/B} \subset \underline{\mathbb{F}}_T^P(B)$, and hence an equivalence whenever P is atomic.

4. In particular, all functoriality of $\mathcal{A}_T^P$ is inherited from $\underline{\mathbb{F}}_T^P$, i.e. restrictions are given by pullback, norms by postcomposition, and analogous to $\underline{\mathbb{F}}_T^P$ also $\mathcal{A}_T^P$ encodes the non-parametrized cocartesian monoidal structure on each of its fibers $\mathcal{A}_T^P(X) = (\mathbb{F}_T^P)_{/X}$.

Proof. For ease of notation we let $\mathbb{F}_{T,T,P}$ denote the adequate triple $(\mathbb{F}_T, \mathbb{F}_T, \mathbb{F}_T^P)$. Moreover, recall that $\mathsf{Fun}_{\mathsf{AdTrip}}(X,Y) \subseteq \mathsf{Fun}(X,Y)$ is full on the morphisms of adequate triples, and can be equipped with the structure of an adequate triple. The label $\mathsf{Fun}^{\{0\leftarrow 1\}^{-1}}$ denotes the full subcategory on those functors which invert the backwards morphism $0 \leftarrow 1$ in $\mathrm{Tw}(\Delta^1)$. Then one checks that

$$\mathsf{Fun}^{\{0\leftarrow 1\}^{-1}}_{\mathsf{AdTrip}}(\mathrm{Tw}^r(\Delta^1), X) = \mathsf{Fun}_{\mathsf{AdTrip}}(\Delta^1_{=,\mathsf{all}}, X)$$

where $(\Delta^1_{=,\mathsf{all}}$ is the adequate triple on Δ^1 with backwards morphisms identities and forwards morphisms everything. It is straightforward to verify from the definition of the adequate triple structure on $\mathsf{Fun}_{\mathsf{AdTrip}}$ that

$$\mathrm{Span}(\mathsf{Fun}_{\mathsf{AdTrip}}(\Delta^1_{=,\mathsf{all}}, \mathbb{F}_{T,T,P})) = \mathrm{Span}_{\mathsf{ct},P}(\mathrm{Ar}_P(\mathbb{F}_T)).$$

Now we have the following diagram

$$\begin{tikzcd}
\mathrm{Ar}_{\mathsf{act}}(\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)) \arrow[r, "\simeq"] \arrow[d, hook] & \mathrm{Span}(\mathsf{Fun}^{\{0\leftarrow 1\}^{-1}}_{\mathsf{AdTrip}}(\mathrm{Tw}^r(\Delta^1), \mathbb{F}_{T,T,P})) \arrow[r, "\gamma^*", "\simeq"'] \arrow[d, hook] & \mathrm{Span}(\mathsf{Fun}_{\mathsf{AdTrip}}((\Delta^1, (\Delta^1)^{\simeq}, \Delta^1), \mathbb{F}_{T,T,P})) \arrow[dd, equal] \\
\mathsf{Fun}(\Delta^1, \mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)) \arrow[r, "\simeq"'] \arrow[d, "t"'] & \mathrm{Span}(\mathsf{Fun}_{\mathsf{AdTrip}}(\mathrm{Tw}^r(\Delta^1), \mathbb{F}_{T,T,P})) \arrow[d, "\mathrm{Span}(\mathrm{ev}_2)"] & \\
\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T) \arrow[r, equal] & \mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T) & \mathrm{Span}_{\mathsf{ct},P}(\mathrm{Ar}_P(\mathbb{F}_T)) \arrow[l, "\mathrm{Span}(t)"]
\end{tikzcd}$$

which shows the desired equivalence $\mathrm{Ar}_{\mathsf{act}}(\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)) \simeq \mathrm{Span}_{\mathsf{ct},P}(\mathrm{Ar}_P(\mathbb{F}_T))$ over $\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)$. In $\mathsf{Fun}(\Delta^1, \mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)) = \mathrm{Span}(\mathsf{Fun}_{\mathsf{AdTrip}}(\mathrm{Tw}^r\,\Delta^1, \mathbb{F}_{T,T,P}))$ a morphism from $X'' \leftarrow X \xrightarrow{P} X'$ to $Z'' \leftarrow Z \xrightarrow{P} Z'$ are diagrams of the form

$$\begin{tikzcd}
X'' & Y'' \arrow[l] \arrow[r, "P"] & Z'' \\
X \arrow[u] \arrow[d, "P"'] & Y \arrow[u] \arrow[l] \arrow[r, "P"] \arrow[d, "P"] \arrow[ur, phantom, "\urcorner", very near start] \arrow[dl, phantom, "\llcorner", very near start] & Z \arrow[u] \arrow[d, "P"] \\
X' & Y' \arrow[l] \arrow[r, "P"'] & Z'
\end{tikzcd}$$

and the full subcategories consist of those object where all upwards maps are equivalences, and hence we're basically only considering morphisms

$$\begin{tikzcd}
X \arrow[d, "P"'] & Y \arrow[l] \arrow[r, "P"] \arrow[d, "P"] \arrow[dl, phantom, "\llcorner", very near start] & Z \arrow[d, "P"] \\
X' & Y' \arrow[l] \arrow[r, "P"'] & Z'
\end{tikzcd}$$

Regarding the addenda:

1. The description of morphisms in $\int \mathcal{A}_T^P$ is now clear. The description of the cocartesian morphisms can be obtained via both of the above models for $\int \mathcal{A}_T^P$. Using $\mathrm{Ar}_{\mathsf{act}}(\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T))$, it follows from the fact that the active/forwards morphisms form the right class of an orthogonal factorization system on $\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)$ by [HHLN23b, 4.9], and then using [LNP22, Proposition 6.7]. On the other hand, $\mathrm{Span}_{\mathsf{ct},P}(\mathrm{Ar}_P(\mathbb{F}_T)) \subset \mathrm{Span}_{\mathsf{ct},tP}(\mathrm{Ar}_P(\mathbb{F}_T))$ is the inclusion of a wide subcategory over $\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)$ and in fact both have the same $\mathrm{Span}(t)$-cocartesian morphisms, compare Example 2.25.

2. From the above description of cocartesian morphisms in $\int \mathcal{A}_T^P$ it is clear that the identity section gives a P-commutative algebra A_T^P as claimed. Now by the adjunction equivalence (4) there exists a unique P-symmetric monoidal T-functor F with $R \simeq (\int F)_*(A_T^P)$, as desired (note that $\mathrm{CAlg}_T^P(F)$ is given by postcomposing with $\int F$).

3. This follows from Lemma A.9(3c).

4. The diagram gives us a natural transformation $\mathcal{A}_T^P \Rightarrow \underline{\mathbb{F}}_T^{P-\sqcup}$, which is pointwise the inclusion of a wide subcategory. The space of such natural transformations is contractible by Lemma D.3. Alternatively, we already know that both cocartesian unstraightenings have the same cocartesian morphisms. $\square$

Warning 2.30. Let us remark on a potentially confusing issue regarding the previous corollary. Namely, if $P \subset T$ is orbital, then $(\mathbb{F}_T^P)_{/-} \subset \underline{\mathbb{F}}_T^P$ is a pointwise wide subcategory. The latter admits finite P-coproducts, and the former has a P-symmetric monoidal structure which is fiberwise the cocartesian one, and the norms $p_\otimes$ agree with the restrictions of the left adjoints $p_!$ (i.e. postcomposition on the slices) encoding the existence of p-coproducts in $\underline{\mathbb{F}}_T^P$, c.f. Example 2.25. Hence one might come to expect that also $(\mathbb{F}_T^P)_{/-}$ admits finite P-coproducts, and that $\mathcal{A}_T^P$ is encoding the P-cocartesian P-symmetric monoidal structure. However, for non-atomic P, this will generally be false: Going through the proof that $\underline{\mathbb{F}}_T^P$ admits finite P-coproducts (see [CLL23a, 4.2.16]), every step also works for $(\mathbb{F}_T^P)_{/-}$, except that the unit transformation of the adjunction $p_! \dashv p^*$ will not restrict to the wide subcategory $(\mathbb{F}_T^P)_{/-}$. Indeed, for $p : A \to B$, it is pulled back from the diagonal $A \to A \times_B A$, and neither this diagonal nor the pullback need generally be morphisms in $\mathbb{F}_T^P$.

2.3 Modules

The aim of this subsection is to construct parametrized module categories; given a P-symmetric monoidal T-category $\mathcal{C}$ and an algebra $R \in \mathrm{CAlg}_T^P(\mathcal{C})$, we want to understand when we can construct a P-symmetric monoidal T-category of modules $\underline{\mathsf{Mod}}_R(\mathcal{C}) \in \mathsf{Mack}_T^P(\mathsf{Cat})$ which at $B \in T^{\mathsf{op}}$ is $\mathsf{Mod}_{R(B)}(\mathcal{C}(B))$. Analogously to the non-parametrized case, this will be possible whenever $\mathcal{C}$ is compatible with geometric realizations in a sense to be defined below. We begin with some recollections on module categories.

Given a symmetric monoidal category $\mathcal{C}$ and a commutative algebra $A \in \mathrm{CAlg}(\mathcal{C})$, we have a category $\mathsf{Mod}_A(\mathcal{C})$ of A-modules in $\mathcal{C}$. Formally, one can define an operad $\mathcal{CM}^{\otimes}$ for modules over a commutative algebra as in [LNP22, Definition A.1]. Specifically, a $\mathcal{CM}$-algebra in a symmetric monoidal category $\mathcal{C}$ is a pair (A, M) where A is a commutative algebra and M an A-module. It was shown in [Gla14, Proposition 7] or [Hin, Lemma B.1.1] that for any operad $\mathcal{O}$ there is an equivalence $\mathsf{Mod}^{\mathsf{Fin}_*}(\mathcal{O}) \simeq \mathrm{Alg}_{\mathcal{CM}}(\mathcal{O})$, where the former category is defined in [Lur17, Section 3.3.3]. There is a forgetful functor $\mathrm{Alg}_{\mathcal{CM}}(\mathcal{C}) \to \mathrm{CAlg}(\mathcal{C})$ and given $A \in \mathrm{CAlg}(\mathcal{C})$ one sets $\mathsf{Mod}_A(\mathcal{C}) \coloneqq \mathrm{Alg}_{\mathcal{CM}}(\mathcal{C}) \times_{\mathrm{CAlg}(\mathcal{C})} \{A\}$. This is an analogue of Lurie's operad $\mathcal{LM}^{\otimes}$ characterizing left modules over associative algebras.

Given a map of commutative algebras $f : A \to B$, every B-module can be viewed as an A-module via restriction, inducing a forgetful functor $f^* : \mathsf{Mod}_B(\mathcal{C}) \to \mathsf{Mod}_A(\mathcal{C})$. If the monoidal structure on $\mathcal{C}$ is compatible with geometric realizations[8] then we obtain an induced symmetric monoidal structure on $\mathsf{Mod}_A(\mathcal{C})$ via the relative tensor product $\otimes_A$ (cf. [Lur17, 4.5.2.1]) and a symmetric monoidal left adjoint $B \otimes_A - : \mathsf{Mod}_A(\mathcal{C}) \to \mathsf{Mod}_B(\mathcal{C})$ to f^*, see [Lur17, 4.6.2.17]. In fact, the restriction functors can generally be assembled into a functor $\mathrm{CAlg}(\mathcal{C})^{\mathsf{op}} \to \mathsf{Cat},\ A \mapsto \mathsf{Mod}_A(\mathcal{C})$, see [Lur17, 4.2.3.2]. In the case that $\mathcal{C}$ is compatible with geometric realizations, we can then take pointwise left adjoints and obtain the functor

$$\mathsf{Mod}_{\bullet}(\mathcal{C}) : \mathrm{CAlg}(\mathcal{C}) \to \mathsf{Cat},\ (f : A \to B) \mapsto B \otimes_A -.$$

as done in [Lur17, 4.5.3.1]. This is the functoriality we are interested in.

In our case, we start with a P-symmetric monoidal T-category $\mathcal{C}$ compatible with geometric realizations (to be defined below, cf. Definition 2.32) and an algebra $R \in \mathrm{CAlg}_T^P(\mathcal{C})$, and we want to construct a P-symmetric monoidal T-category $\underline{\mathsf{Mod}}_R(\mathcal{C}) \in \mathsf{Mack}_T^P(\mathsf{Cat})$ which at $B \in T$ is $\mathsf{Mod}_{R(B)}(\mathcal{C}(B))$. Note however that for this we need to know the functoriality of module categories not only in the algebra, but actually in pairs $(\mathcal{D}, A)$ where $\mathcal{D} \in \mathsf{CMon}(\mathsf{Cat})$ and $A \in \mathrm{CAlg}(\mathcal{D})$. This has been investigated by Lurie in [Lur17, Section 4.8.3-4.8.5], however we will opt to use the technology of [LNP22, Theorem 5.10, Appendix A] instead.

Let $\mathcal{I}$ be a category and $\mathcal{I}^{\sqcup}$ the cocartesian ∞-operad as defined in [Lur17, Section 2.4.3]. There is a canonical functor $\ell : \mathcal{I} \times \mathsf{Fin}_* \to \mathcal{I}^{\sqcup}$ sending (i, n_+) to the constant tuple $(i, \ldots, i)$ with n entries. Given an $\mathcal{I}^{\sqcup}$-monoidal category $\mathcal{C}^{\otimes} \to \mathcal{I}^{\sqcup}$, we can pull back along ℓ, straighten the cocartesian fibration, and curry to obtain $\mathcal{I} \to \mathsf{Fun}(\mathsf{Fin}_*, \mathsf{Cat})$, which will then by assumption of $\mathcal{I}^{\sqcup}$-monoidality factor through $\mathsf{CMon}(\mathsf{Cat})$[9].

Proposition 2.31 ([DG22, A.12])**.** Let $\mathsf{Cat}_{\mathcal{I}^{\sqcup}}^{\otimes}$ denote the category of $\mathcal{I}^{\sqcup}$-monoidal categories. The above construction furnishes an equivalence

$$\mathsf{Cat}_{\mathcal{I}^{\sqcup}}^{\otimes} \simeq \mathsf{Fun}(\mathcal{I}, \mathsf{CMon}(\mathsf{Cat})).$$

Let $\mathcal{K}$ be a collection of categories. We say that an $\mathcal{I}^{\sqcup}$-monoidal category $\mathcal{C}^{\otimes} \to \mathcal{I}^{\sqcup}$ is compatible

[8] So $\mathcal{C}$ admits geometric realizations and $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves them separately in each variable.
[9] By Lemma D.4 this agrees with the construction described in [LNP22, Construcion 5.4].

with $\mathcal{K}$-indexed colimits if it is so in the sense of [Lur17, 3.1.1.18]. Concretely, this means that each $\mathcal{C}_i$ admits $\mathcal{K}$-indexed colimits, the tensor product $\otimes : \mathcal{C}_i \times \mathcal{C}_i \to \mathcal{C}_i$ preserves $\mathcal{K}$-indexed colimits separately in each variable, and each pushforward $(i \to j)_\otimes : \mathcal{C}_i \to \mathcal{C}_j$ preserves $\mathcal{K}$-indexed colimits. In the case $\mathcal{K} = \{\Delta^{\mathsf{op}}\}$ we say that $\mathcal{C}^\otimes$ is compatible with geometric realizations. This is needed for relative tensor products to exist, so that categories of modules in $\mathcal{C}$ admit a symmetric monoidal structure and covariant functoriality in the algebra. We will need an analogous definition for P-symmetric monoidal T-categories.

Definition 2.32. Let $P \subset T$ be orbital and $\mathcal{K}$ a collection of categories. Given a P-symmetric monoidal T-category $\mathcal{C} \in \mathsf{Mack}_T^P(\mathsf{Cat})$, we say that $\mathcal{C}$ is compatible with $\mathcal{K}$-indexed colimits if the following conditions are satisfied:

1. For every $B \in T$, the category $\mathcal{C}(B)$ admits $\mathcal{K}$-indexed colimits, and the tensor product $\otimes_B : \mathcal{C}(B) \times \mathcal{C}(B) \to \mathcal{C}(B)$ preserves $\mathcal{K}$-indexed colimits in both variable separately.
2. For every $f : A \to B$ in T, the restriction $f^* : \mathcal{C}(B) \to \mathcal{C}(A)$ preserves $\mathcal{K}$-indexed colimits.
3. For every $p : A \to B$ in P, the norm $p_\otimes : \mathcal{C}(A) \to \mathcal{C}(B)$ preserves $\mathcal{K}$-indexed colimits.

We let $\mathsf{Mack}_T^P(\mathsf{Cat}; \mathcal{K}) \subset \mathsf{Mack}_T^P(\mathsf{Cat})$ denote the subcategory on those P-symmetric monoidal T-categories compatible with $\mathcal{K}$-indexed colimits, and morphisms $f : \mathcal{C} \to \mathcal{D}$ so that each $f(B) : \mathcal{C}(B) \to \mathcal{D}(B)$ preserves $\mathcal{K}$-indexed colimits for $B \in T^{\mathsf{op}}$.

Lemma 2.33. Let $P \subset T$ be orbital, $\mathcal{K}$ a collection of categories and $f : \mathcal{C} \to \mathcal{D}$ in $\mathsf{Mack}_T^P(\mathsf{Cat}; \mathcal{K})$.

1. The underlying T-category $\mathcal{C}$ admits fiberwise $\mathcal{K}$-colimits, i.e. $\mathcal{C}|_{\mathbb{F}_T^{\mathsf{op}}} : \mathbb{F}_T^{\mathsf{op}} \xrightarrow{\times} \mathsf{Cat}$ factors through $\mathsf{Cat}(K)$, and $f_X : \mathcal{C}(X) \to \mathcal{D}(X)$ preserves $\mathcal{K}$-indexed colimits for every $X \in \mathbb{F}_T$.
2. If $\mathcal{K}$ consists entirely of sifted categories, then $\mathsf{Mack}_T^P(\mathsf{Cat}; \mathcal{K}) \simeq \mathsf{Mack}_T^P(\mathsf{Cat}(\mathcal{K}))$, where $\mathsf{Cat}(\mathcal{K}) \subset \mathsf{Cat}$ is the subcategory on categories admitting $\mathcal{K}$-indexed colimits and functors preserving them.
3. Let $\mathcal{C}^\otimes \to \mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)^\sqcup$ be the $\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)^\sqcup$-monoidal category corresponding to $\mathcal{C}$ under the equivalence from Proposition 2.31. Then $\mathcal{C}^\otimes$ is compatible with $\mathcal{K}$-indexed colimits.
4. $\mathsf{Mack}_T^P(\mathsf{Cat}; \mathcal{K}) \subset \mathsf{Mack}_T^P(\mathsf{Cat})$ is closed under limits.

Proof. Recall that colimits in limits of categories are computed pointwise. For products, this follows from the statement for functor categories, and for pullbacks this is [Lur09, 5.4.5.5]. Moreover, a functor preserves all limits if and only if it preserves products and pullbacks by [Lur09, 4.4.2.7]. Finally, (by definition) colimits indexed on sifted categories commute with finite products, see [Lur09, Section 5.5.8]. From this all claims follow easily. □

We come to the main theorem we will use for the construction of parametrized module categories. Because we need some of the tools from their proof later on, we also describe their construction.

Construction/Theorem 2.34 ([LNP22, Theorem 5.10, Appendix A])**.** Let $\mathcal{C}^\otimes \to \mathcal{I}^\sqcup$ be a $\mathcal{I}^\sqcup$-monoidal category compatible with geometric realizations. By Construction 5.11 of op.cit., there is for every operad $\mathcal{O}$ an $\mathcal{I}^\sqcup$-monoidal category $\mathrm{Alg}_{\mathcal{O}^\otimes/\mathcal{I}^\sqcup}(\mathcal{C})^\otimes \to \mathcal{I}^\sqcup$ which corresponds to the functor $\mathcal{I} \to \mathsf{CMon}(\mathsf{Cat}), i \mapsto \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}_i)^\otimes$. Recall the commutative module operad $\mathcal{CM}^\otimes$ from Definition A.1 of op.cit., which admits an inclusion $\mathsf{Fin}_* \to \mathcal{CM}^\otimes$. By [Lur17, 3.2.4.3(3)] this induces an $\mathcal{I}^\sqcup$-monoidal functor

$$p_{\mathcal{I}} : \mathrm{Alg}_{\mathcal{CM}^\otimes/\mathcal{I}^\sqcup}(\mathcal{C})^\otimes \to \mathrm{Alg}_{\mathsf{Fin}_*/\mathcal{I}^\sqcup}(\mathcal{C})^\otimes.$$

which in the fiber over $\{x_j\}_{j\in J} \in \mathcal{I}^\sqcup$ is given by the product $\prod_{j\in J} \mathsf{Mod}(\mathcal{C}_{x_j}) \to \prod_{j\in J} \mathrm{CAlg}(\mathcal{C}_{x_j})$ of the cocartesian fibration from [Lur17, 4.5.3.1]. Moreover, by [LNP22, Lemma A.9] $p_{\mathcal{I}}$ is a cocartesian fibration, with cocartesian morphisms those given by the composition of a cocartesian morphism over $\mathcal{I}^\sqcup$ followed by a fiberwise cocartesian morphism. Chasing through the definitions, this gives in particular that the cocartesian pushforward of a morphism of algebras $F : A \to B$ lying over $f : i \to j$ in $\mathcal{I}$ is given by

$$\mathsf{Mod}_A(\mathcal{C}_i) \xrightarrow{f_\otimes} \mathsf{Mod}_{f_\otimes A}(\mathcal{C}_j) \xrightarrow{B\otimes_{f_\otimes A} -} \mathsf{Mod}_B(\mathcal{C}_j). \tag{5}$$

For $R \in \mathrm{Alg}_{\mathcal{I}^\sqcup/\mathcal{I}^\sqcup}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{I}^\sqcup}(\mathrm{Alg}_{\mathsf{Fin}_*/\mathcal{I}^\sqcup}(\mathcal{C}))$ one defines the parametrized module category $\mathsf{Mod}_{R_\bullet}(\mathcal{C}_\bullet) : \mathcal{I} \to \mathsf{CMon}(\mathsf{Cat})$ by applying the equivalence from Proposition 2.31 to the left vertical in

$$\begin{tikzcd}
\int \mathsf{Mod}_{R_\bullet}(\mathcal{C}_\bullet) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Alg}_{\mathcal{CM}^\otimes/\mathcal{I}^\sqcup}(\mathcal{C})^\otimes \arrow[d, "p_{\mathcal{I}}"] \\
\mathcal{I}^\sqcup \arrow[r, "R"] & \mathrm{Alg}_{\mathsf{Fin}_*/\mathcal{I}^\sqcup}(\mathcal{C})^\otimes
\end{tikzcd}$$

The functoriality is that of (5). Moreover, there is a map of $\mathcal{I}^\sqcup$-monoidal categories

$$P : \mathrm{Alg}_{\mathcal{CM}^\otimes/\mathcal{I}^\sqcup}(\mathcal{C})^\otimes \to \mathrm{Alg}_{\mathsf{Fin}_*/\mathcal{I}^\sqcup}(\mathcal{C})^\otimes \times_{\mathcal{I}^\sqcup} \mathcal{C}^\otimes$$

picking out the commutative algebra object and underlying object of the module. We have $p_{\mathcal{I}} = \mathrm{pr}_{\mathrm{Alg}} \circ P$. Their proof also shows that P admits an $\mathcal{I}^\sqcup$-monoidal relative left adjoint F. This yields a $\mathcal{I}^\sqcup$-monoidal functor $F \circ (R, \mathrm{id}) : \mathcal{C}^\otimes \to \int \mathsf{Mod}_{R_\bullet}(\mathcal{C}_\bullet)$ with operadic fiber over $i \in \mathcal{I}$ given by the symmetric monoidal free R_i-module functors $R_i \otimes - : \mathcal{C}_i \to \mathsf{Mod}_{R_i}(\mathcal{C}_i)$.

Corollary 2.35. In the context of Construction/Theorem 2.34, suppose that $\mathcal{I}$ admits finite products, that $\mathcal{C}_\bullet : \mathcal{I} \xrightarrow{\times} \mathsf{CMon}(\mathsf{Cat})$ preserves them, and that R is cocartesian on all projections in $\mathcal{I}$. Then $\mathsf{Mod}_{R_\bullet}(\mathcal{C}_\bullet) : \mathcal{I} \xrightarrow{\times} \mathsf{CMon}(\mathsf{Cat})$ also preserves finite products.

Proof. Consider the following commutative square

$$\begin{tikzcd} \mathsf{Mod}_{R_{i\times j}}(\mathcal{C}_{i\times j}) \arrow[r, "\simeq"] \arrow[d] & \mathsf{Mod}_{(\mathcal{C}_{\mathrm{pr}_i} R_{i\times j},\, \mathcal{C}_{\mathrm{pr}_j} R_{i\times j})}(\mathcal{C}_i \times \mathcal{C}_j) \arrow[d, "\simeq"] \\ \mathsf{Mod}_{R_i}(\mathcal{C}_i) \times \mathsf{Mod}_{R_j}(\mathcal{C}_j) & \mathsf{Mod}_{\mathcal{C}_{\mathrm{pr}_i} R_{i\times j}}(\mathcal{C}_i) \times \mathsf{Mod}_{\mathcal{C}_{\mathrm{pr}_j} R_{i\times j}}(\mathcal{C}_j) \arrow[l] \end{tikzcd}$$

The left vertical map is induced by the functoriality of $\mathsf{Mod}_{R_\bullet}(\mathcal{C}_\bullet)$ applied to pr_i and pr_j. The top horizontal equivalence is induced by the symmetric monoidal equivalence $(\mathcal{C}_{\mathrm{pr}_i}, \mathcal{C}_{\mathrm{pr}_j}) : \mathcal{C}_{i\times j} \to \mathcal{C}_i \times \mathcal{C}_j$ from the assumption that $\mathcal{C}$ preserves finite products, the right vertical equivalence comes from the fact that generally $\mathrm{Alg}_{\mathcal{O}}(-) : \mathsf{Op} \to \mathsf{Cat}$ preserves limits (and $\mathsf{Mod}_A(\mathcal{C}) = \{A\} \times_{\mathrm{CAlg}(\mathcal{C})} \mathrm{Alg}_{\mathcal{CM}}(\mathcal{C})$). The bottom map is now given on the first component by $R_i \otimes_{\mathcal{C}_{\mathrm{pr}_i} R_{i\times j}} -$ and analogously for the second. But if R is cocartesian on pr_i then the comparison map $\mathcal{C}_{\mathrm{pr}_i} R_{i\times j} \to R_i$ is an equivalence, and so by assumption the bottom horizontal functor is also an equivalence. so we conclude the claim. By 2-out-of-3 also the left vertical functor is an equivalence, as desired. $\square$

Algebras $R \in \mathrm{CAlg}_T^P(\mathcal{C})$ can be pulled back to algebras in $\mathrm{Alg}_{\mathcal{I}^\sqcup/\mathcal{I}^\sqcup}(\mathcal{C}^\otimes)$ which are automatically cocartesian on all projections in $\mathcal{I} = \mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)$. More generally, let S be a semiadditive category and $S^\sqcup \to \mathsf{Fin}_*$ encode the cocartesian monoidal structure and $\ell : S \times \mathsf{Fin}_* \to S^\sqcup, (s, n_+) \mapsto (s, \dots, s)$ the canonical map over Fin_* as in [Lur17, Section 2.4.3]. Now by semiadditivity of S this monoidal structure is also cartesian, so there exists a symmetric monoidal equivalence $S^\sqcup \simeq S^\times$ whose underlying functor is homotopic to the identity by [Lur17, 2.4.1.8]. In particular, we have a cartesian structure $\pi : S^\sqcup \to S$ in the sense of [Lur17, 2.4.1.1], whose underlying functor is the identity. Concretely, this means that π inverts active maps and gives equivalences $\prod_{i=1}^n \pi(\rho^i) : \pi(s_1, \dots, s_n) \to \bigoplus_{i=1}^n s_i$ where $\rho^i : (s_1, \dots, s_n) \to (s_i)$ is the cocartesian lift of the inert morphism $n_+ \to 1_+$ sending $i \mapsto 1$ and everything else to the basepoint. In particular, π sends $\rho^i : (s_1, \dots, s_n) \to (s_i)$ to $\mathrm{pr}_i : \bigoplus_{j=1}^n s_j \to s_i$.

Lemma 2.36. Let S be semiadditive and $f : S \xrightarrow{\times} \mathsf{Cat}$ preserve finite products. We have cartesian squares where all vertical maps are cocartesian fibrations

$$\begin{tikzcd} \int f \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \int f\pi\ell \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \int f\pi \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \int f \arrow[d] \\ S \arrow[r, "{(-,1)}"] & S \times \mathsf{Fin}_* \arrow[r, "\ell"] & S^\sqcup \arrow[r, "\pi"] & S \arrow[r, "f"] & \mathsf{Cat} \end{tikzcd}$$

Here $\int f\pi \to S^\sqcup$ is the $S^\sqcup$-monoidal category encoding the lift $S \xrightarrow{\times} \mathsf{CMon}(\mathsf{Cat})$ of f through $U : \mathsf{CMon}(\mathsf{Cat}) \to \mathsf{Cat}$. Moreover, precomposition with π induces a functor

$$\mathsf{Fun}^{\mathrm{pr-cc}}_{/S}(S, \textstyle\int f) \to \mathsf{Fun}^{\mathsf{int},\mathrm{pr-cc}}_{/S^\sqcup}(S^\sqcup, \textstyle\int f\pi)$$

from sections which are cocartesian on all projections $\mathrm{pr}_i : \bigoplus_{j=1}^n s_j \to s_i$ to sections which are cocartesian on all inert morphisms and all projections in S.

Proof. Note that $\pi \circ \ell \circ (-,1)$ is homotopic to the identity, which gives the diagram. To see that $\int f\pi \to S^{\sqcup}$ is $S^{\sqcup}$-monoidal, we need to show that the cocartesian maps $\rho^i(s_1,\dots,s_n) \to s_i$ in $S^{\sqcup}$ induce equivalences $(\int f\pi)_{(s_1,\dots,s_n)} \to \prod_{i=1}^n (\int f\pi)_{c_i}$. This follows from the fact that f preserves finite products and π sends the ρ^i to the projections as mentioned above.

Now given a section $R \in \mathsf{Fun}^{\mathrm{pr-cc}}_{/S}(S, \int f)$, we see that $R\pi$ is cocartesian on all inert morphisms ρ^i. Moreover, the restriction of $R\pi$ to S (i.e. along $\ell\circ(-,1)$) just gives back R and is hence also cocartesian on projections in S. It follows from Lemma D.1 and the fact that projections out of a product are jointly conservative that $R\pi$ is actually cocartesian on all fiberwise projections. □

Theorem 2.37. Let $P \subset T$ be orbital and $\mathcal{C} \in \mathsf{Mack}_T^P(\mathsf{Cat}; \{\Delta^{\mathsf{op}}\})$ a P-symmetric monoidal T-category compatible with geometric realizations. Then there exists a functor

$$\underline{\mathsf{Mod}}_{(-)}(\mathcal{C}) : \mathrm{CAlg}_T^P(\mathcal{C}) \to \mathsf{Mack}_T^P(\mathsf{Cat}), R \mapsto \underline{\mathsf{Mod}}_R(\mathcal{C}) = \mathsf{Mod}_{R_\bullet}(\mathcal{C}_\bullet).$$

A morphism $f : R \to S$ is sent to $S \otimes_R - : \underline{\mathsf{Mod}}_R(\mathcal{C}) \to \underline{\mathsf{Mod}}_S(\mathcal{C})$ which at $B \in T^{\mathsf{op}}$ is given by the symmetric monoidal left adjoint $S(B) \otimes_{R(B)} - : \mathsf{Mod}_{R(B)}(\mathcal{C}(B)) \to \mathsf{Mod}_{S(B)}(\mathcal{C}(B))$. Moreover, for every $R \in \mathrm{CAlg}_T^P(\mathcal{C})$:

1. there is a P-symmetric monoidal free R-module functor $R \otimes - : \mathcal{C} \to \underline{\mathsf{Mod}}_R(\mathcal{C})$,
2. and a *lax* P-symmetric monoidal forgetful functor $U : \underline{\mathsf{Mod}}_R(\mathcal{C}) \to \mathcal{C}$, i.e. a morphism $U : \int \underline{\mathsf{Mod}}_R(\mathcal{C}) \to \int \mathcal{C}$ in Fbrs_T^P.

Proof. For ease of notation we let $\mathcal{I} := \mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)$. Let $\Phi_{\mathcal{I}} : \mathsf{Cat}^{\otimes}_{\mathcal{I}^{\sqcup}} \simeq \mathsf{Fun}(\mathcal{I}, \mathsf{CMon}(\mathsf{Cat}))$ denote the equivalence from Proposition 2.31, and let $\mathcal{C}^{\otimes} \to \mathcal{I}^{\sqcup}$ be the $\mathcal{I}^{\sqcup}$-monoidal category corresponding to $\mathcal{C}$ under this equivalence, which is compatible with geometric realizations by Lemma 2.33. Recall the forgetful $\mathcal{I}^{\sqcup}$-monoidal functor P and its $\mathcal{I}^{\sqcup}$-monoidal left adjoint F, as well as $p_{\mathcal{I}} = \mathrm{pr}_{\mathrm{Alg}} \circ P$ from Construction/Theorem 2.34. The cartesian structure $\pi : \mathcal{I}^{\sqcup} \to \mathcal{I}$ induces a functor

$$\pi^* : \mathrm{CAlg}_T^P(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{I}^{\sqcup}/\mathcal{I}^{\sqcup}}(\mathcal{C}^{\otimes}) \simeq \mathrm{Alg}_{\mathcal{I}^{\sqcup}}(\mathrm{Alg}_{\mathsf{Fin}_*/\mathcal{I}^{\sqcup}}(\mathcal{C}^{\otimes})^{\otimes}).$$

Now it follows from Corollary 2.35 and Lemma 2.36 that for $R \in \mathrm{CAlg}_T^P(\mathcal{C})$ the functor $\mathsf{Mod}_{R_\bullet}(\mathcal{C}_\bullet) := \Phi_{\mathcal{I}}((\pi^*R)^* p_{\mathcal{I}}) \in \mathsf{Fun}(\mathcal{I}, \mathsf{CMon}(\mathsf{Cat}))$ already lands in $\mathsf{Mack}_T^P(\mathsf{CMon}(\mathsf{Cat})) \simeq \mathsf{Mack}_T^P(\mathsf{Cat})$. So in our case we do not actually need to separately encode the monoidal structures using the cocartesian operad $\mathcal{I}^{\sqcup}$ on $\mathcal{I}$. Since the equivalence $\mathrm{Cocart}(-) \simeq \mathsf{Fun}(-, \mathsf{Cat})$ is natural, i.e. pulling back the cocartesian fibration and then straightening is the same as first straightening and then precomposing, we can thus consider

$$\underline{\mathsf{Mod}}_R(\mathcal{C}) : \mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T) = \mathcal{I} \to \mathcal{I}^{\sqcup} \xrightarrow{\pi^* R} \mathrm{Alg}_{\mathsf{Fin}_*/\mathcal{I}^{\sqcup}}(\mathcal{C}^{\otimes})^{\otimes} \xrightarrow{\mathrm{St}^{\mathsf{cc}}(p_{\mathcal{I}})} \mathsf{Cat}$$

which is equivalent to $\mathsf{Mod}_{R_\bullet}(\mathcal{C}_\bullet) \in \mathsf{Mack}_T^P(\mathsf{Cat})$. This is functorial in R via

$$\mathrm{CAlg}_T^P(\mathcal{C}) \xrightarrow{\pi^*} \mathrm{Alg}_{\mathcal{I}^\sqcup}(\mathrm{Alg}_{\mathsf{Fin}_*/\mathcal{I}^\sqcup}(\mathcal{C}^\otimes)^\otimes) \to \mathsf{Fun}(\mathcal{I}^\sqcup, \mathrm{Alg}_{\mathsf{Fin}_*/\mathcal{I}^\sqcup}(\mathcal{C}^\otimes)^\otimes) \to \mathsf{Fun}(\mathcal{I}, \mathsf{Cat})$$

where the middle arrow is forgetful and the last one precomposes with $\mathcal{I} \to \mathcal{I}^\sqcup$ and postcomposes with $\mathrm{St}^{\mathsf{cc}}(p_\mathcal{I})$. By the above arguments this factors through $\mathsf{Mack}_T^P(\mathsf{Cat})$, and gives the desired functor $\underline{\mathsf{Mod}}_{(-)}(\mathcal{C})$. From this it is also clear that a morphism $R \to S$ in $\mathrm{CAlg}_T^P(\mathcal{C})$ induces the claimed P-symmetric monoidal T-functor $S \otimes_R - : \underline{\mathsf{Mod}}_R(\mathcal{C}) \to \underline{\mathsf{Mod}}_S(\mathcal{C})$, as we are just whiskering $\mathrm{St}^{\mathsf{cc}}(p_\mathcal{I})$ with $\pi^*(R \to S)$, and hence pointwise have the usual functoriality of $\mathsf{Mod}(\mathcal{C}(B)) : \mathrm{CAlg}(\mathcal{C}(B)) \to \mathsf{Cat}$ of [Lur17, 4.5.3.1].

To construct the free and forgetful functors, where we don't need functoriality in R, it is easier to work with the fibrations. We consider the diagram

$$\begin{tikzcd}
\int \mathsf{Mod}_{R_\bullet}(\mathcal{C}_\bullet) \arrow[rr] \arrow[d, "U"] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathrm{Alg}_{\mathcal{CM}^\otimes/\mathcal{I}^\sqcup}(\mathcal{C})^\otimes \arrow[d, "P"] & \\
\mathcal{C}^\otimes \arrow[u, dashed, bend left, "R\otimes -"] \arrow[rr, "{(R,\mathrm{id})}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathrm{Alg}_{\mathsf{Fin}_*/\mathcal{I}^\sqcup}(\mathcal{C})^\otimes \times_{\mathcal{I}^\sqcup} \mathcal{C}^\otimes \arrow[u, dashed, bend left, "F"] \arrow[r] \arrow[d, "\mathrm{pr}_{\mathrm{Alg}}"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}^\otimes \arrow[d] \\
\mathcal{I}^\sqcup \arrow[rr, "R"] \arrow[rrr, equal, bend right] & & \mathrm{Alg}_{\mathsf{Fin}_*/\mathcal{I}^\sqcup}(\mathcal{C})^\otimes \arrow[r] & \mathcal{I}^\sqcup
\end{tikzcd}$$

The solid diagram commutes, and as in Construction/Theorem 2.34 the dashed arrow $R \otimes - : \mathcal{C}^\otimes \to \int \mathsf{Mod}_{R_\bullet}(\mathcal{C}_\bullet)$ induced via $F \circ (R, \mathrm{id})$ is $\mathcal{I}^\sqcup$-monoidal and fiberwise given by the symmetric monoidal left adjoint $R_i \otimes - : \mathcal{C}_i \to \mathsf{Mod}_{R_i}(\mathcal{C}_i)$. The forgetful $U : \int \mathsf{Mod}_{R_\bullet}(\mathcal{C}_\bullet) \to \mathcal{C}^\otimes$ preserves cocartesian lifts of those morphisms which are sent to cocartesian morphisms by R. As mentioned above, because $\mathcal{I} = \mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)$ is semiadditive and $\mathcal{C}$ is a Mackey functor, we obtain the P-symmetric monoidal T-categories $\underline{\mathsf{Mod}}_R(\mathcal{C})$ and $\mathcal{C}$ also as straightenings of the pullback along $i : \mathcal{I} \to \mathcal{I}^\sqcup$. So we have a morphism $i^*U : \int \underline{\mathsf{Mod}}_R(\mathcal{C}) \to \int \mathcal{C}$ over $\mathcal{I}$, which preserves cocartesian lifts of backwards morphisms, i.e. a lax P-symmetric monoidal functor $U : \underline{\mathsf{Mod}}_R(\mathcal{C}) \to \mathcal{C}$ as claimed in (2). □

Lemma 2.38. Consider a collection of categories $\mathcal{K}$ with $\Delta^{\mathsf{op}} \in \mathcal{K}$. If $\mathcal{C} \in \mathsf{Mack}_T^P(\mathsf{Cat}; \mathcal{K})$, then also the functor $\underline{\mathsf{Mod}}_{(-)}(\mathcal{C}) : \mathrm{CAlg}_T^P(\mathcal{C}) \to \mathsf{Mack}_T^P(\mathsf{Cat})$ factors through $\mathsf{Mack}_T^P(\mathsf{Cat}; \mathcal{K})$.

Proof. Fix $R \in \mathrm{CAlg}_T^P(\mathcal{C})$. By [Lur17, 4.2.3.5] each $\mathsf{Mod}_{R(B)}(\mathcal{C}(B))$ admits $\mathcal{K}$-indexed colimits which are computed in $\mathcal{C}(B)$, and by [Lur17, 4.4.2.15, 4.5.2.1] the symmetric monoidal structure on it is also compatible with $\mathcal{K}$-indexed colimits. Given a morphism $f : A \to B$ in T, we know by assumption that $f^* : \mathcal{C}(B) \to \mathcal{C}(A)$ preserves $\mathcal{K}$-indexed colimits and that R is cocartesian on the backwards morphism associated to f. In particular, the canonical comparison map $f^*R(B) \to R(A)$ is an equivalence, and $\underline{\mathsf{Mod}}_R(\mathcal{C})$ sends $B \xleftarrow{f} A = A$ to the functor $\mathsf{Mod}(f^*) : \mathsf{Mod}_{R(B)}(\mathcal{C}(B)) \to \mathsf{Mod}_{R(A)}(\mathcal{C}(A))$, c.f. Construction/Theorem 2.34. This functor preserves $\mathcal{K}$-indexed colimits since f^* does, and $\mathcal{K}$-indexed colimits in these module categories are computed in the underlying categories. Similarly, if $p : A \to B$ is in P, then the associated norm $\underline{\mathsf{Mod}}_R(\mathcal{C})(p)$ factors as $\mathsf{Mod}(p_\otimes)$ followed by the left adjoint $R(B) \otimes_{p_\otimes R(A)} -$, both of which preserve $\mathcal{K}$-indexed colimits. This proves that $\underline{\mathsf{Mod}}_R(\mathcal{C}) \in \mathsf{Mack}_T^P(\mathsf{Cat}; \mathcal{K})$. Finally, we saw above that any morphism of algebras $f : R \to S$ in $\mathrm{CAlg}_T^P(\mathcal{C})$ induces the pointwise left adjoint $S \otimes_R -$ which is thus a morphism in $\mathsf{Mack}_T^P(\mathsf{Cat}; \mathcal{K})$. $\square$

Lemma 2.39. Let $\mathcal{K}$ be as in the previous lemma and consider a functor as in Lemma 2.16(2) so that we have an induced strong Segal morphism $i : \mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T) \to \mathrm{Span}_{\mathsf{all},Q}(\mathbb{F}_S)$ which preserves finite products. Given $\mathcal{C} \in \mathsf{Mack}_S^Q(\mathsf{Cat}; \mathcal{K})$, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{CAlg}_S^Q(\mathcal{C}) \arrow[r, "\underline{\mathsf{Mod}}_{(-)}(\mathcal{C})"] \arrow[d, "i^*"'] & \mathsf{Mack}_S^Q(\mathsf{Cat}; \mathcal{K}) \arrow[d, "i^*"] \\
\mathrm{CAlg}_T^P(i^*\mathcal{C}) \arrow[r, "\underline{\mathsf{Mod}}_{(-)}(i^*\mathcal{C})"] & \mathsf{Mack}_T^P(\mathsf{Cat}; \mathcal{K})
\end{tikzcd}$$

Proof. By the construction of $\underline{\mathsf{Mod}}_{(-)}(\mathcal{C})$ from the proof of Theorem 2.37 and Lemma 2.38 it suffices to show that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{CAlg}_S^Q(\mathcal{C}) \arrow[r] \arrow[dd, "i^*"'] \arrow[rr, bend left, "\underline{\mathsf{Mod}}_{(-)}(\mathcal{C})"] & \mathsf{Fun}(\mathrm{Span}_{\mathsf{all},Q}(\mathbb{F}_S)^{\sqcup}, \mathrm{Alg}_{\mathsf{Fin}_*/\,\mathrm{Span}_{\mathsf{all},Q}(\mathbb{F}_S)^{\sqcup}}(\mathcal{C}^{\otimes})^{\otimes}) \arrow[r] \arrow[d, "i^*"] & \mathsf{Fun}(\mathrm{Span}_{\mathsf{all},Q}(\mathbb{F}_S), \mathsf{Cat}) \arrow[dd, "i^*"] \\
& \mathsf{Fun}(\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)^{\sqcup}, \mathrm{Alg}_{\mathsf{Fin}_*/\,\mathrm{Span}_{\mathsf{all},Q}(\mathbb{F}_S)^{\sqcup}}(\mathcal{C}^{\otimes})^{\otimes}) \arrow[dr] & \\
\mathrm{CAlg}_T^P(i^*\mathcal{C}) \arrow[r] \arrow[rr, bend right, "\underline{\mathsf{Mod}}_{(-)}(i^*\mathcal{C})"'] & \mathsf{Fun}(\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)^{\sqcup}, \mathrm{Alg}_{\mathsf{Fin}_*/\,\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)^{\sqcup}}(i^*\mathcal{C}^{\otimes})^{\otimes}) \arrow[r] \arrow[u] & \mathsf{Fun}(\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T), \mathsf{Cat})
\end{tikzcd}$$

Here the vertical bottom middle functor is given by postcomposition with the middle vertical in the following diagram. Everything except the bottom right triangle clearly commutes, and commutativity of said triangle is via cocartesian unstraightening equivalent the following left square being cartesian:

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{CM}^{\otimes}/\,\mathrm{Span}_{\mathsf{all},Q}(\mathbb{F}_S)^{\sqcup}}(\mathcal{C}^{\otimes})^{\otimes} \arrow[r, "p_{\mathrm{Span}_{\mathsf{all},Q}(\mathbb{F}_S)}"] & \mathrm{Alg}_{\mathsf{Fin}_*/\,\mathrm{Span}_{\mathsf{all},Q}(\mathbb{F}_S)^{\sqcup}}(\mathcal{C}^{\otimes})^{\otimes} \arrow[r] & \mathrm{Span}_{\mathsf{all},Q}(\mathbb{F}_S)^{\sqcup} \\
\mathrm{Alg}_{\mathcal{CM}^{\otimes}/\,\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)^{\sqcup}}(i^*\mathcal{C}^{\otimes})^{\otimes} \arrow[r, "p_{\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)}"'] \arrow[u] & \mathrm{Alg}_{\mathsf{Fin}_*/\,\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)^{\sqcup}}(i^*\mathcal{C}^{\otimes})^{\otimes} \arrow[r] \arrow[u] & \mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)^{\sqcup} \arrow[u, "i"']
\end{tikzcd}$$

This in turn follows from pullback pasting; using the universal property of $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C}^{\otimes})^{\otimes} \to \mathcal{O}^{\otimes}$ from [Lur17, 3.2.4.1] one checks that the right square and whole rectangle are cartesian. $\square$

2.4 Comparison with P-commutative T-monoids and the formalism of Nardin-Shah

Let $\mathcal{C}$ be a category with finite products. Call a functor $F : \mathbb{F}_* \to \mathcal{C}$ semiadditive if for each $n \geq 0$ the Segal morphisms $\rho^i : n_+ \to 1_+$ induce an equivalence $F(n_+) \simeq \prod_{i=1}^n F(1_+)$. This is precisely Lurie's definition of a commutative monoid in $\mathcal{C}$, see [Lur17, Secction 2.4.2]. In the parametrized setting the analogous definitions and results were covered in [CLL23a, Section 4]. The main goal of this section is that their T-category $\underline{\mathsf{CMon}}_T^P(\underline{\mathcal{E}}_T)$ of P-commutative T-monoids in the T-category $\underline{\mathcal{E}}_T$ of T-objects in $\mathcal{E}$ agrees with the T-category $\underline{\mathsf{Mack}}_T^P(\mathcal{E})$, see Theorem 2.54. The non-parametrized version with more restrictive hypotheses on $P \subset T$ and with a different proof has already appeared as [NS22, Theorem 2.3.9, Theorem 2.4.14]. Moreover, we show in Corollary 2.59 that the theory of T-∞-operads of Nardin-Shah agrees with the theory of fibrous patterns over $\mathrm{Span}(\mathbb{F}_T; T)$, and in particular the notions of T-symmetric monoidal T-category and T-commutative algebras agree. The only result relevant for the other sections of this text is the non-parametrized version of Corollary 2.56 in the equivariant and global context, which were already known before, see Remark 2.57.

Let us begin with a recollection on the necessary notions from parametrized category theory, following [CLL23a]. Fix a base category T (the reader is invited to think of $T = \mathsf{Orb}_G$, which was the original motivation for this theory). Recall from Appendix A our notation and related results on the free finite-coproduct completion $\mathbb{F}_T$ of T, also called the category of finite T-sets. The category of T-categories is defined as $\mathsf{Cat}_T \coloneqq \mathsf{Fun}(T^{\mathsf{op}}, \mathsf{Cat})$. Since Cat is complete, we can (and will) limit extend to $\mathsf{Cat}_T \simeq \mathsf{Fun}^\times(\mathbb{F}_T^{\mathsf{op}}, \mathsf{Cat}) \simeq \mathsf{Fun}^R(\mathrm{PSh}(T)^{\mathsf{op}}, \mathsf{Cat})$. So we can view T-categories as Cat-valued sheaves on the presheaf-topos $\mathrm{PSh}(T)$, and more generally one can replace $\mathrm{PSh}(T)$ by any topos $\mathcal{B}$; this is studied under the name of internal higher category theory in a series of papers by Martini and Wolf, see e.g. [MW24]. As the name suggests, one can do essentially all of higher category also in this setting of parametrized / internal higher category theory; there are corresponding notions of adjunctions, (co)limits, Kan extensions, fibrations, (un)straightening, etc. In the case of presheaf topoi, this agrees with the theory built by Barwick, Dotto, Glasman, Nardin and Shah in a series of papers beginning with [BDG+16a]. More recently, Nardin and Shah have also built a parametrized theory of operads in [NS22]. We recommend [CLL23a, Section 2] for an introduction to parametrized higher category theory, and we will largely follow their notation.

Definition 2.40. Consider the adjunction $\mathrm{const} : \mathsf{Cat} \rightleftarrows \mathsf{Cat}_T : \Gamma \coloneqq \lim_{T^{\mathsf{op}}}$. We call $\Gamma\mathcal{C}$ the underlying category of the T-category $\mathcal{C}$.

Remark 2.41. The category Cat_T inherits cartesian-closedness from Cat. We denote the internal hom by $\underline{\mathsf{Fun}}_T$. Its underlying category Fun_T agrees with the usual mapping category $\mathrm{Nat}_{T^{\mathsf{op}}}$ of the $(\infty, 2)$-categorical structure inherited from Cat, see Remark D.5.

Definition/Lemma 2.42. For categories T and $\mathcal{E}$, we define the cofree T-category on $\mathcal{E}$, also called the T-category of T-objects in $\mathcal{E}$ by

$$\underline{\mathcal{E}}_T : T^{\mathsf{op}} \to \mathsf{Cat},\ B \mapsto \mathsf{Fun}((T_{/B})^{\mathsf{op}}, \mathcal{E}).$$

Here the functoriality is via the postcomposition functoriality for $T_{/\bullet} = \mathrm{St}^{\mathsf{cc}}(t : \mathrm{Ar}(T) \to T)$. Clearly this is functorial in $\mathcal{E}$, and assembles into a functor $\underline{(-)}_T : \mathsf{Cat} \to \mathsf{Cat}_T$.

Recall that for a functor $F : \mathcal{C} \to \mathsf{Cat}$ we denote the total category of the cocartesian unstraightening of F by $\int F$. The following lemma explains the "cofree" terminology of the above definition.

Lemma 2.43 ([CLL23a, Lemma 2.2.13])**.** For a category T, we have an adjunction of $(\infty, 2)$-categories $\int : \mathsf{Cat}_T \rightleftarrows \mathsf{Cat} : \underline{(-)}_T$. In particular we have an equivalence $\mathsf{Fun}(\int \mathcal{C},\ \mathcal{E}) \simeq \mathsf{Fun}_T(\mathcal{C}, \underline{\mathcal{E}}_T)$ natural in $\mathcal{C} \in \mathsf{Cat}_T^{\mathsf{op}}$ and $\mathcal{E} \in \mathsf{Cat}$.

A standard computation yields $\Gamma\underline{\mathcal{E}}_T \simeq \mathsf{Fun}(T^{\mathsf{op}}, \mathcal{E})$. In particular, we obtain a T-categorical lift $\underline{\mathsf{Cat}}_T$ of Cat_T, which we call the T-category of T-categories. One also considers T-spaces $\underline{\mathsf{Spc}}_T \subseteq \underline{\mathsf{Cat}}_T$. By [Lur09, 5.1.6.12] we have an equivalence $\underline{\mathsf{Spc}}_T(A) = \mathrm{PSh}(T_{/A}) \simeq \mathrm{PSh}(T)_{/\text{よ}A}$. It was shown in [CLL23a, 2.1.16] that this is natural in A for the pullback functoriality $\mathrm{PSh}(T)_{/-} = \mathrm{Un}^{\mathsf{ct}}(t : \mathrm{Ar}(\mathrm{PSh}(T)) \to \mathrm{PSh}(T))|_{T^{\mathsf{op}}}$, thus giving an equivalence of T-categories $\underline{\mathsf{Spc}}_T \simeq \mathrm{PSh}(T)_{/-}$. As shown in [CLL23b, Theorem 5.5], in the case of $T = \mathsf{Orb}$ the Orb-category $\underline{\mathsf{Spc}}_{\mathsf{Orb}}$ sends $G \mapsto \mathsf{Spc}^G$ with the usual restriction functoriality.[10] Generally, given $f : X \to Y$ in $\mathbb{F}_T$ and a T-category $\mathcal{C}$ we will denote $f^* \coloneqq \mathcal{C}(f) : \mathcal{C}(Y) \to \mathcal{C}(X)$. The most important example for us was already discussed in Definition 2.24

Definition 2.44 ([CLL23a, 2.3.13])**.** A T-category $\mathcal{C}$ is said to admit finite P-coproducts if

1. It admits fiberwise finite coproducts, meaning each $\mathcal{C}(X)$ admits finite coproducts, and each $f^* : \mathcal{C}(Y) \to \mathcal{C}(X)$ preserves them for $f : X \to Y$ in $\mathbb{F}_T$.
2. For every $p : X \to Y$ in $\mathbb{F}_T^P$ the restriction p^* admits a left adjoint $p_!$. Moreover, for every pullback
$$\begin{tikzcd} A' \arrow[r, "\alpha"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "p"] \\ B' \arrow[r, "\beta"'] & B \end{tikzcd} \tag{6}$$
with $A, B, B' \in T$ and p in P, the Beck-Chevalley transformation $p'_!\alpha^* \Rightarrow \beta^* p_!$ is an equivalence.

A T-functor $F : \mathcal{C} \to \mathcal{D}$ preserves finite P-coproducts if each $F_X : \mathcal{C}(X) \to \mathcal{D}(X)$ preserves finite coproducts, and for each $p : X \to Y$ in $\mathbb{F}_T^P$ the Beck-Chevalley transformation $p_! F_X \Rightarrow F_Y p_!$ is an

[10] More formally, using language of the later sections, one constructs a natural equivalence between $\underline{\mathsf{Spc}}_{\mathsf{Orb}}$ and the parametrized Dwyer-Kan localization of $\mathrm{Bor}_{\mathsf{Orb}}(\mathsf{sSet}) : \mathsf{Orb}^{\mathsf{op}} \to \mathsf{Cat}$, $G \mapsto \mathsf{Fun}(BG, \mathsf{sSet})$ which inverts the G-weak equivalences at level G. The analogous result for the global category of G-global spaces was shown in [CLL23a, Theorem 3.3.1].

equivalence. We remark that by [CLL23a, 4.2.14] it suffices to consider $f : A \to B$ in T or $p : A \to B$ in P in the above definitions. Dually one defines the existence and preservation of finite P-products, where we denote the right adjoint of p^* by p_*.

For example, it was shown in [CLL23a, 4.2.17] that $\underline{\mathbb{F}}_T^P$ is the free T-category admitting finite P-coproducts on one generator. The left adjoints are given by postcomposition $p_! : (\mathbb{F}_T)_{/X} \to (\mathbb{F}_T)_{/Y}$, and we described the resulting cocartesian P-symmetric monoidal structure on $\underline{\mathbb{F}}_T^P$ in Example 2.25. In the G-equivariant case $P = T = \mathsf{Orb}_G$, one can think of a general $p_!$ as a coproduct indexed by a finite G-set. For example, we know from above that the G-category of finite G-sets admits finite G-coproducts. Given $p : X \to Y$ in $\mathbb{F}_G$ decomposing into $\coprod_{i=1}^n$ with $p_i : \coprod_{j=1}^{n_i} G/H_{ij} \to G/K_i$, we see that $p_!$ is given by the composite

$$\underline{\mathbb{F}}_G(X) \simeq \prod_{i=1}^{n}\prod_{j=1}^{n_i} \mathbb{F}_{H_{ij}} \xrightarrow{\prod_{i=1}^n \prod_{j=1}^n \mathrm{Ind}_{H_{ij}}^{K_i}} \prod_{i=1}^{n} \mathbb{F}_{K_i}^{n_i} \xrightarrow{\prod_{i=1}^n \coprod_{n_i}} \prod_{i=1}^{n} \mathbb{F}_{K_i} \simeq \underline{\mathbb{F}}_G(Y)$$

For this reason it is common to write $\coprod_{K/H} \coloneqq \mathrm{Ind}_H^K \coloneqq (G/H \to G/K)_! : \mathcal{C}(H) \to \mathcal{C}(K)$ for a G-category $\mathcal{C}$ admitting finite G-coproducts.

Lemma 2.45. The cofree functor $\underline{(-)}_T$ restricts to $\mathsf{Cat}^\times \to \mathsf{Cat}_T^{P-\times}$, where the latter category is the subcategory of Cat_T on T-categories admitting finite P-products and T-functors preserving them.

Proof. It is clear that it factors through $\mathsf{Cat}_T^\times \coloneqq \mathsf{Fun}(T^{\mathsf{op}}, \mathsf{Cat}^\times)$, the subcategory on T-categories admitting fiberwise finite products and T-functors preserving them, giving point(1) of Definition 2.44. For $\mathcal{E} \in \mathsf{Cat}^\times$ we can pointwise limit-extend to identify $\underline{\mathcal{E}}_T \simeq \mathsf{Fun}^\times(((\mathbb{F}_T)_{/\bullet})^{\mathsf{op}}, \mathcal{E})$. Let $p : X \to Y$ be in $\mathbb{F}_T^P$. Then for $Z \to Y$ in $(\mathbb{F}_T)_{/Y}$ the pullback $X \times_Y Z$ exists, and hence $(\mathbb{F}_T)_{/X} \times_{(\mathbb{F}_T)_{/Y}} (\mathbb{F}_T)_{/Z} \simeq (\mathbb{F}_T)_{/X\times_Y Z}$ has a final object. So given $\Phi \in \underline{\mathcal{E}}_T(X)$, it follows that the right Kan extension along $((\mathbb{F}_T)_{/X})^{\mathsf{op}} \to (\mathbb{F}_T)_{/Y})^{\mathsf{op}}$ exists in $\mathsf{Fun}((\mathbb{F}_T)_{/Y})^{\mathsf{op}}, \mathcal{E})$, and is given by $(p_*\Phi)(Z \to Y) \simeq \Phi(X \times_Y Z)$. By Lemma A.4 $\mathbb{F}_T$ is extensive so $X \times_Y -$ preserves finite coproducts and hence $p_*\Phi \in \underline{\mathcal{E}}_T(Y)$, showing that right Kan extension assembles into a right adjoint $p_* \vdash p^*$. Now consider the square Eq. (6). By Lemma C.2 the resulting Beck-Chevalley map $\beta p_* \Rightarrow p'_*\alpha$ is given at $(X' \to B')$ in $(\mathbb{F}_T)_{/B'}$ by restricting along the equivalence $((\mathbb{F}_T)_{/X'\times_{B'}A'})^{\mathsf{op}} \xrightarrow{\simeq} ((\mathbb{F}_T)_{/X'\times_B A})^{\mathsf{op}}$ in the pointwise limit formulas, hence is an equivalence itself. Finally, note that the right Kan extensions p_* are "absolute in $\mathsf{Cat}^\times$", meaning that the canonical comparison map $F \circ p_*\Phi \Rightarrow p_*(F \circ \Phi)$ is an equivalence for any $F : \mathcal{E} \xrightarrow{\times} \mathcal{F}$, since pointwise we just evaluate at some object in the source of Φ. This proves that $\underline{\mathbb{F}}_T : \underline{\mathcal{E}}_T \to \underline{\mathcal{F}}_T$ preserves finite P-products, as the relevant Beck-Chevalley map $\underline{F}_T p_*^{\underline{\mathcal{E}}_T} \Rightarrow p_*^{\underline{\mathcal{F}}_T} \underline{F}_T$ evaluated at $\Phi \in \underline{\mathcal{E}}_T(X)$ is precisely the above comparison map $F \circ p_*\Phi \xRightarrow{\simeq} p_*(F \circ \Phi)$. ◻

Definition 2.46 ([CLL23a, Definition 4.1.1])**.** A T-category $\mathcal{C}$ admits a T-final object if it does so fiberwise,i.e. if it factors through the category Cat^* of categories admitting a final object and functors preserving it. Analogously $\mathcal{C}$ is T-pointed if it is so fiberwise, and any $\mathcal{C}$ admitting a T-final object can be made T-pointed by postcomposing with the left adjoint $\mathsf{Cat}^* \to \mathsf{Cat}^{\mathrm{pt}}, \mathcal{D} \mapsto \mathcal{D}_{*/}$.

Example 2.47. Clearly $\underline{\mathbb{F}}_T^P$ admits the T-final object given in $\underline{\mathbb{F}}_T^P(X)$ by id_X. Hence we can define the T-category of finite pointed P-sets $\underline{\mathbb{F}}_{T,*}^P := (-)_* \circ \underline{\mathbb{F}}_T^P$, which still admits finite P-coproducts, compare [CLL23a, Section 4.7].

If $P \subset T$ is even *atomic* orbital and $\mathcal{C}$ is a pointed T-category admitting finite P-coproducts and finite P-products, then one can define norm maps $\mathrm{Nm}_p : p_! \Rightarrow p_*$ for every $p : X \to Y$ in $\mathbb{F}_T^P$, see [CLL23a, Section 4.3]. In the non-parametrized case $P = T = *$, these are the canonical comparison maps from a finite coproduct to a finite product $\coprod \Rightarrow \prod$ available in any pointed category admitting finite (co)products. In the G-equivariant case as above, the Norm map for $p : G/H \to G/K$ is a Wirthmüller-type map $\mathrm{Ind}_H^K \Rightarrow \mathrm{Coind}_H^K$. One then says $\mathcal{C}$ is P-semiadditive if all these norm maps are equivalences. This recovers the usual notion of semiadditivity in the non-parametrized case, and the notion of "G-semiadditivity", i.e. furthermore having Wirthmüller isomorphisms, in the G-equivariant case.[11] We will be more interested in a relative version; given a T-functor $F : \mathcal{C} \to \mathcal{D}$ such that $\mathcal{C}$ admits finite P-coproducts and $\mathcal{D}$ admits finite P-products, then one can define a norm $\mathrm{Nm}_p^F : F(Y)p_! \Rightarrow p_*F(X)$, see [CLL23a, Section 4.6].

Definition 2.48. In the above situation, we say that F is P-semiadditive if all the norm maps Nm_p^F are equivalences. This forms a full T-subcategory $\underline{\mathsf{Fun}}_T^{P-\oplus}(\mathcal{C}, \mathcal{D}) \subseteq \underline{\mathsf{Fun}}_T(\mathcal{C}, \mathcal{D})$, see [CLL23a, 4.6.6]. In view of Example 2.47 one can define the T-category of P-commutative T-monoids in $\mathcal{D}$

$$\underline{\mathsf{CMon}}_T^P(\mathcal{D}) := \underline{\mathsf{Fun}}_T^{P-\oplus}(\underline{\mathbb{F}}_{T,*}^P, \mathcal{D}).$$

The first step in proving the equivalence $\underline{\mathsf{CMon}}_T^P(\mathcal{E}_T) \simeq \underline{\mathsf{Mack}}_T^P(\mathcal{E})$ will be to apply the adjunction equivalence from Lemma 2.43. Therefore our first goal will be to find a good description of the cocartesian unstraightening of $\underline{\mathbb{F}}_{T,*}^P$. We can use [HHLN23b, Theorem 3.9] for this, since for atomic orbital $P \subset T$ we have an equivalence of T-categories $\underline{\mathbb{F}}_{T,*}^P \simeq \mathrm{Span}_{\mathsf{si},\mathsf{all}}(\underline{\mathbb{F}}_T^P)$ by Lemma A.10, analogously to the well-known equivalence $\mathbb{F}_* \simeq \mathrm{Span}_{\mathsf{si},\mathsf{all}}(\mathbb{F})$.

Definition 2.49. Using the notation of Definition 2.24, let $\mathrm{Ar}_P(\mathbb{F}_T)_{\mathsf{tdeg}} \subset \mathrm{Ar}_P(\mathbb{F}_T)$ respectively $\mathrm{Ar}_P(\mathbb{F}_T)_{\mathsf{si}} \subset \mathrm{Ar}_P(\mathbb{F}_T)$ be the wide subcategory on those morphisms $(V \to X) \to (W \to Y)$ with degenerate target projection respectively where the comparison map to the pullback $V \to X \times_Y W$ is a summand inclusion.

Analogously, one can define $\mathrm{Ar}_P((F_T)_{/X}) \subseteq \mathrm{Ar}((\mathbb{F}_T)_{/X})$ as the full subcategory on the morphisms in $\underline{\mathbb{F}}_T^P(X)$, i.e. on those which forget to morphisms in $\mathbb{F}_T^P$. Again we have wide subcategories $\mathrm{Ar}_P((\mathbb{F}_T)_{/X})_{\mathsf{si}}$ and $\mathrm{Ar}_P((\mathbb{F}_T)_{/X})_{\mathsf{tdeg}}$ of $\mathrm{Ar}_P((\mathbb{F}_T)_{/X})$.

Lemma 2.50. Let $P \subset T$ be atomic orbital and $X \in \mathbb{F}_T$. Denote by $(\mathbb{F}_T)_{/\bullet}$ the covariant postcomposition-functoriality of the slice.

[11] See the introduction of [CLL23a] for a discussion of this.

1. We have cartesian squares

$$\begin{tikzcd} \mathrm{Ar}_P((\mathbb{F}_T)_{/X})_{\mathsf{si}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Ar}_P((\mathbb{F}_T)_{/X}) \arrow[r, "t"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (\mathbb{F}_T)_{/X} \arrow[d, "\pi_X"] \\ \mathrm{Ar}_P(\mathbb{F}_T)_{\mathsf{si}} \arrow[r] & \mathrm{Ar}_P(\mathbb{F}_T) \arrow[r, "t"] & \mathbb{F}_T \end{tikzcd}$$

and likewise for the $(-)_{\mathsf{tdeg}}$ subcategories. These categories assemble into adequate triples $\mathrm{Ar}_P(\mathbb{F}_T)_{\mathsf{si},\mathsf{tdeg}}$ and $\mathrm{Ar}_P((\mathbb{F}_T)_{/X})_{\mathsf{si},\mathsf{tdeg}}$. The postcomposition functoriality $(\mathbb{F}_T)_{/\bullet}$ and forgetful functors $\pi_\bullet$ induce a natural transformation $\mathrm{Ar}_P(\pi_\bullet) : \mathrm{Ar}_P((\mathbb{F}_T)_{/\bullet})_{\mathsf{si},\mathsf{tdeg}} \Rightarrow \mathrm{const}\,\mathrm{Ar}_P(\mathbb{F}_T)_{\mathsf{si},\mathsf{tdeg}}$ of functors $\mathbb{F}_T \to \mathsf{AdTrip}$.

2. The functor induced by the target projection

$$\mathrm{Span}(t) : \mathrm{Span}_{\mathsf{si},\mathsf{tdeg}}(\mathrm{Ar}_P(\mathbb{F}_T)) \to \mathrm{Span}_{\mathsf{all},\simeq}(\mathbb{F}_T) \simeq \mathbb{F}_T^{\mathsf{op}}$$

is a cocartesian fibration classified by the T-category $\underline{\mathbb{F}}_{T,*}^P$. Similarly for $(\mathbb{F}_T)_{/X}$ in place of $\mathbb{F}_T$.

3. The source projection $s : \mathrm{Ar}_P(\mathbb{F}_T) \to \mathbb{F}_T$ factors through $\mathbb{F}_T^P$, and overall we have a commutative diagram of natural transformations $\mathbb{F}_T \to \mathsf{Cat}$

$$\begin{tikzcd} \mathrm{Span}_{\mathsf{all},P}((\mathbb{F}_T)_{/\bullet}) \arrow[d, "\mathrm{Span}(\pi_\bullet)"'] & \mathrm{Span}_{\mathsf{si},\mathsf{tdeg}}(\mathrm{Ar}_P((\mathbb{F}_T)_{/\bullet})) \arrow[l, "\mathrm{Span}(s)"'] \arrow[r, "\mathrm{Span}(t)"] \arrow[d, "\mathrm{Span}(\mathrm{Ar}_P(\pi_{/\bullet}))"'] \arrow[dr, phantom, "\lrcorner", very near start] & ((\mathbb{F}_T)_{/\bullet})^{\mathsf{op}} \arrow[d, "\pi_\bullet"] \\ \mathrm{const}\,\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T) & \mathrm{const}\,\mathrm{Span}_{\mathsf{si},\mathsf{tdeg}}(\mathrm{Ar}_P(\mathbb{F}_T)) \arrow[l, "\mathrm{Span}(s)_*"] \arrow[r, "\mathrm{Span}(t)"'] & \mathrm{const}\,\mathbb{F}_T^{\mathsf{op}} \end{tikzcd}$$

The right square is cartesian, and all vertical transformations exhibit their target as the colimit of their source.

Proof. In the first claim, the right square and the left square for the $(-)_{\mathsf{tdeg}}$ subcategories are obviously cartesian, and for the pictured left square this follows from the fact that π_X creates coproducts and pullbacks, and in particular a morphism f in $(\mathbb{F}_T)_{/X}$ is a summand inclusion if and only if $\pi_X(f)$ is. Moreover, pullbacks in Ar_P are computed pointwise, and both summand inclusions and equivalences are stable under pullback, which shows that we have the desired adequate triples

$$\mathrm{Ar}_P(\mathbb{F}_T)_{\mathsf{si},\mathsf{tdeg}} := (\mathrm{Ar}_P(\mathbb{F}_T), \mathrm{Ar}_P(\mathbb{F}_T)_{\mathsf{si}}, \mathrm{Ar}_P(\mathbb{F}_T)_{\mathsf{tdeg}})$$

and likewise for $\mathrm{Ar}_P((\mathbb{F}_T)_{/X})_{\mathsf{si},\mathsf{tdeg}}$, with π_X and all postcomposition functors $(\mathbb{F}_T)_{/X} \to (\mathbb{F}_T)_{/Y}$ upgrading to morphisms of adequate triples. Hence we have constructed

$$\mathrm{Ar}_P((\mathbb{F}_T)_{/\bullet})_{\mathsf{si},\mathsf{tdeg}} : \mathbb{F}_T \to \mathsf{AdTrip} \qquad \text{and} \qquad \pi_\bullet : \mathrm{Ar}_P((\mathbb{F}_T)_{/\bullet})_{\mathsf{si},\mathsf{tdeg}} \Rightarrow \mathrm{const}\,\mathrm{Ar}_P(\mathbb{F}_T)_{\mathsf{si},\mathsf{tdeg}}.$$

Postcomposing / whiskering with Span, we obtain the right cartesian square in (3), since limits in functor categories and AdTrip are computed pointwise, and Span is right adjoint by Theorem 2.11.

For (2), recall from Lemma A.10 that we have an equivalence of T-categories

$$\underline{\mathbb{F}}^P_{T,*} \simeq \mathrm{Span}_{\mathsf{si,all}}(\underline{\mathbb{F}}^P_T) \coloneqq \mathrm{Span} \circ (\underline{\mathbb{F}}^P_T, (\underline{\mathbb{F}}^P_T)^{\mathsf{si}}, \underline{\mathbb{F}}^P_T).$$

Now the claimed map is precisely the formula for the cocartesian unstraightening given by [HHLN23b, Theorem 3.9], since the cartesian unstraightening of $\underline{\mathbb{F}}^P_T$ is given by $t : \mathrm{Ar}_P(\mathbb{F}_T) \to \mathbb{F}_T$ (cf. Definition 2.24) and analogously $t : \mathrm{Ar}_P(\mathbb{F}_T)_{\mathsf{si}} \to \mathbb{F}_T$ is the cartesian unstraightening of $(\underline{\mathbb{F}}^P_T)^{\mathsf{si}}$.

The first claim of (3) follows from atomicity of P and Lemma A.9(3). The right vertical transformation in the diagram of (3) always exhibits $\mathbb{F}^{\mathsf{op}}_T$ as the colimit of $((\mathbb{F}_T)_{/\bullet})^{\mathsf{op}}$. The right-pointing horizontal morphisms are pointwise cocartesian fibrations. Basechange along cocartesian fibrations preserves colimits by [NS18, Lemma A.16], so we conclude that the middle vertical exhibits $\mathrm{Span}_{\mathsf{si,tdeg}}(\mathrm{Ar}_P(F_T))$ as the colimit of $\mathrm{Span}_{\mathsf{si,tdeg}}(\mathrm{Ar}_P((\mathbb{F}_T)_{/\bullet}))$. For the left vertical transformation this was already shown in Lemma 2.17. By the same argument as in (1) the source projections assemble into the claimed natural transformations, and commutativity of the left square is clear. □

For $X \in \mathbb{F}_T$ we denote by $\underline{X} \coloneqq \mathbb{F}_T(-, X) \in \mathsf{Fun}^{\times}(\mathbb{F}^{\mathsf{op}}_T, \mathsf{Cat}) = \mathsf{Cat}_T$ the T-space associated to X. By [CLL23a, Corollary 2.2.9] there is an equivalence $\underline{\mathsf{Fun}}_T(\mathcal{C}, \mathcal{D})(X) \simeq \mathsf{Fun}_T(\underline{X} \times \mathcal{C}, \mathcal{D})$ natural in $(X, \mathcal{C}, \mathcal{D}) \in \mathbb{F}^{\mathsf{op}}_T \times \mathsf{Cat}^{\mathsf{op}}_T \times \mathsf{Cat}_T$. Moreover, we now have equivalences natural in $X \in \mathbb{F}_T$:

$$\begin{aligned}
\textstyle\int(\underline{X} \times \underline{\mathbb{F}}^P_{T,*}) &\simeq \textstyle\int \underline{X} \times_{\mathbb{F}^{\mathsf{op}}_T} \int \underline{\mathbb{F}}^P_{T,*} \\
&\simeq ((\mathbb{F}_T)_{/X})^{\mathsf{op}} \times_{\mathbb{F}^{\mathsf{op}}_T} \mathrm{Span}_{\mathsf{si,tdeg}}(\mathrm{Ar}_P(\mathbb{F}_T)) \\
&\simeq \mathrm{Span}_{\mathsf{si,tdeg}}(\mathrm{Ar}_P((\mathbb{F}_T)_{/X})).
\end{aligned} \tag{7}$$

Combining this with Lemma 2.43, we get an equivalence of categories natural in $\mathcal{E} \in \mathsf{Cat}$ and $X \in \mathbb{F}^{\mathsf{op}}_T$:

$$\mathsf{Fun}(\mathrm{Span}_{\mathsf{si,tdeg}}(\mathrm{Ar}_P((\mathbb{F}_T)_{/X})), \mathcal{E}) \simeq \underline{\mathsf{Fun}}_T(\underline{\mathbb{F}}^P_{T,*}, \underline{\mathcal{E}}_T)(X). \tag{8}$$

If $\mathcal{E} \in \mathsf{Cat}^{\times}$, then by Lemma 2.45 it makes sense to ask what the full T-subcategory

$$\underline{\mathsf{CMon}}^P_T(\underline{\mathcal{E}}_T) \coloneqq \underline{\mathsf{Fun}}^{P-\oplus}_T(\underline{\mathbb{F}}^P_{T,*}, \underline{\mathcal{E}}_T) \subseteq \underline{\mathsf{Fun}}_T(\underline{\mathbb{F}}^P_{T,*}, \underline{\mathcal{E}}_T)$$

on P-commutative T-monoids corresponds to on the right side of (8). To answer this, recall from Example 2.12 how augmented adequate triples give us algebraic patterns based on span categories. We will consider the pattern $\mathrm{Span}_{\mathsf{si,tdeg}}(\mathrm{Ar}_P((\mathbb{F}_T)_{/\bullet}); T_{/\bullet})$ where we view $T_{/\bullet} \subseteq \mathrm{Ar}_P((\mathbb{F}_T)_{/\bullet})$ via the identity section.

Proposition 2.51. Let $P \subset T$ be atomic orbital and $X \in \mathbb{F}^P_T$. The adjunction equivalence (8) restricts to an equivalence on full subcategories

$$\mathrm{Seg}(\mathrm{Span}_{\mathsf{si},\mathsf{tdeg}}(\mathrm{Ar}_P((\mathbb{F}_T)_{/X}); T_{/X}), \mathcal{E}) \simeq \underline{\mathsf{CMon}}_T^P(\underline{\mathcal{E}}_T)(X).$$

Proof. Let $F \in \underline{\mathsf{Fun}}_T(\underline{\mathbb{F}}_{T,*}^P, \underline{\mathcal{E}}_T)(X)$ be given by $F : \underline{X} \times \underline{\mathbb{F}}_{T,*}^P \to \underline{\mathcal{E}}_T$ and denote by

$$\widetilde{F} : \mathrm{Span}_{\mathsf{si},\mathsf{tdeg}}(\mathrm{Ar}_P((\mathbb{F}_T)_{/X})) \simeq \textstyle\int(\underline{X} \times \underline{\mathbb{F}}_{T,*}^P) \to \mathcal{E}$$

its adjoint under $\int \dashv \underline{(-)}_T$, where we use the natural equivalences from Eq. (7). Under this identification of the cocartesian unstraightening of $\underline{X} \times \underline{\mathbb{F}}_{T,*}^P$, it was shown in [CLL23a, 4.9.8, 4.9.9] that $F \in \underline{\mathsf{CMon}}_T^P(\underline{\mathcal{E}}_T)(X)$ if and only if both of the following conditions hold:

(a) For each $(f : Y \to X) \in ((\mathbb{F}_T)_{/X})^{\mathsf{op}}$, the restriction of $\widetilde{F}$ to the fiber over f

$$\widetilde{F} : \mathrm{Span}_{\mathsf{si},\mathsf{all}}((\pi_X^* \underline{\mathbb{F}}_T^P)(f)) = \mathrm{Span}_{\mathsf{si},\mathsf{all}}(\underline{\mathbb{F}}_T^P(Y)) = \mathrm{Span}_{\mathsf{si},\mathsf{all}}((\mathbb{F}_T^P)_{/Y}) \to \mathcal{E}$$

is semiadditive in the usual sense, the analogous conditions which say that a functor out of $\mathrm{Span}_{\mathsf{si},\mathsf{all}}(\mathbb{F}) \simeq \mathbb{F}_*$ is a commutative monoid in Lurie's sense. More concretely, for $p : V \to Y$ and $q : W \to Y$ in $(\mathbb{F}_T^P)_{/Y}$ we have the spans $\mathrm{pr}_p : p \sqcup q \leftarrow p = p$ and likewise for pr_q, where $p \sqcup q \coloneqq V \sqcup W \to Y$ is the coproduct in $(\mathbb{F}_T^P)_{/Y}$. Then the above restriction is semiadditive if and only if

$$\widetilde{F}(\mathrm{pr}_p) \times \widetilde{F}(\mathrm{pr}_q) : \widetilde{F}(p \sqcup q) \to \widetilde{F}(p) \times \widetilde{F}(q)$$

is an equivalence.

(b) For every $f : (Y \to X) \in ((\mathbb{F}_T)_{/X})^{\mathsf{op}}$ and all $p : Z \to Y$ in $(\mathbb{F}_T^P)_{/Y}$, the span (backwards morphism) $\rho_p : p \to \mathrm{id}_Z$ in $\mathrm{Span}_{\mathsf{si},\mathsf{tdeg}}(\mathrm{Ar}_P((\mathbb{F}_T)_{/X}))$ given by

$$\begin{tikzcd}
Z \arrow[r, equal] \arrow[d, "p"'] & Z \arrow[r, equal] \arrow[d, equal] & Z \arrow[d, equal] \\
Y \arrow[dr, "f"'] & Z \arrow[l, "p"'] \arrow[r, equal] \arrow[d, dashed, "fp"] & Z \arrow[dl, dashed, "fp"] \\
 & X &
\end{tikzcd}$$

is inverted by $\widetilde{F}$. Note that the domain of ρ_p lies in the fiber over $f \in ((\mathbb{F}_T)_{/X})^{\mathsf{op}}$ and the target in the fiber over $fp \in ((\mathbb{F}_T)_{/X})^{\mathsf{op}}$. Note also that ρ_p really is a morphism in $\mathrm{Span}_{\mathsf{si},\mathsf{tdeg}}(\mathrm{Ar}_P((\mathbb{F}_T)_{/X}))$ because $\delta_p : Z \to Z \times_Y Z$ is a summand inclusion by Lemma A.9, using that P is atomic.

Also, under our explicit model for the cocartesian unstraightening of $\underline{X} \times \underline{\mathbb{F}}_{T,*}^P$, these spans precisely correspond to the morphisms ρ_p defined in [CLL23a, 4.9.7].

It thus remains to see that a functor $\Phi : \mathrm{Span}_{\mathsf{si},\mathsf{tdeg}}(\mathrm{Ar}_P((\mathbb{F}_T)_{/X}); T_{/X}) \to \mathcal{E}$ is a Segal object if and only if it satisfies the conditions (a) and (b) mentioned above. Using the equivalence $\mathbb{F}_{T_{/X}} \simeq (\mathbb{F}_T)_{/X}$ we can replace $P \subset T$ by $\pi_X^{-1}(P) \subset T_{/X}$, compare Lemma A.9. Hence we will show that $\Phi : \mathrm{Span}_{\mathsf{si},\mathsf{tdeg}}(\mathrm{Ar}_P(\mathbb{F}_T); T) \to \mathcal{E}$ is a Segal object if and only if

(a') Φ is fiberwise semiadditive, i.e. for all $X \in (\mathbb{F}_T)^{\mathsf{op}}$ and $p : Y \to X, q : Z \to X$ in $(\mathbb{F}_T^P)_{/X}$, the morphism

$$\Phi(\mathrm{pr}_p) \times \Phi(\mathrm{pr}_q) : \Phi(p \sqcup q) \to \Phi(p) \times \Phi(q)$$

is an equivalence.

(b') For every $p : Y \to X$ in $\mathbb{F}_T^P$, the span / backwards morphism $\rho_p : p \to \mathrm{id}_Y$ is inverted by Φ.

So let $p : Y \to X$ in $\mathbb{F}_T^P$ have coproduct decomposition $p = \coprod_{i=1}^n p_i$ where each $p_i : Y_i \to X$ lies in $P_{/X}$. Using Lemma 2.52 below, the source projection induces an equivalence

$$T \times_{\mathrm{Ar}_P(\mathbb{F}_T)_{\mathsf{si}}} (\mathrm{Ar}_P(\mathbb{F}_T)_{\mathsf{si}})_{/p} \underset{\simeq}{\xrightarrow{s}} T \times_{\mathbb{F}_T} (\mathbb{F}_T)_{/Y} = T_{/Y}. \tag{9}$$

So by Example 2.12, Φ is a Segal object for the pattern in question if and only for each such p the canonical map

$$\Phi(p) \to \lim_{(A \to Y) \in (T_{/Y})^{\mathsf{op}}} \Phi(\mathrm{id}_A) \tag{10}$$

is an equivalence. More specifically, an object $W : A \to Y$ in $T_{/Y}$ corresponds under the equivalence in (9) to the following object φ_W in the source of (9):

$$\begin{tikzcd} A \arrow[r, "W"] \arrow[d, equal] & Y \arrow[d, "p"] \\ A \arrow[r, "pW"'] & X \end{tikzcd}$$

where $A \simeq A \times_Y Y \to A \times_X Y$ is a summand inclusion by Lemma A.9(4). Then projecting to the component at W in (10) gives the map $\Phi(p) \to \Phi(\mathrm{id}_A)$ induced by the span $p \xleftarrow{\varphi_W} \mathrm{id}_A = \mathrm{id}_A$ in $\mathrm{Span}_{\mathsf{si},\mathsf{tdeg}}(\mathrm{Ar}_P(\mathbb{F}_T))$. But since $T_{/Y} \simeq \coprod_{i=1}^n T_{/Y_i}$ by Lemma A.4, the discrete subcategory on the summand inclusions $Y_i \to Y$ is final in $T_{/Y}$. So taking $W = (Y_i \to Y)$ we can factor the associated span $p \xleftarrow{\varphi_W} \mathrm{id}_{Y_i}$ as the backwards morphisms $\mathrm{pr}_{p_i} : p \leftarrow p_i$ followed by $\rho_{p_i} : p_i \leftarrow \mathrm{id}_{Y_i}$:

$$\begin{tikzcd} Y \arrow[d, "p"'] \arrow[dr, phantom, "\varphi_W"] & Y_i \arrow[l, hook', "W"'] \arrow[d, equal] \\ X & Y_i \arrow[l, "pW"] \end{tikzcd} \quad = \quad \begin{tikzcd} Y \arrow[d, "p"'] \arrow[dr, phantom, "\mathrm{pr}_{p_i}"] & Y_i \arrow[l, hook', "W"'] \arrow[d, dashed, "p_i"] \arrow[r, equal] \arrow[dr, phantom, "\rho_{p_i}"] & Y_i \arrow[d, equal] \\ X \arrow[r, equal] & X & Y_i \arrow[l, "p_i"] \end{tikzcd}$$

Under these considerations, the Segal condition (10) then reduces to the map

$$\prod_{i=1}^{n} \Phi(\rho_{p_i} \circ \mathrm{pr}_{p_i}) \ : \ \Phi(p) \to \prod_{i=1}^{n} \Phi(\mathrm{id}_{Y_i}) \tag{11}$$

being an equivalence (for all such p). Note that if Φ is fiberwise semiadditive, i.e. satisfies (a'), then by induction $\prod_{i=1}^{n} \Phi(\mathrm{pr}_{p_i})$ is an equivalence, and (b') gives that each $\Phi(\rho_{p_i})$ is an equivalence. This shows that if Φ satisfies (a') and (b'), then Φ is a Segal object. Conversely, with $p = p_i$ the Segal condition (11) requires the map

$$\Phi(\rho_{p_i}) : \Phi(p_i) \to \Phi(\mathrm{id}_{Y_i})$$

to be an equivalence, so this gives (b'). Moreover, now every $\Phi(\rho_{p_i})$ in (11) is an equivalence, so that by 2-out-of-3 we get that $\prod_{i=1} \Phi(\mathrm{pr}_{p_i}) : \Phi(p) \to \prod_{i=1}^{n} \Phi(p_i)$ is also one, giving the fiberwise semiadditivity required in (a'). $\square$

Lemma 2.52. Let $P \subset T$ be atomic orbital. For $p : Y \to X$ in $\mathbb{F}_T^P$, the inclusion $\mathrm{Ar}_P(\mathbb{F}_T)_{\mathsf{si}} \subset \mathrm{Ar}_P(\mathbb{F}_T)$ induces an equivalence

$$T \times_{\mathrm{Ar}_P(\mathbb{F}_T)_{\mathsf{si}}} (\mathrm{Ar}_P(\mathbb{F}_T)_{\mathsf{si}})_{/p} \xrightarrow{\simeq} T \times_{\mathrm{Ar}_P(\mathbb{F}_T)} (\mathrm{Ar}_P(\mathbb{F}_T))_{/p}.$$

Here the map $T \to \mathrm{Ar}_P(\mathbb{F}_T)_{\mathsf{si}} \subset \mathrm{Ar}_P(\mathbb{F}_T)$ is induced by the identity section $T \to \mathrm{Ar}(\mathbb{F}_T)$. Furthermore, the source projection $\mathrm{Ar}_P(\mathbb{F}_T) \to \mathbb{F}_T$ induces an equivalence

$$s : T \times_{\mathrm{Ar}_P(\mathbb{F}_T)} \mathrm{Ar}_P(\mathbb{F}_T)_{/p} \xrightarrow{\simeq} T \times_{\mathbb{F}_T} (\mathbb{F}_T)_{/Y} = T_{/Y}.$$

Proof. An object in $T \times_{\mathrm{Ar}_P(\mathbb{F}_T)} \mathrm{Ar}_P(\mathbb{F}_T)_{/p}$, i.e. a morphism $\mathrm{id}_A \to p$ inside $\mathrm{Ar}_P(\mathbb{F}_T)$ is the outer square in

$$\begin{tikzcd}
A \arrow[rr] \arrow[dd, equal] \arrow[dr] & & Y \arrow[dd, "p"] \\
& A \times_X Y \arrow[ur] \arrow[dl] \arrow[dr, phantom, "\lrcorner", very near start] & \\
A \arrow[rr] & & X
\end{tikzcd}$$

The morphism $A \simeq A \times_Y Y \to A \times_X Y$ is a summand inclusion by atomicity of P and Lemma A.9(4), and hence the morphism $\mathrm{id}_A \to p$ automatically lies inside $\mathrm{Ar}_P(\mathbb{F}_T)_{\mathsf{si}}$. Thus $T \times_{\mathrm{Ar}_P(\mathbb{F}_T)_{\mathsf{si}}} (\mathrm{Ar}_P(\mathbb{F}_T)_{\mathsf{si}})_{/p} \subset T \times_{\mathrm{Ar}_P(\mathbb{F}_T)} \mathrm{Ar}_P(\mathbb{F}_T)_{/p}$ is a wide subcategory. Moreover, given a morphism $\mathrm{id}_A \to \mathrm{id}_B$ over p in the latter category, we note that $\mathrm{id}_A \to p$ and $\mathrm{id}_B \to p$ and of course $\mathrm{id}_A \to \mathrm{id}_B$ itself already lie in $\mathrm{Ar}_P(\mathbb{F}_T)_{\mathsf{si}}$, hence the first equivalence follows.

For the second equivalence, note that the source map s is essentially surjective, and on mapping spaces it is given by

$$\{\mathrm{id}_A \to p\} \times_{\mathrm{Ar}_P(\mathbb{F}_T)(\mathrm{id}_A,p)} \mathrm{Ar}_P(\mathbb{F}_T)(\mathrm{id}_A,\mathrm{id}_B) \to \{A \to Y\} \times_{\mathbb{F}_T(A,Y)} \mathbb{F}_T(A,B) \tag{12}$$

This is an equivalence as s is right adjoint to the fully faithful identity section $\mathrm{id}_{(-)} : \mathbb{F}_T \hookrightarrow \mathrm{Ar}_P(\mathbb{F}_T)$. □

Proposition 2.53. Let $P \subset T$ atomic orbital. Then the natural transformation

$$\mathrm{Span}(s) : \mathrm{Span}_{\mathsf{si},\mathsf{tdeg}}(\mathrm{Ar}_P((\mathbb{F}_T)_{/\bullet})) \Rightarrow \mathrm{Span}_{\mathsf{all},P}((\mathbb{F}_T)_{/\bullet})$$

from Lemma 2.50 induces an equivalence with inverse given by right Kan extension

$$\mathrm{Span}(s)^* : \underline{\mathsf{Mack}}^P_T(\mathcal{E})(X) \xrightarrow{\simeq} \mathrm{Seg}(\mathrm{Span}_{\mathsf{si},\mathsf{tdeg}}(\mathrm{Ar}_P((\mathbb{F}_T)_{/X}); T_{/X}), \mathcal{E})$$

natural in $X \in \mathbb{F}_T^{\mathsf{op}}$ and $\mathcal{E} \in \mathsf{Cat}^\times$. Moreover, it induces an equivalence of $(\infty,2)$-categories

$$\mathrm{Span}(s)^* : \mathsf{Fbrs}^P_T \xrightarrow{\simeq} \mathsf{Fbrs}(\mathrm{Span}_{\mathsf{si},\mathsf{tdeg}}(\mathrm{Ar}_P(\mathbb{F}_T); T)).$$

Proof. The naturality claims are clear. Now fix $X \in \mathbb{F}_T^{\mathsf{op}}$ and $\mathcal{E} \in \mathsf{Cat}^\times$. By Example 2.15 the source category is the category of Segal objects for the pattern $\mathrm{Span}_{\mathsf{all},P}((\mathbb{F}_T)_{/X}; T_{/X})$, and by Lemma A.9 we can replace $P \subset T$ with $\pi_X^{-1}(P) \subset T_{/X}$ and show that

$$\mathrm{Span}(s) : \mathrm{Span}_{\mathsf{si},\mathsf{tdeg}}(\mathrm{Ar}_P(\mathbb{F}_T); T) \to \mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T; T)$$

satisfies the assumptions of Theorem 2.13(2,3). Concretely, we must verify the following points:

1. $\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)$ is soundly extendable. This was done in Lemma 2.16.
2. For $p = (Y \to X) \in \mathrm{Ar}_P(\mathbb{F}_T)$, the map $s : T \times_{\mathrm{Ar}_P(\mathbb{F}_T)_{\mathsf{si}}} (\mathrm{Ar}_P(\mathbb{F}_T)_{\mathsf{si}})_{/p} \to T \times_{\mathbb{F}_T} (\mathbb{F}_T)_{/X}$ is right cofinal. We showed in Lemma 2.52 that it is even an equivalence.
3. $\mathrm{Span}(s)$ induces an equivalence on elementary objects. This is clear as $T \xrightarrow{\mathrm{id}_-} \mathrm{Ar}_P(\mathbb{F}_T) \xrightarrow{s} T$ is even the identity.
4. For $p = (Y \to X) \in \mathrm{Ar}_P(\mathbb{F}_T)$, the map $s : (\mathrm{Ar}_P(\mathbb{F}_T)_{\mathsf{tdeg}})_{/p} \to (\mathbb{F}_T^P)_{/Y}$ induces an equivalence on maximal subgroupoids. In fact it already does so on the level of categories; note that $\mathrm{Ar}_P(\mathbb{F}_T)_{\mathsf{tdeg}} \simeq \mathbb{F}_T^{\simeq} \times_{\mathbb{F}_T^P} \mathrm{Ar}(\mathbb{F}_T^P) \simeq \coprod_{X \in \mathbb{F}_T} (\mathbb{F}_T^P)_{/X}$ by left-cancellability of $\mathbb{F}_T^P \subset \mathbb{F}_T$ (see Lemma A.9). Thus we obtain equivalences

$$(\mathrm{Ar}_P(\mathbb{F}_T)_{\mathsf{tdeg}})_{/p} \simeq (\mathbb{F}_T^{\simeq} \times_{\mathbb{F}_T^P} \mathrm{Ar}(\mathbb{F}_T^P))_{/p} \simeq ((\mathbb{F}_T^P)_{/X})_{/p}$$

and under these the source map corresponds to the equivalence $((\mathbb{F}_T^P)_{/X})_{/p} \simeq (\mathbb{F}_T^P)_{/Y}$.

$\square$

Putting everything together, we obtain the theorem promised at the beginning of this section.

Theorem 2.54. Let $P \subset T$ be atomic orbital. There are equivalences

$$\underline{\mathsf{Mack}}_T^P(\mathcal{E})(X) \xrightarrow{\mathrm{Span}(s)^*} \mathrm{Seg}(\mathrm{Span}_{\mathsf{si},\mathsf{tdeg}}(\mathrm{Ar}_P((\mathbb{F}_T)_{/X}); T_{/X}), \mathcal{E}) \simeq \underline{\mathsf{CMon}}_T^P(\underline{\mathcal{E}}_T)(X)$$

natural in $X \in \mathbb{F}_T^{\mathsf{op}}$ and $\mathcal{E} \in \mathsf{Cat}^\times$. Here the second one is induced by the adjunction Lemma 2.43 and Eq. (7). Passing to underlying categories, this gives the natural equivalence

$$\mathsf{Mack}_T^P(\mathcal{E}) \xrightarrow{\mathrm{Span}(s)^*} \mathrm{Seg}(\mathrm{Span}_{\mathsf{si},\mathsf{tdeg}}(\mathrm{Ar}_P(\mathbb{F}_T); T), \mathcal{E}) \simeq \mathsf{CMon}_T^P(\underline{\mathcal{E}}_T).$$

Proof. The claimed natural equivalence follows immediately from Proposition 2.53 and Proposition 2.51 with naturality of the latter inherited from Eq. (8). Passage to underlying categories yields the claimed equivalence by Lemma 2.17 and Lemma 2.50. $\square$

Example 2.55. Consider $\mathsf{Orb} \subset \mathsf{Glo}$ and $\mathcal{E} \in \mathsf{Cat}^\times$ as well as a global Mackey functor $\Phi \in \mathsf{Mack}_{\mathsf{Glo}}^{\mathsf{Orb}}(\mathcal{E})$ with corresponding $X \in \mathsf{CMon}_{\mathsf{Glo}}^{\mathsf{Orb}}(\underline{\mathcal{E}}_{\mathsf{Glo}})$ under the above equivalence. In this example we sketch how to determine X in terms of Φ. It was shown in [CLL23a, 5.2.4] that $\underline{\mathbb{F}}_{\mathsf{Glo},*}^{\mathsf{Orb}}$ is the global Borelification of finite pointed sets $\mathrm{Bor}_{\mathsf{Glo}}(\mathbb{F}_*)$ in the sense of Section 3.3, i.e. the global category sending G to the category of finite pointed G-sets $\mathbb{F}_{G,*}$ with restriction functoriality. By [CLL23a, 2.2.16] and our explicit model for the cocartesian unstraightening of $\underline{\mathbb{F}}_{\mathsf{Glo},*}^{\mathsf{Orb}}$, we obtain for each $\alpha : G \to K$ in Glo a commutative diagram

$$\begin{tikzcd}
\mathbb{F}_{K,*} \arrow[r, "X_K"] \arrow[d, "\alpha^*"'] & \mathsf{Fun}((\mathsf{Glo}_{/K})^{\mathsf{op}}, \mathcal{E}) \arrow[d, "((\mathsf{Glo}_{/\alpha})^{\mathsf{op}})^*"'] \arrow[drr, "\mathrm{ev}_\alpha"] & & \\
\mathbb{F}_{G,*} \arrow[r, "X_G"] \arrow[d, "\simeq"'] & \mathsf{Fun}((\mathsf{Glo}_{/G})^{\mathsf{op}}, \mathcal{E}) \arrow[rr, "\mathrm{ev}_{\mathrm{id}_G}"] & & \mathcal{E} \\
\mathrm{Span}_{\mathsf{si},\mathsf{all}}(\mathbb{F}_G) \arrow[r, "i_G"'] & \mathrm{Span}_{\mathsf{si},\mathsf{tdeg}}(\mathrm{Ar}_{\mathsf{Orb}}(\mathbb{F}_{\mathsf{Glo}})) \arrow[rr, "\mathrm{Span}(s)"'] \arrow[urr, "\widetilde{X}"] & & \mathrm{Span}_{\mathsf{all},\mathsf{Orb}}(\mathbb{F}_{\mathsf{Glo}}) \arrow[u, "\Phi"']
\end{tikzcd}$$

where $\widetilde{X} = \Phi \circ \mathrm{Span}(s)$ is adjoint to X via Lemma 2.43 and i_G is the inclusion of the fiber over G. The bottom horizontal composite is then the inclusion induced by $\mathsf{Orb}_G \simeq \mathsf{Orb}_{/G} \to \mathsf{Orb} \subset \mathsf{Glo}$ sending G/H to H. Given an injection $j : H \to K$, then $j_! : \underline{\mathbb{F}}_{\mathsf{Glo},*}^{\mathsf{Orb}}(H) \to \underline{\mathbb{F}}_{\mathsf{Glo},*}^{\mathsf{Orb}}(K)$ is Ind_H^K under the above equivalence, and $j_* : \underline{\mathcal{E}}_{\mathsf{Glo}}(H) \to \underline{\mathcal{E}}_{\mathsf{Glo}}(K)$ sends $F : (\mathsf{Glo}_{/H})^{\mathsf{op}} \to \mathcal{E}$ to $F(H \times_K -)$ in $\mathsf{Fun}^\times(((\mathbb{F}_{\mathsf{Glo}})_{/H})^{\mathsf{op}}, \mathcal{E}) \simeq \underline{\mathcal{E}}_{\mathsf{Glo}}(H)$ by the proof of Lemma 2.45. By Orb-semiadditivity of X, we have an equivalence $X_K \circ \mathrm{Ind}_H^K \simeq X_H \circ j_*$. In particular, for $\alpha : G \to K$ we get

$$X_K(K/H_+)(G \to K) \simeq X_H(H/H_+)(H \times_K G \to H) \simeq \Phi(H \times_K G).$$

We leave the explicit determination of X_K respectively $X_K(K/H_+)$ on morphisms to the reader. The above considerations similarly allow one to express Φ in terms of X.

Corollary 2.56. Let $P \subset T$ be atomic orbital. Then there is an equivalence of T-categories

$$\underline{\mathsf{Sp}}_T^P \simeq \underline{\mathsf{Mack}}_T^P(\mathsf{Sp})$$

where the left hand side is the T-category of P-genuine T-spectra defined in [CLL23a, 6.2.5].

Proof. By definition, we have $\underline{\mathsf{Sp}}_T^P := \mathsf{Sp}(\underline{\mathsf{CMon}}_T^P(\underline{\mathsf{Spc}}_T))$. The claim then follows from the natural equivalence $\mathsf{Sp}(\underline{\mathsf{Mack}}_T^P(\mathsf{Spc})) \simeq \underline{\mathsf{Mack}}_T^P(\mathsf{Sp})$, which pointwise follows from [GGN13, B.3], and the naturality is clear. $\square$

Remark 2.57. In the special case of $(T, P) = (\mathsf{Orb}_G, \mathsf{Orb}_G)$ for a finite group G, this is the famous Guillou-May/Barwick spectral Mackey-functor description of Sp^G which we discussed in Example 2.9. In the case of $(T, P) = (\mathsf{Glo}, \mathsf{Orb})$, the category $\underline{\mathsf{Sp}}_{\mathsf{Glo}}^{\mathsf{Orb}}$ is also called the global category of global spectra. This is because the main theorem of [CLL23a] shows that this free globally presentable equivariantly stable global category on one generator $\underline{\mathsf{Sp}}_{\mathsf{Glo}}^{\mathsf{Orb}}$ sends a finite group G to the category of G-global spectra $\mathsf{Sp}^{G\text{-}\mathsf{gl}}$ of Lenz [Len21], and a group homomorphism $\alpha : K \to G$ to the restriction $\alpha^* : \mathsf{Sp}^{G\text{-}\mathsf{gl}} \to \mathsf{Sp}^{K\text{-}\mathsf{gl}}$. As mentioned in the Introduction, the category of G-global spectra $\mathsf{Sp}^{G\text{-}\mathsf{gl}}$ can be seen as a common generalization of $\mathsf{Sp}^{\mathsf{gl}}$ and Sp^G; we have a forgetful functor $\mathrm{res}_e^G : \mathsf{Sp}^{G\text{-}\mathsf{gl}} \to \mathsf{Sp}^{\mathsf{gl}}$ and two fully faithful inclusions $\mathsf{Sp}^G \subseteq \mathsf{Sp}^{G\text{-}\mathsf{gl}}$ exhibiting Sp^G as both a left- and right Bousfield localization of $\mathsf{Sp}^{G\text{-}\mathsf{gl}}$. We will elaborate on the underlying models of these categories in Section 4.2. Using the above, we obtain a spectral Mackey functor description of G-global spectra functorial in G:

$$\underline{\mathsf{Sp}}_{\mathsf{Glo}}^{\mathsf{Orb}} \simeq \underline{\mathsf{Mack}}_{\mathsf{Glo}}^{\mathsf{Orb}}(\mathsf{Sp}) = \mathsf{Fun}^\times(\mathrm{Span}_{\mathsf{all},\mathsf{Orb}}((\mathbb{F}_{\mathsf{Glo}})_{/\bullet}), \mathsf{Sp}).$$

For $G = e$, a more explicit equivalence $\mathsf{Sp}^{\mathsf{gl}} \simeq \mathsf{Fun}^\times(\mathrm{Span}_{\mathsf{all},\mathsf{Orb}}(\mathbb{F}_{\mathsf{Glo}}), \mathsf{Sp})$ was already constructed by Lenz using global algebraic K-theory in [Len22, Theorem A]. He shows that the equivalence sends a global spectrum X to the global Mackey functor of its genuine fixed points $(X^G)_{G \in \mathsf{Glo}^{\mathsf{op}}}$ with restrictions and transfers corresponding via representability to the restrictions and transfers on the equivariant homotopy groups π_0^G, see Section 6 of op. cit.

Remark 2.58. Let $P = T$ be atomic orbital. By [NS22, Remark 2.3.8], the category of T-symmetric monoidal T-categories $\mathbf{Cat}_T^{\otimes}$ of Nardin-Shah agrees with $\mathsf{CMon}_T(\underline{\mathsf{Cat}}_T) \simeq \mathsf{Mack}_T(\mathsf{Cat})$. Hence their notion of T-symmetric monoidal T-category agrees with ours.

Now the proof [BHS22, 5.2.14] that fibrous patterns for $\mathrm{Span}(G)$ agree with the G-∞-operads of Nardin-Shah immediately adapts to this general case.

Corollary 2.59. Let $P = T$ be atomic orbital. Recall from Lemma 2.50 that $t : \mathrm{Span}_{\mathsf{si},\mathsf{tdeg}}(\mathrm{Ar}(\mathbb{F}_T)) \to \mathbb{F}_T$ is the cocartesian unstraightening of $\underline{\mathbb{F}}_{T,*}$. Then $\mathsf{Fbrs}(\mathrm{Span}_{\mathsf{si},\mathsf{tdeg}}(\mathrm{Ar}(\mathbb{F}_T; T))$ and the category Op_T

of T-∞-operads defined in [NS22] form the same subcategory of $\mathsf{Cat}_{/\int \underline{\mathbb{F}}_{T,*}}$. In particular the source projection from Proposition 2.53 induces an equivalence $\mathsf{Fbrs}_T \xrightarrow[\simeq]{\mathrm{Span}(s)^*} \mathsf{Op}_T$ which is compatible with the mapping categories in that for $\mathcal{O}, \mathcal{P} \in \mathsf{Fbrs}_T$ we have

$$\mathrm{Span}(s)^* : \mathsf{Fun}_{\mathsf{Fbrs}_T}(\mathcal{O}, \mathcal{P}) \xrightarrow{\simeq} \mathbf{Alg}_T(s^*\mathcal{O}, s^*\mathcal{P}),$$

where the right hand side the category of algebras defined in [NS22, 2.2.1]. In particular if $\mathcal{C} \in \mathsf{Mack}_T(\mathsf{Cat})$ is a T-symmetric monoidal category, we obtain an induced equivalence

$$\mathrm{CAlg}_T(\mathcal{C}) \coloneqq \mathsf{Fun}_{\mathsf{Fbrs}_T}(\mathrm{Span}(\mathbb{F}_T), \textstyle\int \mathcal{C}) \xrightarrow[\simeq]{s^*} \mathbf{CAlg}_T\left(s^* \textstyle\int \mathcal{C}\right).$$

Proof. We have already seen in Proposition 2.53 that $\mathrm{Span}(s)$ induces an equivalence compatible with the mapping categories

$$\mathrm{Span}(s)^* : \mathsf{Fbrs}_T \xrightarrow{\simeq} \mathsf{Fbrs}(\mathrm{Span}(\mathrm{Ar}_{\mathsf{si},\mathsf{tdeg}}(\mathbb{F}_T); T)).$$

By [NS22, Definition 2.1.3], their inert morphisms in $\int \underline{\mathbb{F}}_{T,*}$ are precisely the backwards morphisms in $\mathrm{Span}_{\mathsf{si},\mathsf{tdeg}}(\mathrm{Ar}(\mathbb{F}_T))$. Arguing as in [BHS22, 5.2.12], we see that these inert morphisms contain all cocartesian lifts of $t : \int \underline{\mathbb{F}}_{T,*} \to \mathbb{F}_T^{\mathsf{op}}$, and we obtain that Op_T agrees with the category of weak Segal fibrations on $\int \underline{\mathbb{F}}_{T,*}$. Next we show that $\mathrm{Span}_{\mathsf{si},\mathsf{tdeg}}(\mathbb{F}_T; T)$ is a sound pattern, so that by [BHS22, Proposition 4.1.7] fibrous patterns on it agree with weak Segal fibrations, which then gives the first equivalence $\mathrm{Span}(s)^* : \mathsf{Fbrs}_T \simeq \mathsf{Op}_T$. By [BHS22, Proposition 3.3.23] it suffices to show

1. $(\mathrm{Ar}(\mathbb{F}_T)_{\mathsf{si}})_{/f} \to \mathrm{Ar}(\mathbb{F}_T)_{/f}$ is fully faithful for every $f : X \to Y$ in $\mathbb{F}_T$. This follows from left-cancellability of summand inclusions in $\mathbb{F}_T$, see Lemma A.4.
2. The inclusion $T \times_{\mathrm{Ar}(\mathbb{F}_T)_{\mathsf{si}}} (\mathrm{Ar}(\mathbb{F}_T)_{\mathsf{si}})_{/f} \to T \times_{\mathrm{Ar}(\mathbb{F}_T)} \mathrm{Ar}(\mathbb{F}_T)_{/f}$ is cofinal for every $f : X \to Y$ in $\mathbb{F}_T$. In fact, it is even an equivalence by Lemma 2.52.

Since the categories $\mathbf{Alg}_T$ are also defined as full subcategories of $\mathsf{Fun}_{/\int \underline{\mathbb{F}}_{T,*}}$ on functors preserving inert morphisms, the remaining claims follow. $\square$

Remark 2.60 (Comparison of parametrized Mackey functors)**.** In [Nar16, Definition 4.10], Nardin introduced the T-category of spans of finite T-sets for atomic orbital T, which he denoted $\underline{\mathbf{A}}^{\mathit{eff}}(T)$. It was already observed in [HHLN23b, Remark 3.22] that there is an equivalence of T-categories $\underline{\mathbf{A}}^{\mathit{eff}}(T) \simeq \mathrm{Span}(\underline{\mathbb{F}}_T)$, where the right hand side is the composite

$$T^{\mathsf{op}} \xrightarrow{\underline{\mathbb{F}}_T} \mathsf{Cat}^{\mathsf{lex}} \to \mathsf{AdTrip} \xrightarrow{\mathrm{Span}} \mathsf{Cat}$$

with $\mathsf{Cat}^{\mathsf{lex}} \to \mathsf{AdTrip}, \mathcal{C} \mapsto (\mathcal{C}, \mathcal{C}, \mathcal{C})$ from [HHLN23b, Example 2.3(2)]. The T-category $\mathrm{Span}(\underline{\mathbb{F}}_T)$ is fiberwise semiadditive by Lemma B.2. Moreover, using [BH17, Corollary C.21], one sees that $\mathrm{Span}(\underline{\mathbb{F}}_T)$

is even T-semiadditive since $\underline{\mathbb{F}}_T$ admits finite T-coproducts.

Indeed, given $f : X \to Y$ in $\mathbb{F}_T$, we consider the adjunction $f_! : (\mathbb{F}_T)_{/X} \rightleftarrows (\mathbb{F}_T)_{/Y} : f^*$. Recall that the counit is given at $V \to Y$ by the morphism $X \times_Y V \to V$ over Y, and the unit is given at $Z \to X$ by $Z \simeq X \times_{X \times_Y X} (X \times_Y X) \times_X Z \to (X \times_Y Z)$ over X. It follows from pullback pasting that both are cartesian natural transformations. Hence the claim follows from [BH17, Corollary C.21].

One sees directly that the inclusion $j : \underline{\mathbb{F}}_{T,*} \simeq \mathrm{Span}_{\mathsf{si},\mathsf{all}}(\underline{\mathbb{F}}_T) \simeq \mathrm{Span}(\underline{\mathbb{F}}_T)$ preserves finite T-coproducts. It follows from T-semiadditivity and [CLL23a, 4.6.14] that i is a P-commutative monoid in $\mathrm{Span}(\underline{\mathbb{F}}_T)$, which was already noted in [Nar16, Lemma 6.3]. Moreover, Nardin goes on to show in [Nar16, Theorem 6.5] that restriction along j induces an equivalence

$$j^* : \underline{\mathbf{Mack}}_T(\mathcal{C}) \coloneqq \underline{\mathsf{Fun}}_T^{T-\times}(\mathrm{Span}(\underline{\mathbb{F}}_T), \mathcal{C}) \xrightarrow{\simeq} \underline{\mathsf{CMon}}_T(\mathcal{C})$$

natural in $\mathcal{C} \in \mathsf{Cat}_T^{T-\times}$. In particular, considering $\mathcal{C} = \underline{\mathcal{E}}_T$, we obtain an equivalence of T-categories

$$\underline{\mathbf{Mack}}_T(\underline{\mathcal{E}}_T) \xrightarrow[\simeq]{j^*} \underline{\mathsf{CMon}}_T(\underline{\mathcal{E}}_T) \simeq \underline{\mathsf{Mack}}_T(\mathcal{E})$$

natural in $\mathcal{E} \in \mathsf{Cat}^{\times}$.

3 Borel Parametrized Algebra

This section is devoted to studying Borel-inclusions $(T, P) \subseteq (S, Q)$ and their induced "Borelifications" (right Kan extensions) $\mathsf{Mack}_T^P \Rightarrow \mathsf{Mack}_S^Q$. The name is inspired from the special case $(BG, BG) \subseteq (\mathsf{Orb}_G, \mathsf{Orb}_G)$ which has been studied in [Hil24, Section 2.4], in which case the theory specializes to the classical situation in equivariant homotopy theory where one embeds Borel-equivariant objects[12] into genuinely G-equivariant objects. The classical examples of this are $\mathsf{Fun}(BG, \mathsf{Spc}) \subseteq \mathsf{Spc}^G$ or $\mathsf{Fun}(BG, \mathsf{Sp}) \subseteq \mathsf{Sp}^G$. We will focus on the parametrized symmetric monoidal version of this; see Example 3.1 for the main example to keep in mind.

We have three subsections on the general, equivariant and global case. The main result is Proposition 3.7 and its more specific equivariant version Theorem 3.15 which show that commutative algebras in a "Borelified" parametrized symmetric monoidal category $\mathrm{Bor}(\mathcal{C})$ can be identified with commutative algebras in $\mathcal{C}$. These results will be crucial in Section 4 for defining a comparison functor from a 1-category of strictly commutative ring spectra to the ∞-category of commutative algebras in an equivariantly symmetric monoidal global category of equivariant/global spectra.

3.1 General Borel Theory

Before we dive into the generalities, let us begin with the motivating example to keep in mind.

[12] Also known as "objects with G-action" or local systems on BG.

Example 3.1. Given a category $\mathcal{C} \in \mathsf{Cat}$, we can right Kan extend along $\{*\} \subseteq \mathsf{Glo}$ to obtain a global Borel category $\mathrm{Bor}_{\mathsf{Glo}}(\mathcal{C}) \in \mathsf{Cat}_{\mathsf{Glo}}$ which sends $G \mapsto \mathsf{Fun}(BG, \mathcal{C})$ with restriction functoriality. We will see below that the inclusion $(*, *) \subseteq (\mathsf{Glo}, \mathsf{Orb})$ induces a fully faithful inclusion $i : \mathrm{Span}(\mathbb{F}) \hookrightarrow \mathrm{Span}_{\mathsf{all},\mathsf{Orb}}(\mathbb{F}_{\mathsf{Glo}})$, and given a symmetric monoidal category $\mathcal{C} \in \mathsf{Mack}(\mathsf{Cat})$, we may consider its Borelification $\mathrm{Bor}^{\mathsf{Orb}}_{\mathsf{Glo}}(\mathcal{C}) \coloneqq i_*\mathcal{C} \in \mathsf{Mack}^{\mathsf{Orb}}_{\mathsf{Glo}}(\mathsf{Cat})$. This equivariantly symmetric monoidal global category admits the following description:

1. The underlying global category agrees with $\mathrm{Bor}_{\mathsf{Glo}}(U\mathcal{C})$ where $U : \mathsf{Mack}(\mathsf{Cat}) \to \mathsf{Cat}$ forgets the symmetric monoidal structure. In particular we have $\mathrm{Bor}^{\mathsf{Orb}}_{\mathsf{Glo}}(\mathcal{C})(G) = \mathcal{C}^{BG}$ for each $G \in \mathsf{Glo}$. Under this identification any $f : G \to K$ in Glo, viewed as a backwards morphism in $\mathrm{Span}_{\mathsf{all},\mathsf{Orb}}(\mathbb{F}_{\mathsf{Glo}})$, induces the usual restriction $f^* : \mathcal{C}^{BK} \to \mathcal{C}^{BG}$.

2. The restriction along $\mathrm{Span}(\mathbb{F}) \to \mathrm{Span}_{\mathsf{all},\mathsf{Orb}}(\mathbb{F}_{\mathsf{Glo}})$ sending $* \mapsto G$ encodes the pointwise symmetric monoidal structure on $\mathcal{C}^{BG}$.

3. The forwards maps $p : X \to Y$ in $\mathbb{F}^{\mathsf{Orb}}_{\mathsf{Glo}}$ induce the indexed tensor products / multiplicative norms $p_\otimes : \mathcal{C}^X \to \mathcal{C}^Y$ of [HHR16, A.3.2]. In particular, for a subgroup inclusion $p : H \leq G$ this functor $p_\otimes : \mathcal{C}^{BH} \to \mathcal{C}^{BG}$ sends $c \in \mathcal{C}^{BH}$ to $\bigotimes_{G/H} c \in \mathcal{C}^{BG}$.

Importantly, we have an equivalence $\mathrm{CAlg}^{\mathsf{Orb}}_{\mathsf{Glo}}(\mathrm{Bor}^{\mathsf{Orb}}_{\mathsf{Glo}}(\mathcal{C})) \xrightarrow{\simeq} \mathrm{CAlg}(\mathcal{C})$ induced by evaluation at the trivial group, see Proposition 3.7.

Definition 3.2. Let $Q \subset S$ be an orbital subcategory and $T \subseteq S$ a full subcategory. Suppose that

($\star$) For every morphism $f : A \to B$ in Q with $B \in T$, we also have $A \in T$.

Then $P \coloneqq Q \cap T$ is orbital in T and we call $(T, P) \subseteq (S, Q)$ a Borel-inclusion.

Remark 3.3. Note that if $(T, P) \subseteq (S, Q)$ is a Borel-inclusion, then $\mathbb{F}_T \subseteq \mathbb{F}_S$ and $\mathbb{F}^P_T \subseteq \mathbb{F}^S_Q$ are fully faithful, and the following apparently stronger version of ($\star$) follows immediately:

($\star'$) If $f : X \to Y$ lies in $\mathbb{F}^Q_S$ and $Y \in \mathbb{F}_T$, then f already lies in $\mathbb{F}^P_T$ (and $X \in \mathbb{F}_T$).

Example 3.4. Let G be a finite group.

1. $(BG, BG) \subseteq (\mathsf{Orb}_G, \mathsf{Orb}_G)$ is a Borel-inclusion (where $* \mapsto G/e$) which is the classical case mentioned in the introduction.

2. $(\mathsf{Orb}_G, \mathsf{Orb}_G) \subseteq (\mathsf{Glo}_{/G}, \pi_G^{-1}(\mathsf{Orb}))$ is a Borel-inclusion, where $\pi_G : \mathsf{Glo}_{/G} \to \mathsf{Glo}$ is the forgetful functor, and hence $\pi_G^{-1}(\mathsf{Orb})$ denotes the wide subcategory of $\mathsf{Glo}_{/G}$ on the injective morphisms. This extends G-equivariant objects to G-global objects.

3. $(*, *) \subseteq (\mathsf{Glo}, \mathsf{Orb}) \subseteq (\mathsf{Spc}, \mathsf{finfib})$ are Borel-inclusions. Here $\mathsf{finfib} \subset \mathsf{Spc}$ is the subcategory on those morphisms of anima with finite fibers. The composite case $(*, *) \subseteq (\mathsf{Spc}, \mathsf{finfib})$ was investigated in [EH21, Section 3.2].

Lemma 3.5. Let $(T,P) \subseteq (S,Q)$ be a Borel-inclusion. Then the induced functor $\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T) \to \mathrm{Span}_{\mathsf{all},Q}(\mathbb{F}_S)$ is a fully faithful strong Segal morphism of soundly extendable algebraic patterns.

Proof. By Lemma 2.16 it remains to show that i is fully faithful. Recall the formula for mapping spaces in Span-categories from Lemma B.1. We claim that under the given hypotheses we already have an equivalence of categories

$$i : (\mathbb{F}_T)_{/X} \times_{\mathbb{F}_T} (\mathbb{F}_T^P)_{/Y} \to (\mathbb{F}_S)_{/X} \times_{\mathbb{F}_S} (\mathbb{F}_S^Q)_{/Y}$$

for all $X, Y \in \mathbb{F}_T$. Since $T \subseteq S$ is fully faithful, so is $\mathbb{F}_T \subseteq \mathbb{F}_S$ and hence $(\mathbb{F}_T)_{/X} \subseteq (\mathbb{F}_S)_{/X}$ for any $X \in \mathbb{F}_T$. Moreover, the condition $(\star)$ implies that $(\mathbb{F}_T^P)_{/Y} \to (\mathbb{F}_S^Q)_{/Y}$ is an equivalence for all $Y \in \mathbb{F}_T$. This yields essential surjectivity of i, and we get fully faithfulness from the fact that fully faithful functors are closed under limits in $\mathrm{Ar}(\mathsf{Cat})$. □

Let us fix some notation for the following proposition. Consider a Borel-inclusion $i : (T,P) \subseteq (S,Q)$, with induced $i : \mathbb{F}_T \subseteq \mathbb{F}_S$ (cf. Lemma A.4 for fully faithfulness). By the above Lemma we have a fully faithful inclusion $I : \mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T) \subseteq \mathrm{Span}_{\mathsf{all},Q}(\mathbb{F}_S)$. We also consider the inclusions $k : \mathbb{F}_T^{\mathsf{op}} \to \mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)$ and $\ell : \mathbb{F}_S^{\mathsf{op}} \to \mathrm{Span}_{\mathsf{all},Q}(\mathbb{F}_S)$.

Proposition 3.6. With notation as above, suppose that $\mathcal{C}$ is a category with finite products and that the right Kan extension $i_* : \mathsf{Fun}(T^{\mathsf{op}}, \mathcal{C}) \to \mathsf{Fun}(S^{\mathsf{op}}, \mathcal{C})$ exists. Then the right Kan extensions in the following diagram exist, and the Beck-Chevalley transformations make it commute:

$$\begin{tikzcd}
\mathsf{Fun}(T^{\mathsf{op}}, \mathcal{C}) \arrow[d, hook, "i_*"'] & \mathsf{Fun}^\times(\mathbb{F}_T^{\mathsf{op}}, \mathcal{C}) \arrow[l, "\simeq"'] \arrow[d, hook, "i_*"] & \mathsf{Mack}_T^P(\mathcal{C}) \arrow[l, "k^*"'] \arrow[d, hook, "I_*"] \\
\mathsf{Fun}(S^{\mathsf{op}}, \mathcal{C}) & \mathsf{Fun}^\times(\mathbb{F}_S^{\mathsf{op}}, \mathcal{C}) \arrow[l, "\simeq"] & \mathsf{Mack}_S^Q(\mathcal{C}) \arrow[l, "\ell^*"]
\end{tikzcd}$$

In particular, $\Phi : \mathrm{Span}_{\mathsf{all},Q}(\mathbb{F}_S) \xrightarrow{\times} \mathcal{C}$ is right Kan extended from $\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)$ if and only if its restrictions $\Phi|_{\mathbb{F}_S^{\mathsf{op}}}$ respectively $\Phi|_{S^{\mathsf{op}}}$ are right Kan extended from $\mathbb{F}_T^{\mathsf{op}}$ respectively T^{op}.

Proof. In the left square, the horizontal equivalences are restrictions along $j_T : T \subseteq \mathbb{F}_T$ and $j_S : S \subseteq \mathbb{F}_S$, with inverse given by right Kan extension, see Example 2.8. Clearly $i^* j_S^* \simeq j_T^* i^*$ and since the j_T^*, j_S^* are equivalences we get the existence of the right adjoint to $i^* : \mathsf{Fun}^\times(\mathbb{F}_T^{\mathsf{op}}, \mathcal{C}) \to \mathsf{Fun}^\times(\mathbb{F}_S^{\mathsf{op}}, \mathcal{C})$ as well as the equivalence $BC : i^*(j_S)_* \xRightarrow{\simeq} (j_T)_* i^*$. By Definition/Lemma C.5, taking the total mate[13] gives that $BC : j_S^* i_* \Rightarrow i_* j_T^*$ is an equivalence as desired. To see existence and commutativity of the right square, it suffices by Lemma C.2 to show that the following functor is left cofinal for each $X \in \mathbb{F}_S^{\mathsf{op}}$:

$$F : \mathbb{F}_T^{\mathsf{op}} \times_{\mathbb{F}_S^{\mathsf{op}}} (\mathbb{F}_S^{\mathsf{op}})_{X/} \to \mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T) \times_{\mathrm{Span}_{\mathsf{all},Q}(\mathbb{F}_S)} \mathrm{Span}_{\mathsf{all},Q}(\mathbb{F}_S)_{X/}.$$

[13] Note that since j_T^* is an equivalence, its left adjoint, right adjoint and inverse agree. Analogously for j_S^*.

By Lemma B.3 the map $L : (\mathbb{F}_S^{\mathsf{op}})_{X/} \to \mathrm{Span}_{\mathsf{all},Q}(\mathbb{F}_S)_{X/}$ admits a right adjoint R which sends a span $(X \leftarrow B \to A)$ to $X \leftarrow B$. Now on the above full subcategories, L clearly restricts to F, and also R will restrict to a functor right adjoint to F; If $\varphi = (X \leftarrow B \to A)$ is an object in the codomain of F, then $A \in \mathbb{F}_T$ and $B \to A$ in Q, so by definition of a Borel-inclusion we have that B lies in $\mathbb{F}_T$ hence $R(\varphi) = (X \leftarrow B) \in \mathbb{F}_T^{\mathsf{op}} \times_{\mathbb{F}_S^{\mathsf{op}}} (\mathbb{F}_S^{\mathsf{op}})_{X/}$ as well. This proves that F is a right adjoint, hence coinitial, and so we are done.

For the remaining direction of the remark at the end, suppose that $\Phi \in \mathsf{Mack}_S^Q(\mathsf{Cat})$ such that $\ell^*\Phi$ is right Kan extended from $\mathbb{F}_T^{\mathsf{op}}$, i.e. such that $\eta^i_{\ell^*\Phi} : \ell^*\Phi \to i_*i^*\ell^*\Phi$ is an equivalence. We need to show that Φ is right Kan extended from $\mathsf{Mack}_T^P(\mathsf{Cat})$, i.e. that $\eta^I_\Phi : \Phi \to I_*I^*\Phi$ is an equivalence. Note that ℓ^* is conservative as ℓ is essentially surjective, so it suffices to check that $\ell^*\eta^I_\Phi$ is an equivalence. This follows from the following commutative diagram, which exists by Lemma C.3:

$$\begin{tikzcd}
i_*k^*I^*\Phi & i_*i^*\ell^*\Phi \arrow[l, "\simeq"'] \\
\ell^*I_*I^*\Phi \arrow[u, "BC", "\simeq"'] & \ell^*\Phi \arrow[l, "\ell^*\eta^I_\Phi"] \arrow[u, "\simeq", "\eta^i_{\ell^*\Phi}"']
\end{tikzcd}$$

□

In general, the inclusion $i : \mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T) \subseteq \mathrm{Span}_{\mathsf{all},Q}(\mathbb{F}_S)$ coming from a Borel-inclusion will not satisfy the conditions of Theorem 2.13, although the only hypothesis which is not satisfied is that i induces an equivalence on elementary objects. Nevertheless, we have the following related result which essentially also appears as [BH17, Corollary C.19], but we don't understand their proof, hence give a different one.

Proposition 3.7. Let $(T, P) \subseteq (S, Q)$ be a Borel-inclusion and $\mathcal{C} \in \mathsf{Mack}_T^P(\mathsf{Cat})$ with extension $i_*\mathcal{C} \in \mathsf{Mack}_S^Q(\mathsf{Cat})$. Then restriction along i induces an equivalence

$$i^* : \mathrm{CAlg}_S^Q(i_*\mathcal{C}) \xrightarrow{\simeq} \mathrm{CAlg}_T^P(\mathcal{C})$$

Proof. Recall that by definition $\mathrm{CAlg}_S^Q(-) := \mathrm{Sect}^{\mathbb{F}_S^{\mathsf{op}}-\mathsf{cc}}(\textstyle\int -)$. By Theorem 2.28 the following natural composite

$$\mathsf{Fun}_S^{Q-\otimes}(\mathcal{A}_S^Q, \mathcal{D}) \xrightarrow{\int} \mathsf{Fun}_{\mathsf{Fbrs}_S^Q}(\mathrm{Ar}_{\mathsf{act}}(\mathrm{Span}_{\mathsf{all},Q}(\mathbb{F}_S)), \textstyle\int\mathcal{D}) \xrightarrow{(\mathrm{id}_{(-)})^*} \mathrm{CAlg}_T^P(\mathcal{D})$$

is an equivalence of categories for every $\mathcal{D} \in \mathsf{Mack}_S^Q(\mathsf{Cat})$. An analogous statement holds for $\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)$. We saw in Lemma 3.5 that $i : \mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T) \to \mathrm{Span}_{\mathsf{all},Q}(\mathbb{F}_S)$ is a fully faithful strong Segal morphism of soundly extendable algebraic patterns. Hence by [BHS22, 4.2.8] we have a commutative square

$$\begin{tikzcd} \mathsf{Mack}_S^Q(\mathsf{Cat}) \arrow[r, "\int"] \arrow[d, "i^*"'] & \mathsf{Fbrs}_S^Q \arrow[d, "i^*"] \\ \mathsf{Mack}_T^P(\mathsf{Cat}) \arrow[r, "\int"] & \mathsf{Fbrs}_T^P \end{tikzcd}$$

and it follows from [BHS22, 5.3.13, 5.3.15] and the fact that (un)straightening promotes to an equivalence of $(\infty, 2)$-categories (see e.g. [HHLN23a]) that this is a commutative square of $(\infty, 2)$-categories. Using the defining property of a Borel-inclusion, one easily checks that the following square is cartesian:

$$\begin{tikzcd} \mathrm{Ar}_{\mathsf{act}}(\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)) \arrow[r, hook, "\mathrm{Ar}_{\mathsf{act}}(i)"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Ar}_{\mathsf{act}}(\mathrm{Span}_{\mathsf{all},Q}(\mathbb{F}_S)) \arrow[d] \\ \mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T) \arrow[r, hook, "i"'] & \mathrm{Span}_{\mathsf{all},Q}(\mathbb{F}_S) \end{tikzcd}$$

Since the vertical fibrations are classified by $\mathcal{A}_T^P$ respectively $\mathcal{A}_S^Q$ (c.f. Theorem 2.28), this also gives $i^*\mathcal{A}_S^Q \simeq \mathcal{A}_T^P$. Moreover, the identity section $\mathrm{id}_{(-)} : \mathrm{Span}_{\mathsf{all},Q}(\mathbb{F}_S) \to \mathrm{Ar}_{\mathsf{act}}(\mathrm{Span}_{\mathsf{all},Q}(\mathbb{F}_S))$ pulls back along i to the identity section for $\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)$. For $\mathcal{D} \in \mathsf{Mack}_S^Q(\mathsf{Cat})$ we get a commutative diagram

$$\begin{tikzcd} \mathsf{Fun}_S^{Q-\otimes}(\mathcal{A}_S^Q, \mathcal{D}) \arrow[r, "\int"'] \arrow[d, "i^*"] \arrow[rr, bend left, "\simeq"] & \mathsf{Fun}_{\mathsf{Fbrs}_S^Q}(\mathrm{Ar}_{\mathsf{act}}(\mathrm{Span}_{\mathsf{all},Q}(\mathbb{F}_S)), \int \mathcal{D}) \arrow[r, "(\mathrm{id}_{(-)})^*"'] \arrow[d, "i^*"] & \mathrm{CAlg}_S^Q(\mathcal{D}) \arrow[d, "i^*"] \\ \mathsf{Fun}_T^{P-\otimes}(\mathcal{A}_T^P, i^*\mathcal{D}) \arrow[r, "\int"] \arrow[rr, bend right, "\simeq"'] & \mathsf{Fun}_{\mathsf{Fbrs}_T^P}(\mathrm{Ar}_{\mathsf{act}}(\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)), \int i^*\mathcal{D}) \arrow[r, "(\mathrm{id}_{(-)})^*"] & \mathrm{CAlg}_T^P(i^*\mathcal{D}) \end{tikzcd}$$

In the special case where $\mathcal{D} = i_*\mathcal{C}$, the left vertical functor is an equivalence by the adjunction $i^* \dashv i_*$, the fact that i_* is fully faithful so that $\varepsilon : i^*i_* \simeq \mathrm{id}$, and Lemma D.6. Then by commutativity also the right vertical morphism will be an equivalence, as desired. $\square$

Remark 3.8. Instead of the above proof, one could also argue by considering the adjunction $i^* \dashv i_*$ on the level of cocartesian fibrations, e.g. via [Sha23, Example 2.26].

Let us record the following fact shown in the above proof.

Corollary 3.9. Let $i : (T, P) \subseteq (S, Q)$ be Borel-inclusion. Then $i^*\mathcal{A}_S^Q \simeq \mathcal{A}_T^P$

3.2 Equivariant Borel Theory

In this subsection, we want to take a closer look at the Borel inclusion $(BG, BG) \to (\mathsf{Orb}_G, \mathsf{Orb}_G)$ with induced $i : \mathrm{Span}(\mathbb{F}_G^{\mathsf{free}}) \subseteq \mathrm{Span}(G)$ (recall that $BG \subseteq \mathbb{F}_G^{\mathsf{free}}, * \mapsto G/e$ exhibits the latter as free finite-coproduct completion of the former). For ease of notation, we write $\mathsf{Mack}_G \coloneqq \mathsf{Mack}_{\mathsf{Orb}_G}$. We begin by recalling an equivalence $\mathsf{CMon}(\mathcal{E})^{BG} \simeq \mathsf{Mack}_{BG}(\mathcal{E})$ for $\mathcal{E} \in \mathsf{Cat}^{\times}$, which we use to define the G-equivariant Borelification $\mathrm{Bor}_G : \mathsf{CMon}(\mathcal{E})^{BG} \simeq \mathsf{Mack}_{BG}(\mathcal{E}) \xhookrightarrow{i_*} \mathsf{Mack}_G(\mathcal{E})$. The aim of this section is then to determine precisely the functor which induces the equivalence of the following theorem[14], which is also part of a section about G-equivariant Borel theory done completely internal to G-equivariant higher category theory.

Theorem 3.10 ([Hil24, Theorem 2.4.10])**.** For $\mathcal{C} \in \mathsf{CMon}(\mathsf{Cat})^{BG}$ there is a natural equivalence

$$\mathrm{CAlg}_G(\mathrm{Bor}_G(\mathcal{C})) \simeq \mathrm{CAlg}(\mathcal{C})^{hG}.$$

Let us first recall the connection between $\mathrm{Span}(\mathbb{F}_G^{\mathsf{free}})$-Mackey functors and commutative monoids with G-action. The following theorem, originally proven in [Gla17, Theorem A.1], also appears as a special case of [Har20, Theorem 5.29]. For a good exposition, we recommend [Ram].

Theorem 3.11. The inclusion[15] $j : BG \subseteq \mathbb{F}_G^{\mathsf{free}} \to \mathrm{Span}(\mathbb{F}_G^{\mathsf{free}})$ of the object $\{G/e\}$ exhibits $\mathrm{Span}(\mathbb{F}_G^{\mathsf{free}})$ as the free semiadditive category on BG. In other words, for any semiadditive category $\mathcal{C}$, restriction along j induces an equivalence

$$j^* : \mathsf{Mack}_{BG}(\mathcal{C}) \xrightarrow{\simeq} \mathcal{C}^{BG}.$$

Note that without loss of generality we can assume $\mathcal{C} \simeq \mathsf{Mack}(\mathcal{E}) \simeq \mathsf{CMon}(\mathcal{E})$ for some category $\mathcal{E}$ with finite products. In this case, we are interested in the composite equivalence

$$\beth : \mathsf{Mack}_{BG}(\mathcal{E}) \xleftarrow[\simeq]{(\mathrm{ev}_1)_*} \mathsf{Mack}_{BG}(\mathsf{Mack}(\mathcal{E})) \xrightarrow[\simeq]{j^*} \mathsf{Mack}(\mathcal{E})^{BG}.$$

Below, we will give a different description of $\beth$ and another one for $(-)^{hG} \circ \beth$, which will be useful for studying G-commutative algebras in Borel G-symmetric monoidal categories.

Definition 3.12. If $\mathcal{E}$ admits enough limits for the right Kan extension i_* to exist, we define Bor_G as the fully faithful right adjoint

$$\mathrm{Bor}_G : \mathsf{Mack}(\mathcal{E})^{BG} \xrightarrow[\simeq]{\beth^{-1}} \mathsf{Mack}_{BG}(\mathcal{E}) \xhookrightarrow{i_*} \mathsf{Mack}_G(\mathcal{E}).$$

Following [BH17, p.107], let $\mathsf{fold} \subset \mathbb{F}_G^{\mathsf{free}}$ denote the class of maps that are finite sums of fold maps $\nabla : \coprod_n G/e \to G/e$ (where $n = 0$ is allowed). For every $X \in \mathbb{F}_G^{\mathsf{free}}$ the inclusion induces an equivalence

[14] Neither statement nor proof of this theorem immediately tell us the functor which induces the equivalence, as it is proven via a long chain of adjunctions.

[15] Note that $BG \subseteq \mathbb{F}_G^{\mathsf{free}} \to \mathrm{Span}(\mathbb{F}_G^{\mathsf{free}})$ is naturally equivalent to $BG \simeq BG^{\mathsf{op}} \subseteq (\mathbb{F}_G^{\mathsf{free}})^{\mathsf{op}} \to \mathrm{Span}(\mathbb{F}_G^{\mathsf{free}})$

$\mathsf{fold}_{/X} \simeq (\mathbb{F}_G^{\mathsf{free}})_{/X}$. Indeed, note that if gf and g are fold maps then f is also one, and hence $\mathsf{fold}_{/X} \hookrightarrow (\mathbb{F}_G^{\mathsf{free}})_{/X}$ is fully faithful. Since $\mathbb{F}_G^{\mathsf{free}} \simeq BG^{\sqcup}$, all other maps are equivalences, which gives essential surjectivity. For example, the map $g : G/e \simeq G/e$ induces an equivalence $g : g \xrightarrow{\simeq} \mathrm{id}_{G/e}$ in $(\mathbb{F}_G^{\mathsf{free}})_{/(G/e)}$. It then follows from the mapping space formula for span categories Lemma B.1 that the inclusion also induces an equivalence

$$\mathrm{Span}_{\mathsf{all},\mathsf{fold}}(\mathbb{F}_G^{\mathsf{free}}) \xrightarrow{\simeq} \mathrm{Span}(\mathbb{F}_G^{\mathsf{free}}).$$

Thus we can also consider their functor

$$\Theta : BG \times \mathrm{Span}(\mathbb{F}) \subseteq (\mathbb{F}_G^{\mathsf{free}})^{\mathsf{op}} \times \mathrm{Span}(\mathbb{F}) \to \mathrm{Span}(\mathbb{F}_G^{\mathsf{free}}),\ (*, X) \mapsto \coprod_X G/e.$$

Lemma 3.13. For $\mathcal{E} \in \mathsf{Cat}^{\times}$ we have a commutative diagram of equivalences

$$\begin{tikzcd}
\mathsf{Mack}_{BG}(\mathcal{E}) \arrow[d, "\Theta^*"'] \arrow[dr, "\sqsupset"] & \mathsf{Mack}_{BG}(\mathsf{Mack}(\mathcal{E})) \arrow[l, "(\mathrm{ev}_1)_*"'] \arrow[d, "j^*"] \\
\mathsf{Fun}'(BG \times \mathrm{Span}(\mathbb{F}), \mathcal{E}) \arrow[r, "\mathrm{curry}"'] & \mathsf{Mack}(\mathcal{E})^{BG}
\end{tikzcd}$$

Here the bottom left category is the full subcategory of $\mathsf{Fun}(BG \times \mathrm{Span}(\mathbb{F}), \mathcal{E})$ which under currying is equivalent to $\mathsf{Mack}(\mathcal{E})^{BG}$, i.e. those functors which preserve finite products in the second variable. Moreover, the following diagram commutes if $\mathcal{E}$ admits enough limits for i_* and $(-)^{hG}$ to exist:

$$\begin{tikzcd}
\mathsf{Mack}(\mathcal{E}) \arrow[r, "p^*"] \arrow[dr, "\mathrm{infl}_G"'] & \mathsf{Mack}_{BG}(\mathcal{E}) \arrow[r, hook, "i_*"] \arrow[d, "\simeq", "\sqsupset"'] & \mathsf{Mack}_G(\mathcal{E}) \arrow[d, "s^*"] \\
 & \mathsf{Mack}(\mathcal{E})^{BG} \arrow[r, "(-)^{hG}"'] \arrow[ur, "\mathrm{Bor}_G"] & \mathsf{Mack}(\mathcal{E})
\end{tikzcd}$$

Here $i : \mathrm{Span}(\mathbb{F}_G^{\mathsf{free}}) \hookrightarrow \mathrm{Span}(G)$ is the fully faithful inclusion, $p : \mathrm{Span}(\mathbb{F}_G^{\mathsf{free}}) \to \mathrm{Span}(\mathbb{F})$ is induced by $BG \to *$ and $s : \mathrm{Span}(\mathbb{F}) \to \mathrm{Span}(G)$ by the inclusion of G/G.

Proof. For the first diagram, note that Θ^* is an equivalence by [BH17, Lemma C.4] and the fact that we have an equivalence $\mathrm{Span}_{\mathsf{all},\mathsf{fold}}(\mathbb{F}_G^{\mathsf{free}}) \xrightarrow{\simeq} \mathrm{Span}(\mathbb{F}_G^{\mathsf{free}})$. Now the upper triangle commutes by definition of $\sqsupset$, and so it suffices to check the outer square commutes. To this end, note that if

$(\mathrm{id}, 1) : BG \to BG \times \mathrm{Span}(\mathbb{F})$, then $\Theta \circ (\mathrm{id}, 1) = j$, so that we have a commutative diagram

$$\begin{tikzcd}
\mathsf{Mack}(\mathcal{E})^{BG} \arrow[d, equal] & \mathsf{Mack}_{BG}(\mathsf{Mack}(\mathcal{E})) \arrow[l, "j^*"'] \arrow[r, "(\mathrm{ev}_1)_*"] \arrow[d, "\Theta^*"] & \mathsf{Mack}_{BG}(\mathcal{E}) \arrow[d, "\Theta^*"] \\
\mathsf{Mack}(\mathcal{E})^{BG} \arrow[d, equal] & \mathsf{Fun}'(BG \times \mathrm{Span}(\mathbb{F}), \mathsf{Mack}(\mathcal{E})) \arrow[l, "(\mathrm{id},1)^*"'] \arrow[r, "(\mathrm{ev}_1)_*"'] \arrow[d, "\mathrm{curry}"] & \mathsf{Fun}'(BG \times \mathrm{Span}(\mathbb{F}), \mathcal{E}) \arrow[d, "\mathrm{curry}"] \\
\mathsf{Mack}(\mathcal{E})^{BG} & \mathsf{Mack}(\mathsf{Mack}(\mathcal{E}))^{BG} \arrow[l, "(\mathrm{ev}_1)_*"] \arrow[r, "((\mathrm{ev}_1)_*)_*"'] & \mathsf{Mack}(\mathcal{E})^{BG}
\end{tikzcd}$$

The claim then follows from noting that the bottom horizontal composite is the identity. In other words, ev_1 and $(\mathrm{ev}_1)_*$ are naturally equivalent functors $\mathsf{Mack}(\mathsf{Mack}(\mathcal{E})) \to \mathsf{Mack}(\mathcal{E})$. As ev_1 corresponds to the forgetful functor $U : \mathsf{CMon}(\mathcal{E}) \to \mathcal{E}$ under the natural equivalence $\mathsf{Mack}(\mathcal{E}) \simeq \mathsf{CMon}(\mathcal{E})$, hence follows form the fact of it being a right Bousfield localization $\mathsf{Cat}^{\times} \to \mathsf{Cat}^{\oplus}$.

In the second diagram, commutativity of the left triangle follows from $pj : BG \to *$ hence $j^*p^* = \mathrm{infl}_G$ and the right top triangle is simply the definition of Bor_G, so it suffices to show the square commutes. This follows from commutativity of the following diagram:

$$\begin{tikzcd}
\mathsf{Fun}(BG, \mathsf{Mack}(\mathcal{E})) \arrow[d, hook, "b_*"'] & \mathsf{Mack}_{\mathbb{F}_G^{\mathsf{free}}}(\mathsf{Mack}(\mathcal{E})) \arrow[l, "j^*"'] \arrow[r, "(\mathrm{ev}_1)_*"] \arrow[d, hook, "i_*"'] & \mathsf{Mack}_{\mathbb{F}_G^{\mathsf{free}}}(\mathcal{E}) \arrow[d, hook, "i_*"] \\
\mathsf{Fun}(\mathsf{Orb}_G^{\mathsf{op}}, \mathsf{Mack}(\mathcal{E})) \arrow[d, "\mathrm{ev}_{G/G}"'] & \mathsf{Mack}_G(\mathsf{Mack}(\mathcal{E})) \arrow[l, "k^*"'] \arrow[r, "(\mathrm{ev}_1)_*"] \arrow[d, "s^*"'] & \mathsf{Mack}_G(\mathcal{E}) \arrow[d, "s^*"] \\
\mathsf{Mack}(\mathcal{E}) & \mathsf{Mack}(\mathsf{Mack}(\mathcal{E})) \arrow[l, "\mathrm{ev}_1"'] \arrow[r, "(\mathrm{ev}_1)_*"] & \mathsf{Mack}(\mathcal{E})
\end{tikzcd}$$

Here, $b : BG \hookrightarrow \mathsf{Orb}_G^{\mathsf{op}}$ is the fully faithful inclusion of G/e, and $k : \mathsf{Orb}_G^{\mathsf{op}} \subseteq \mathbb{F}_G^{\mathsf{op}} \to \mathrm{Span}(G)$ again the inclusion. Note that by the limit formula for right Kan extensions the left vertical composite is $(-)^{hG}$, and we can identify the bottom horizontal composite equivalence with the identity on $\mathsf{Mack}(\mathcal{E})$, as we did above. All little squares except the top left one obviously commute, and the remaining one does by Proposition 3.6. $\square$

Remark 3.14. Commutativity of the right square in the second diagram tells us in particular that if $X \in \mathsf{Mack}_{BG}(\mathcal{E})$ is a commutative monoid with G-action, then $\mathrm{Bor}_G(X)(G/G) \simeq \mathrm{Bor}_G(X)(G/e)^{hG}$, where the homotopy fixed points are taken with respect to the G-action induced by functoriality from the canonical G-action on G/e.

Theorem 3.15. Let G be a finite group and $s : \mathrm{Span}(\mathbb{F}) \to \mathrm{Span}(G)$ be induced by $* \mapsto G/G$. Restriction along s induces an equivalence natural in $\mathcal{C} \in \mathsf{Mack}(\mathsf{Cat})^{BG}$:

$$\mathrm{ev}_{G/G} = s^* : \mathrm{CAlg}_G(\mathrm{Bor}_G(\mathcal{C})) \xrightarrow{\simeq} \mathrm{CAlg}(\mathcal{C}^{hG}).$$

Proof. Again Theorem 2.28 provides the envelope $\dashv$ straightening adjunctions for the algebraic patterns $\mathrm{Span}(G)$ and $\mathrm{Span}(\mathbb{F})$. Moreover, s is a strong Segal morphism by Lemma 2.16. Since Orb_G is atomic orbital, we know from Lemma 2.29 that $\mathcal{A}_G \simeq \underline{\mathbb{F}}_G^{\sqcup}$ is the G-cocartesian G-symmetric monoidal G-category of finite G-sets defined in Example 2.25, and $s^*\mathcal{A}_G \simeq \mathbb{F}_G^{\sqcup} \in \mathsf{Mack}(\mathsf{Cat})$ encodes the cocartesian symmetric monoidal structure on the category of finite G-sets. We also claim that $\mathcal{A}_G \simeq i_*\mathcal{A}_{BG}$. Indeed, note that by Corollary 3.9 we have $i^*\mathcal{A}_G \simeq \mathcal{A}_{BG}$, and then by the addendum of Proposition 3.6 it suffices to show that $\ell^*\mathcal{A}_G \simeq \underline{\mathbb{F}}_G \in \mathsf{Cat}_G$ is right Kan extended along $BG \to \mathsf{Orb}_G^{\mathsf{op}}$, where $\ell : \mathsf{Orb}_G^{\mathsf{op}} \subseteq \mathbb{F}_G^{\mathsf{op}} \to \mathrm{Span}(G)$. But clearly the G-category of finite G-sets $\underline{\mathbb{F}}_G \simeq \mathsf{Fun}(B(-), \mathbb{F}) \simeq \mathrm{Bor}_G(\mathrm{infl}_G \mathbb{F})$ is right Kan extended in this way from $\mathrm{infl}_G \mathbb{F}$. It follows that in the commutative diagram

$$\begin{tikzcd}
\mathrm{Ar}_{\mathsf{act}}(\mathrm{Span}(\mathbb{F})) \arrow[r, dashed, "\exists!"] \arrow[rr, bend left, "\mathrm{Ar}_{\mathsf{act}}(s)"] \arrow[dr, "t"'] & \mathrm{Ar}_{\mathbb{F}_G \to \mathbb{F}}(\mathrm{Span}(G)) \arrow[r] \arrow[d, "t"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Ar}_{\mathsf{act}}(\mathrm{Span}(G)) \arrow[d, "t"] \\
& \mathrm{Span}(\mathbb{F}) \arrow[r, "s"] & \mathrm{Span}(G)
\end{tikzcd}$$

the pulled back fibration is the cocartesian unstraightening of $\mathbb{F}_G^{\sqcup}$, and the dashed morphism is the unstraightening of the symmetric monoidal inclusions $\mathbb{F}^{\sqcup} \subseteq \mathbb{F}_G^{\sqcup}$. Moreover, it is clear that the pullback of the identity section $\mathrm{id}_{(-)} : \mathrm{Span}(G) \to \mathrm{Ar}_{\mathsf{act}}(\mathrm{Span}(G))$ along s factors as the identity section $\mathrm{id}_{(-)} : \mathrm{Span}(\mathbb{F}) \to \mathrm{Ar}_{\mathsf{act}}(\mathrm{Span}(\mathbb{F}))$ followed by the dashed morphism. Overall, we obtain a commutative diagram

$$\begin{tikzcd}
\mathsf{Fun}_G^{\otimes}(\mathcal{A}_G, \mathrm{Bor}_G(\mathcal{C})) \arrow[r, "\int"] \arrow[d, "s^*"'] \arrow[rr, bend left, "\simeq"] & \mathsf{Fun}_{\mathsf{Fbrs}_G}(\mathrm{Ar}_{\mathsf{act}}(\mathrm{Span}(G)), \int \mathrm{Bor}_G(\mathcal{C})) \arrow[r, "\mathrm{id}_{(-)}^*"] \arrow[d, "s^*"] & \mathrm{CAlg}_G(\mathrm{Bor}_G(\mathcal{C})) \arrow[dd, "s^*"] \\
\mathsf{Fun}^{\otimes}(\mathbb{F}_G^{\sqcup}, \mathcal{C}^{hG}) \arrow[r, "\int"] \arrow[d, "(\mathbb{F}^{\sqcup} \to \mathbb{F}_G^{\sqcup})^*"'] & \mathsf{Fun}_{\mathsf{Fbrs}(\mathrm{Span}(\mathbb{F}))}(\mathrm{Ar}_{\mathbb{F}_G \to \mathbb{F}}(\mathrm{Span}(G)), \int \mathcal{C}^{hG}) \arrow[d, "(\int(\mathbb{F}^{\sqcup} \to \mathbb{F}_G^{\sqcup}))^*"] & \\
\mathsf{Fun}^{\otimes}(F^{\sqcup}, \mathcal{C}^{hG}) \arrow[r, "\int"] \arrow[rr, bend right, "\simeq"'] & \mathsf{Fun}_{\mathsf{Fbrs}(\mathrm{Span}(\mathbb{F}))}(\mathrm{Ar}_{\mathsf{act}}(\mathrm{Span}(\mathbb{F}), \int \mathcal{C}^{hG}) \arrow[r, "\mathrm{id}_{(-)}^*"] & \mathrm{CAlg}(\mathcal{C}^{hG})
\end{tikzcd}$$

Finally, by Lemma 3.13 and Lemma D.6 the following diagram commutes

$$\begin{tikzcd}
\mathsf{Fun}^{\otimes}_{G}(i^*\mathcal{A}_G, \beth^{-1}\mathcal{C}) \arrow[r, "i_*", "\simeq"'] \arrow[d, "\beth"', "\simeq"] & \mathsf{Fun}^{\otimes}_{G}(\mathcal{A}_G, \mathrm{Bor}_G(\mathcal{C})) \arrow[d, "s^*"] \\
\mathsf{Fun}^{\otimes}_{BG}(\mathrm{infl}_G \mathbb{F}^{\sqcup}, \mathcal{C}) \arrow[r, "(-)^{hG}"] \arrow[dr, "\simeq"'] & \mathsf{Fun}^{\otimes}(\mathbb{F}^{\sqcup}_G, \mathcal{C}^{hG}) \arrow[d, "(\mathbb{F}^{\sqcup} \to \mathbb{F}^{\sqcup}_G)^*"] \\
 & \mathsf{Fun}^{\otimes}(\mathbb{F}^{\sqcup}, \mathcal{C}^{hG})
\end{tikzcd}$$

where $\mathsf{Fun}^{\otimes}_{BG} := \mathrm{Nat}_{\mathsf{Mack}(\mathsf{Cat})^{BG}}$ (see Remark D.5) analogously to $\mathsf{Fun}^{\otimes}_{G} = \mathrm{Nat}_{\mathrm{Span}(G)}$ or $\mathsf{Fun}^{\otimes} = \mathrm{Nat}_{\mathrm{Span}(\mathbb{F})}$ from Definition 2.20. The diagonal arrow is an equivalence as it is precisely the adjunction equivalence; $\mathbb{F}^{\sqcup} \to \mathsf{Fun}(BG, \mathbb{F}^{\sqcup}) \simeq \mathbb{F}^{\sqcup}_G$ is the unit for $\mathrm{infl}_G \dashv (-)^{hG}$ at $\mathbb{F}^{\sqcup}$. Thus the left and hence the right vertical morphism in the large diagram above are equivalences, as desired. $\square$

Warning 3.16. Although s is induced by the fully faithful inclusion $\{G/G\} \subseteq \mathsf{Orb}_G$, and s induces the above equivalence, it is *not* the case that $(\{G/G\}, \{G/G\}) \subseteq (\mathsf{Orb}_G, \mathsf{Orb}_G)$ is a Borel-inclusion.

3.3 Global Borel Theory

In this short subsection we collect some lemmas on the Borel-inclusion $i : (*, *) \subseteq (\mathsf{Glo}, \mathsf{Orb})$. As in the equivariant case, this will be important in Section 4 to construct a comparison functor from the 1-category of ultracommutative orthogonal ring spectra to the ∞-category of equivariantly commutative global algebras in an equivariantly symmetric monoidal global category of equivariant or global spectra.

The defining property of Borel-equivariant objects is that equivalences can be checked after forgetting the action. This leads to some interesting consequences for sections of "Borelified" (monoidal) categories. For a category $\mathcal{E}$ with enough limits for the following to make sense, we define

$$\mathrm{Bor}^{\mathsf{Orb}}_{\mathsf{Glo}} := i_* : \mathsf{Mack}(\mathcal{E}) \to \mathsf{Mack}^{\mathsf{Orb}}_{\mathsf{Glo}}(\mathcal{E}).$$

Let us also denote $\mathrm{Bor}_{\mathsf{Glo}} : \mathsf{Cat} \to \mathsf{Cat}_{\mathsf{Glo}}$ the right Kan extension along $i : * \to \mathsf{Glo}^{\mathsf{op}}$, sending a category $\mathcal{C}$ to the global category $G \mapsto \mathcal{C}^{BG}$.

Lemma 3.17. Let $\mathcal{C}$ be a category and $s : \mathsf{Glo}^{\mathsf{op}} \to \int \mathrm{Bor}_{\mathsf{Glo}}(\mathcal{C})$ a section of the cocartesian unstraightening of $\mathrm{Bor}_{\mathsf{Glo}}(\mathcal{C})$. If s is cocartesian on $\mathsf{Orb}^{\mathsf{op}}$, then it is cocartesian on all of $\mathsf{Glo}^{\mathsf{op}}$.

Proof. By right-cancellability of cocartesian morphisms (see Kerodon 01TS), it suffices to show that s is cocartesian on all morphisms $e \leftarrow G$. Note that the inclusion of the unit $G \leftarrow e$ induces the conservative functor $\mathrm{res}^G_e : \mathcal{C}^{BG} \to \mathcal{C}$. It thus follows from Lemma D.1 we can check whether $s(e \leftarrow G) : s(e) \to s(G)$ is cocartesian after postcomposing with the cocartesian map $s(G) \to \mathrm{res}^G_e s(G)$. But by assumption this second map is given by $s(G \leftarrow e) : s(G) \to s(e)$, so the whole composite is simply $s(e \leftarrow G \leftarrow e) = \mathrm{id}_{s(e)}$, which is clearly cocartesian. □

Lemma 3.18. Let $\mathcal{C} \in \mathsf{Mack}(\mathsf{Cat})$ be a symmetric monoidal category and $s \in \mathrm{CAlg}^{\mathsf{Orb}}_{\mathsf{Glo}}(\mathrm{Bor}^{\mathsf{Orb}}_{\mathsf{Glo}}(\mathcal{C}))$ an algebra in its global Borelification. If s is cocartesian on $\mathbb{F}^{\mathsf{op}}_{\mathsf{Orb}}$, then it is cocartesian on all of $\mathbb{F}^{\mathsf{op}}_{\mathsf{Glo}}$, i.e. on all backwards morphisms.

Proof. Restricting along $\mathsf{Glo}^{\mathsf{op}} \to \mathrm{Span}_{\mathsf{all},\mathsf{Orb}}(\mathbb{F}_{\mathsf{Glo}})$, it follows from the previous proposition that s is cocartesian on $\mathsf{Glo}^{\mathsf{op}}$. Now note that $\mathrm{Bor}^{\mathsf{Orb}}_{\mathsf{Glo}}(\mathcal{C})$ preserves finite products and the projections in $\mathrm{Span}_{\mathsf{all},\mathsf{Orb}}(\mathbb{F}_{\mathsf{Glo}})$ are given by backwards summand inclusions, which are contained in $(\mathbb{F}^{\simeq}_{\mathsf{Glo}})^{\mathsf{op}} \subset (\mathbb{F}^{\mathsf{Orb}}_{\mathsf{Glo}})^{\mathsf{op}}$. So s is cocartesian on $\mathsf{Glo}^{\mathsf{op}}$ and all projections, so by Corollary D.2 also cocartesian on $\mathbb{F}^{\mathsf{op}}_{\mathsf{Glo}}$. □

The following Proposition and Corollary tell us how to relate the Borelifications of the equivariant, global and G-global worlds.

Proposition 3.19. Let G be a finite group and $\mathcal{C}$ a category admitting enough limits for the following right Kan extensions to exist. Then both of the following squares (and hence the rectangle) commute via the Beck-Chevalley transformations

$$\begin{tikzcd}
\mathsf{Mack}_{BG}(\mathcal{C}) \arrow[r, equal] \arrow[d, hook, "K_*"'] & \mathsf{Mack}_{BG}(\mathcal{C}) \arrow[d, hook, "I_*"'] & \mathsf{Mack}(\mathcal{C}) \arrow[l, "P^*"'] \arrow[d, hook, "J_*"] \\
\mathsf{Mack}_G(\mathcal{C}) & \mathsf{Mack}^{\pi_G^{-1}(\mathsf{Orb})}_{\mathsf{Glo}_{/G}}(\mathcal{C}) \arrow[l, "L^*"] & \mathsf{Mack}^{\mathsf{Orb}}_{\mathsf{Glo}}(\mathcal{C}) \arrow[l, "\Pi_G^*"]
\end{tikzcd}$$

Here $I = LK$ and K, L and J are induced by corresponding Borel-inclusions from Example 3.4. Moreover $P : \mathrm{Span}(\mathbb{F}^{\mathsf{free}}_G) \to \mathrm{Span}(\mathbb{F})$ respectively Π_G are induced by the functors $BG \to *$ respectively $\pi_G : \mathsf{Glo}_{/G} \to \mathsf{Glo}$.

Proof. Recall that $\mathrm{Span} : \mathsf{AdTrip} \to \mathsf{Cat}$ is a right adjoint, and limits in AdTrip are computed pointwise in Cat (cf. [HHLN23b, Lemma 2.4]). Thus we have cartesian squares

$$\begin{tikzcd}
\mathrm{Span}(\mathbb{F}^{\mathsf{free}}_G) \arrow[r, equal] \arrow[d, hook, "K"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Span}(\mathbb{F}^{\mathsf{free}}_G) \arrow[r, "P"] \arrow[d, hook, "I"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Span}(\mathbb{F}) \arrow[d, hook, "J"] \\
\mathrm{Span}(G) \arrow[r, hook, "L"] \arrow[rr, bend right, "Q"'] & \mathrm{Span}_{\mathsf{all},\pi_G^{-1}(\mathsf{Orb})}(\mathbb{F}_{\mathsf{Glo}_{/G}}) \arrow[r, "\Pi_G"] & \mathrm{Span}_{\mathsf{all},\mathsf{Orb}}(\mathbb{F}_{\mathsf{Glo}})
\end{tikzcd}$$

Since Beck-Chevalley transformations compose by Lemma C.1 it suffices to show that both little squares induce an invertible Beck-Chevalley transformation. For the left square, this follows from Lemma C.4.

For the right square, note that forgetting the algebra structure, i.e. restricting along the inclusion of backwards morphisms induces a commutative cube

$$\begin{tikzcd}
& \mathsf{Mack}_{BG}(\mathcal{C}) \arrow[dl] & & \mathsf{Mack}(\mathcal{C}) \arrow[ll, "P^*"'] \arrow[dl] \\
\mathsf{Fun}(BG, \mathcal{C}) & & \mathcal{C} \arrow[ll, "p^*"'] & \\
& \mathsf{Mack}^{\pi_G^{-1}(\mathsf{Orb})}_{\mathsf{Glo}_{/G}}(\mathcal{C}) \arrow[uu, "I^*"] \arrow[dl] & & \mathsf{Mack}^{\mathsf{Orb}}_{\mathsf{Glo}}(\mathcal{C}) \arrow[uu, "J^*"'] \arrow[ll, "\Pi_G^*"'] \arrow[dl] \\
\mathsf{Fun}((\mathsf{Glo}_{/G})^{\mathsf{op}}, \mathcal{C}) \arrow[uu, "i^*"] & & \mathsf{Fun}(\mathsf{Glo}^{\mathsf{op}}, \mathcal{C}) \arrow[ll, "\pi_G^*"'] \arrow[uu, "j^*"] &
\end{tikzcd}$$

Since Beck-Chevalley transformations compose by Lemma C.1 and the left and right faces are right-adjointable by Proposition 3.6, it suffices to prove that the front face is right-adjointable. But this is clear from Lemma C.2 applied to the cartesian square

$$\begin{tikzcd}
BG \simeq \mathsf{Glo}(e, G) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner"] & * \arrow[d, hook] \\
(\mathsf{Glo}_{/G})^{\mathsf{op}} \arrow[r] & \mathsf{Glo}^{\mathsf{op}}
\end{tikzcd}$$

as for every $\alpha : K \to G$ in $\mathsf{Glo}_{/G}$, the map in question is pulled back from the canonical equivalence

$$(\mathsf{Glo}_{/G})^{\mathsf{op}}_{\alpha/} \simeq ((\mathsf{Glo}_{/G})_{/\alpha})^{\mathsf{op}} \xrightarrow{\simeq} (\mathsf{Glo}_{/K})^{\mathsf{op}} \simeq \mathsf{Glo}^{\mathsf{op}}_{K/}.$$

□

Remark 3.20. Given a symmetric monoidal category $\mathcal{C}$, we thus obtain a canonical equivalence $\mathrm{Bor}_G(\mathrm{infl}_G\,\mathcal{C}) \simeq Q^* \,\mathrm{Bor}_{\mathsf{Glo}}(\mathcal{C})$. The right hand side is how the Borel G-symmetric monoidal category on a symmetric monoidal category was defined in [EH21, 3.4.17].

4 Construction of Main Examples

In this section we construct the main examples of parametrized monoidal structures on spectra. Specifically, we will define the equivariantly symmetric monoidal global categories of equivariant respectively global spectra $\underline{\mathsf{Sp}}^{\otimes}, \underline{\mathsf{Sp}}^{\otimes}_{\mathsf{Glo}} \in \mathsf{Mack}^{\mathsf{Orb}}_{\mathsf{Glo}}(\widehat{\mathsf{Cat}}(\mathsf{sift}))$, where $\underline{\mathsf{Sp}}^{\otimes}$ and $\underline{\mathsf{Sp}}^{\otimes}_{\mathsf{Glo}}$ encode restrictions, multiplicative norms and symmetric monoidal structures of G-equivariant and G-global spectra respectively. The way we approach this is to first construct a parametrized monoidal structure on the relevant model categories, and then derive / pointwise localize this structure.

We start in Section 4.1 by recalling the relevant background on Dwyer-Kan localization in families, and then focus on the actual construction of examples in Section 4.2.

4.1 Dwyer-Kan Localization in Families

In this subsection we being by recalling the functorial Dwyer-Kan localization $\mathsf{DK} : \mathsf{Cat}^{\dagger} \to \mathsf{Cat}$ of marked categories. We will need more precise technology for our examples, and recall a weaker but sufficient version of the left-derivable cocartesian fibrations developed in [NS18, Appendix A].

Definition 4.1. A marked category is a category $\mathcal{C}$ equipped with a collection of edges $W \subseteq \mathrm{Ar}(\mathcal{C})$ that is stable under homotopy, composition and contains all equivalences. A functor of marked categories is marked if it preserves the marking. Let $\mathsf{Fun}^{\dagger}(\mathcal{C}, \mathcal{D}) \subseteq \mathsf{Fun}(\mathcal{C}, \mathcal{D})$ be full on the marked functors. In [Lur17, 4.1.7.1] Lurie constructs the category $\mathsf{Cat}^{\dagger}$ of marked categories with mapping categories $\mathsf{Fun}^{\dagger}$. There is a canonical fully faithful functor $(-)^{\flat} : \mathsf{Cat} \to \mathsf{Cat}^{\dagger}$ equipping a category with the minimal marking containing all equivalences.

Equivalently, we can think of a marked category $(\mathcal{C}, W)$ as the inclusion of a wide subcategory $W \subset \mathcal{C}$, of $\mathsf{Cat}^{\dagger}$ as the full subcategory on said inclusions, and of $(-)^{\flat}$ as $\mathcal{C} \mapsto (\mathcal{C}^{\simeq} \subset \mathcal{C})$.

Proposition 4.2 ([Lur17, 4.1.7.2])**.** The functor $(-)^{\flat} : \mathsf{Cat} \hookrightarrow \mathsf{Cat}^{\dagger}$ admits a left adjoint $\mathsf{DK} : \mathsf{Cat}^{\dagger} \to \mathsf{Cat}$ which preserves finite products. For any marked category $(\mathcal{C}, W)$, the unit transformation $\eta_{(\mathcal{C}, W)} : (\mathcal{C}, W) \to \mathsf{DK}(\mathcal{C}, W)^{\flat}$ exhibits $\mathsf{DK}(\mathcal{C}, W)$ as Dwyer-Kan localization of $\mathcal{C}$ at W.

Often, our functors don't preserve the marking strictly, but can still be derived. The canonical example of this is that of a left Quillen functor of model categories $F : \mathcal{M} \to \mathcal{N}$. Here F will generally not preserve the weak equivalences, but nevertheless there exists a left derived functor $\mathbb{L}F : \mathbf{Ho}(\mathcal{M}) \to \mathbf{Ho}(\mathcal{N})$ on the localizations. Analogously to this 1-categorical case, the left Quillen functor derives to a left adjoint functor of the underlying ∞-categories (given by first taking the nerve and then Dwyer-Kan localizing at the weak equivalences), by [Hin, Proposition 1.5.1]. Both in the 1-categorical as in the ∞-categorical case the left derived functor is given by an (absolute) right Kan extension of $\gamma_{\mathcal{N}} \circ F$ along $\gamma_{\mathcal{M}}$, where γ denotes either the 1- or ∞-categorical Dwyer-Kan localization at the weak equivalences, compare [NS18, Example A.10]. A general theory of such left-derivable cocartesian fibrations has been developed in [NS18, Appendix A]. The setting is that we are given a functor $F : \mathcal{C} \to \mathsf{Cat}$ and a marking on each category Fc, although Ff need not preserve the marking for a morphism f in $\mathcal{C}$. Given that F sends each morphism to a left-derivable functor, and these left-derivations are compatible under composition, the usual Dwyer-Kan localization of the total category $\int F$ of the cocartesian unstraightening of F at the fiberwise marking is then again a cocartesian fibration classified by a functor

$$\mathcal{C} \to \mathsf{Cat}, \ (f : c \to d) \mapsto (\mathbf{L}(Ff) : Fc[W_c^{-1}] \to Fd[W_d^{-1}]).$$

All our examples come from model categories, where it is much simpler to construct the left derivation of a left Quillen functor. So instead of the general left-derivable cocartesian fibrations of [NS18, Appendix A], we work with the notion of left-deformable cocartesian fibration introduced below. While these require one to provide a lot more data, namely a left-deformation on each category Fc, they make it a lot easier to verify that all the examples we are interested in are left-derivable.

Definition 4.3. A left-deformation on a marked category $\mathcal{C}$ is an endofunctor $Q : \mathcal{C} \to \mathcal{C}$ together with a pointwise marked natural transformation $q : Q \Rightarrow \mathrm{id}_{\mathcal{C}}$. Denote by $\mathcal{C}_Q \subseteq \mathcal{C}$ the essential image of Q. If $\mathcal{D}$ is another such category with left-deformation (P, p), then a functor $F : \mathcal{C} \to \mathcal{D}$ is compatible with the given left-deformations if it restricts to a marked functor $\mathcal{C}_Q \to \mathcal{D}_P$.

Lemma 4.4 ([NS18, Example A.10])**.** Let $\mathcal{C}, \mathcal{D}$ be marked categories equipped with left-deformations (P, p) and (Q, q). Suppose that $F : \mathcal{C} \to \mathcal{D}$ is compatible with the deformations. Then the transformation

$$\mathsf{DK}(FP)\gamma_{\mathcal{C}} \simeq \gamma_{\mathcal{D}} FP \xRightarrow{\gamma_{\mathcal{D}} Fp} \gamma_{\mathcal{D}} F$$

exhibits $\mathsf{DK}(FP) : \mathcal{C}[W_{\mathcal{C}}^{-1}] \to \mathcal{D}[W_{\mathcal{D}}^{-1}]$ as the left-derivation $\mathbf{L}F$ of F, i.e. as an absolute right Kan extension of $\gamma_{\mathcal{D}} F$ along $\gamma_{\mathcal{C}}$.

Definition 4.5. Let $p : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration and suppose that each fiber $\mathcal{E}_c$ is equipped with a marking and a left-deformation (Q_c, q_c) with essential image $\mathcal{E}_c^Q \coloneqq \mathrm{Im}\, Q_c \subseteq \mathcal{E}_c$. Then p is left-deformable if for every morphism f in $\mathcal{C}$, the pushforward $f_!$ is compatible with the left-deformations.

Note that if $K \to \mathcal{C}$ is any functor then $\mathcal{E} \times_{\mathcal{C}} K \to K$ canonically inherits the structure of a left-deformable cocartesian fibration.

Theorem 4.6. Let $p : \mathcal{E} \to \mathcal{C}$ be a left deformable cocartesian fibration.

1. By the universal property of Dwyer-Kan localization, we obtain a commuting triangle

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "\gamma"] \arrow[dr, "p"'] & & \mathcal{E}[W^{-1}] \arrow[dl, "q"] \\ & \mathcal{C} & \end{tikzcd}$$

   Then q is also a cocartesian fibration and γ preserves cocartesian lifts of morphisms $f : c \to d$ whose pushforward $f_! : \mathcal{E}_c \to \mathcal{E}_d$ preserves the marked edges.

2. The above triangle is stable under pullback along functors $K \to \mathcal{C}$. Specifically, for any such functor, the morphism $\mathcal{E} \times_{\mathcal{C}} K \to \mathcal{E}[W^{-1}] \times_{\mathcal{C}} K$ exhibits the latter as Dwyer-Kan localization of $\mathcal{E} \times_{\mathcal{C}} K$ at the edges mapped into W. In particular, we can identify $\gamma_c : \mathcal{E}_c \to \mathcal{E}[W^{-1}]_c$ with the Dwyer-Kan localization $\mathcal{E}_c \to \mathcal{E}_c[W_c^{-1}]$.

3. For every morphism $f : c \to d$ in $\mathcal{C}$, the associated diagram

$$\begin{tikzcd} \mathcal{E}_c \arrow[r, "f_!^p"] \arrow[d, "\gamma_c"'] & \mathcal{E}_d \arrow[d, "\gamma_d"] \\ \mathcal{E}[W^{-1}]_c \arrow[r, "f_!^q"'] \arrow[ur, Rightarrow] & \mathcal{E}[W^{-1}]_d \end{tikzcd}$$

exhibits $f_!^q$ as the left derived functor $\mathbf{L}f_!^p = \mathsf{DK}(f_!^p Q_c)$ of $f_!^p$, i.e. as the absolute right Kan extension of $\gamma_d f_!^p$ along γ_c. The 2-cell is the natural transformation obtained from the commuting triangle above, and it is invertible if $f_!^p$ preserves marked edges.

Proof. By [NS18, Theorem A.14, A.15], it suffices to verify that a left-deformable cocartesian fibration is left-derivable in the sense of [NS18, Definition A.8]. Being left-derivable is a local property, and can be checked on all 2-simplices of $\mathcal{C}$ separately, so that we may assume $\mathcal{C} = \Delta^2$, with p classified by a diagram

$$\begin{tikzcd} & \mathcal{N} \arrow[dr, "G"] & \\ \mathcal{M} \arrow[ur, "F"] \arrow[rr, "H"'] & & \mathcal{O} \end{tikzcd}$$

Let (P, p) and (Q, q) be the given left-deformations on $\mathcal{M}$ and $\mathcal{N}$. By assumption and Lemma 4.4 F, G, H are left-derivable with $\mathbf{L}F \simeq \mathsf{DK}(FP)$, $\mathbf{L}G \simeq \mathsf{DK}(GQ)$ and $\mathbf{L}H \simeq \mathsf{DK}(HP)$. It remains to check compatibility of these left derivations with respect to composition. The canonical morphism $\mathbf{L}G \circ \mathbf{L}F \Rightarrow \mathbf{L}H$ is then identified with the following equivalence, as desired:

$$\mathsf{DK}(GQ) \circ \mathsf{DK}(FP) \simeq \mathsf{DK}(GQFP) \xRightarrow[\simeq]{\mathsf{DK}(GqFP)} \mathsf{DK}(GFP) \overset{\simeq}{\Longrightarrow} \mathsf{DK}(HP).$$

This uses that FP has image in $\mathcal{N}_Q$ and thus $GqFP$ is pointwise marked. $\square$

Proposition 4.7. Let $p^0 : \mathcal{E}^0 \to \mathcal{C}$ and $p^1 : \mathcal{E}^1 \to \mathcal{C}$ be left-deformable cocartesian fibrations and $f : a \to b$ a morphism in $\mathcal{C}$. Let $\alpha : \mathcal{E}^0 \to \mathcal{E}^1$ be a marked functor over $\mathcal{C}$ which preserves cocartesian lifts of f. Suppose that one of the following two conditions holds:

1. The pushforward $f_!^i : \mathcal{E}_a^i \to \mathcal{E}_b^i$ preserves the marking for $i = 0, 1$.[16]

2. $\alpha|_{\mathcal{E}_a^0} : \mathcal{E}_a^0 \to \mathcal{E}_a^1$ and $\alpha|_{\mathcal{E}_b^0} : \mathcal{E}_b^0 \to \mathcal{E}_b^1$ are compatible with the deformations.

Then $\mathsf{DK}(\alpha) : \mathsf{DK}(\mathcal{E}^0) \to \mathsf{DK}(\mathcal{E}^1)$ also preserves cocartesian lifts of f. In particular, if α preserves all cocartesian edges and $\alpha|_{\mathcal{E}_c^0} : \mathcal{E}_c^0 \to \mathcal{E}_c^1$ is compatible with the deformations for all $c \in \mathcal{C}$, then $\mathsf{DK}(\alpha)$ preserves all cocartesian edges.

[16] This case also works in the generality of left-derivable fibrations, with the same proof.

Proof. We will first show (1) and then the addendum. In view of Theorem 4.6(2), point (2) then follows by applying the addendum to the situation obtained by pulling back along $\Delta^1 \xrightarrow{f} \mathcal{C}$.

1. Since α preserves the marking, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{E}^0 \arrow[rr, "\alpha"] \arrow[dr, "p^0"] \arrow[dd, "\gamma^0"'] & & \mathcal{E}^1 \arrow[dl, "p^1"'] \arrow[dd, "\gamma^1"] \\
& \mathcal{C} & \\
\mathsf{DK}(\mathcal{E}^0) \arrow[rr, "\mathsf{DK}(\alpha)"'] \arrow[ur, "q^0"] & & \mathsf{DK}(\mathcal{E}^1) \arrow[ul, "q^1"']
\end{tikzcd}$$

   Note that by Theorem 4.6(1) and the assumption, γ^0 and γ^1 preserve cocartesian lifts of f. Now recall that γ^0 and γ^1 are essentially surjective[17] so that by uniqueness of cocartesian lifts with a given source, we see that every q^0-cocartesian lift of f is the image under γ^0 of a p^0-cocartesian lift of f. The claim follows by commutativity of the diagram.

2. We show the addendum, so assume that α preserves all cocartesian edges and is fiberwise compatible with the deformations. By Lemma D.4 there is an equivalence $\mathrm{Cocart}(\Delta^1 \times \mathcal{C}) \simeq \mathrm{Ar}(\mathrm{Cocart}(\mathcal{C}))$ given by straightening in Δ^1. Postcomposing with the forgetful $\mathrm{Ar}(\mathrm{Cocart}(\mathcal{C})) \to \mathrm{Ar}(\mathsf{Cat})$, the composite is given by sending a cocartesian fibration $p : \mathcal{A} \to \Delta^1 \times \mathcal{C}$ to $\mathrm{St}^{\mathsf{cc}}(\mathrm{pr} \circ p)$ where $\mathrm{pr} : \Delta^1 \times \mathcal{C} \to \Delta^1$. In particular, there is a unique cocartesian fibration $p : \mathcal{A} \to \Delta^1 \times \mathcal{C}$ corresponding to the morphism of cocartesian fibrations α, and as underlying functors $\mathcal{E}^0 \to \mathcal{E}^1$ we have $\mathrm{St}^{\mathsf{cc}}(\mathrm{pr} \circ p) \simeq \alpha$. Analogously, we will show that we can obtain $\mathbf{L}\alpha$ in such a way, which then yields that it preserves cocartesian morphisms.

   Pulling back p along $\{i\} \times \mathcal{C} \to \Delta^1 \times \mathcal{C}$ gives back p^i, and the pushforward for $(0, c) \to (1, c)$ is precisely given by $\alpha_c : \mathcal{E}^0_c \to \mathcal{E}^1_c$. It follows that p, equipped with the fiberwise left-deformations from $\mathcal{E}^0$ and $\mathcal{E}^1$, is again a left-deformable cocartesian fibration. Moreover, $\mathrm{pr} \circ p : \mathcal{A} \to \Delta^1$ is left-derivable as $\mathcal{E}^0 \to \mathcal{E}^1$ even preserves the marking by assumption. Finally, note that left-deriving and postcomposing with pr commute on the nose. Hence the left-derivation of $\mathrm{pr} \circ p$, which by Theorem 4.6(3) precisely encodes $\mathsf{DK}(\alpha)$, agrees with the morphism of cocartesian fibrations over $\mathcal{C}$ encoded by the left-derivation of p.

$\square$

Let $U : \mathsf{Cat}^\dagger \to \mathsf{Cat}$ be the forgetful functor, and $\eta : \mathrm{id} \Rightarrow (-)^\flat \circ \mathsf{DK}$ the unit of the adjunction $\mathsf{DK} \dashv (-)^\flat$ from Proposition 4.2. This gives us a natural transformation $U\eta : U \Rightarrow \mathsf{DK}$ which at $(\mathcal{C}, W)$ exhibits $\mathcal{C}[W^{-1}]$ as the Dwyer-Kan localization of $\mathcal{C}$ at W.

[17] In fact, by [Cis19, Remark 7.1.4] we can always choose the Dwyer-Kan localization in such a way that it is the identity on objects, although this may sometimes be misleading. For example, although one can think of Spc as a localization of topological spaces at weak homotopy equivalences, one should not think of its objects as topological spaces, but rather as homotopy types.

Corollary 4.8. Given $F : \mathcal{C} \to \mathsf{Cat}^\dagger$, denote by $W \subset \int UF$ the usual fiberwise marked edges in the cocartesian unstraightening of $UF : \mathcal{C} \to \mathsf{Cat}$. Then we have a commutative diagram of cocartesian fibrations over $\mathcal{C}$

$$\begin{tikzcd} & \int UF \arrow[dl, "\gamma"'] \arrow[dr, "\int U\eta"] \arrow[dd] & \\ (\int UF)[W^{-1}] \arrow[rr, "\varphi", "\simeq"'] \arrow[dr] & & \int \mathsf{DK} \circ F \arrow[dl] \\ & \mathcal{C} & \end{tikzcd}$$

In particular, the cocartesian unstraightening of $\mathsf{DK} \circ F$ agrees with the left-derivation of the canonical left-deformable fibration $\int UF$ where all left-deformations are taken to be the identity.

Proof. Clearly $\int U\eta$ and hence its composite to $\mathcal{C}$ also invert W, so by the universal property of Dwyer-Kan localization the diagram exists and commutes. Also $\int U\eta : \int UF \to \int \mathsf{DK} \circ F$ is a morphism of cocartesian fibrations. The same holds for $\gamma : \int UF \to (\int UF)[W^{-1}]$ by the above theorem, as for each $f : c \to d$ in $\mathcal{C}$ the functor $Ff = f_!$ is marked. Then by Proposition 4.7 also φ is a morphism of cocartesian fibrations, so that we can check whether φ is an equivalence fiberwise. But when we pull the whole diagram back along $\{c\} \to \mathcal{C}$, it follows from Theorem 4.6(2) that both γ_c and $(\int U\eta)_c$ exhibit their target as the Dwyer-Kan localization of UFc at the W_c, with φ_c being the unique equivalence between them. □

Corollary 4.9. Let $p : \mathcal{E} \to \mathcal{C}$ be a left-deformable cocartesian fibration and $\mathcal{E}' \subseteq \mathcal{E}$ be any full subcategory with $\operatorname{Im} Q_c \subseteq \mathcal{E}'_c \subseteq \mathcal{E}_c$ for all $c \in \mathcal{C}$. Then $p' = p|_{\mathcal{E}'} : \mathcal{E}' \to \mathcal{C}$ is a left-deformable cocartesian fibration and the inclusion $i : \mathcal{E}' \subseteq \mathcal{E}$ induces an equivalence $\mathcal{E}'[W^{-1}] \xrightarrow{\simeq} \mathcal{E}[W^{-1}]$ of cocartesian fibrations over $\mathcal{C}$.

Proof. By 2-out-of-3 it suffices to consider the case that $\mathcal{E}'$ is the minimal such category with $\mathcal{E}'_c = \operatorname{Im} Q_c$ for each $c \in \mathcal{C}$. It follows from Proposition 4.7 that $\mathcal{E}'[W^{-1}] \to \mathcal{E}[W^{-1}]$ is a morphism of cocartesian fibrations over $\mathcal{C}$, so that we can check the equivalence fiberwise. But by Theorem 4.6(2) we know that over $c \in \mathcal{C}$ this is given by $\mathsf{DK}(i_c) : (\operatorname{Im} Q_c)[W_c^{-1}] \to \mathcal{E}_c[W_c^{-1}]$, which is an equivalence with inverse induced by Q_c. □

In the context of commutative monoids or symmetric monoidal categories we prefer to work with $\mathrm{Span}(\mathbb{F})$ as opposed to $\mathbb{F}_* \simeq \mathrm{Span}_{\mathsf{inj},\mathsf{all}}(\mathbb{F}) \subseteq \mathrm{Span}(\mathbb{F})$. We use the above results to adapt and provide more details on [NS18, Example A.13].

Example 4.10. Let $\mathcal{M}$ be a monoidal model category with functorial cofibrant replacement and cofibrant unit. Let $p : \mathcal{M}^\otimes \to \mathrm{Span}(\mathbb{F})$ denote the cocartesian unstraightening of the associated categorical Mackey functor. We equip $\mathcal{M}^\otimes$ with weak equivalences $W^\otimes$ given fiberwise by $W^n \subset \mathcal{M}^n$, and the left deformations given by pointwise functorial cofibrant replacement (Q^n, q^n). Then p is left-deformable, and its left derivation $q : \mathcal{M}^{\otimes}[(W^{\otimes})^{-1}] \to \mathrm{Span}(\mathbb{F})$ straightens to a categorical Mackey functor encoding the symmetric monoidal structure on the underlying ∞-category of $\mathcal{M}$.

Proof. Backwards injections, surjections and forwards injections induce projections, diagonals and unit maps which by assumption all are strictly homotopical and restrict to categories of cofibrant objects. Forwards surjections induce the left Quillen multifunctors given by the monoidal structure $\otimes : \mathcal{M}^n \to \mathcal{M}$. This proves that p is a left-deformable cocartesian fibration. Let $\rho^i = (n \leftarrow \{i\} = \{i\})$ induce $\rho^i_! = \mathrm{pr}_i : \mathcal{M}^n = \mathcal{M}^{\otimes}_n \to \mathcal{M}^{\otimes}_1 = \mathcal{M}$. Using that $\rho^i_!$ is strictly homotopical together with Theorem 4.6(2,3) and the fact that by Proposition 4.2 Dwyer-Kan localization preserves finite products we obtain a commutative diagram

$$\begin{tikzcd}
& \mathcal{M}^n \arrow[rr, "\mathrm{pr}_i"] \arrow[dd, "\gamma^n"] & & \mathcal{M} \arrow[dd, "\gamma"] \\
\mathcal{M}^{\otimes}_n \arrow[ur, equal] \arrow[rr, "\rho^i_!"] \arrow[dd, "\gamma_n"] & & \mathcal{M}^{\otimes}_1 \arrow[ur, equal] \arrow[dd, "\gamma_1"] & \\
& \mathcal{M}[W^{-1}]^n \arrow[rr, "\mathrm{pr}_i"] & & \mathcal{M}[W^{-1}] \\
\mathcal{M}^{\otimes}[(W^{\otimes})^{-1}]_n \arrow[ur, equal] \arrow[rr, "\rho^i_!"] & & \mathcal{M}^{\otimes}[(W^{\otimes})^{-1}]_1 \arrow[ur, equal] &
\end{tikzcd}$$

In particular, the straightening of q preserves finite products, hence is a categorical Mackey functor.

Finally, it is clear that pulling back p, q and $\gamma : \mathcal{M}^{\otimes} \to \mathcal{M}^{\otimes}[(W^{\otimes})^{-1}]$ along $\mathbb{F}_* \simeq \mathrm{Span}_{\mathsf{inj,all}}(\mathbb{F}) \to \mathrm{Span}(\mathbb{F})$ recovers [NS18, Example A.13], and the usual notion of the symmetric monoidal category underlying $\mathcal{M}$. But by Corollary 2.14 this pullback induces the equivalence $\mathsf{Mack}(\mathsf{Cat}) \simeq \mathsf{CMon}(\mathsf{Cat})$. $\square$

4.2 Models for G-equivariant and G-global Spectra

In this subsection, we will finally construct the parametrized symmetric monoidal structure on equivariant and global spectra $\underline{\mathsf{Sp}}^{\otimes}, \underline{\mathsf{Sp}}^{\otimes}_{\mathsf{Glo}}$ as well as comparison from strictly commutative ring spectra in a suitable model category to the category of parametrized commutative algebras for $\underline{\mathsf{Sp}}^{\otimes}$ and $\underline{\mathsf{Sp}}^{\otimes}_{\mathsf{Glo}}$. Concretely, we will follow the articles [CLL23a, CLL23b] and use the 1-category of G-symmetric spectra (based on simplicial sets) as underlying 1-category of our model categories for G-equivariant and G-global spectra. We begin by summarizing the relevant model-categorical results from [Hau17, Hau19, Len21, LS23]. We let $(\mathsf{Sp}^{\Sigma}, \wedge, \mathbb{S})$ denote the closed symmetric monoidal 1-category of symmetric spectra based on simplicial sets. Moreover, we implicitly identify (symmetric monoidal) 1-categories with their nerves. For a finite group G, the category $G\mathsf{Sp}^{\Sigma} \coloneqq \mathsf{Fun}(BG, \mathsf{Sp}^{\Sigma})$

of symmetric spectra with G-action with its pointwise monoidal structure is the closed symmetric monoidal 1-category of G-symmetric spectra $G\mathsf{Sp}^\Sigma$ from [Hau17].

Theorem 4.11 (G-Equivariant Model Structures)**.** For a finite group G, the category $G\mathsf{Sp}^\Sigma$ admits a flat and a projective G-equivariant model structure with the following properties.

1. Both model structures are combinatorial, stable, proper, simplicial, monoidal with cofibrant unit $\mathbb{S}_G$, and satisfy the monoid axiom.

2. Both model structures have as weak equivalences the G-stable weak equivalences as defined in [Hau17, Definition 2.35]. In both cases, the underlying presentably symmetric monoidal ∞-category is Sp^G. Moreover, a morphism $f : X \to Y$ of G-symmetric spectra is a G-equivariant flat/projective cofibration if and only if its underlying morphism of symmetric spectra $\mathrm{res}^G_e f$ is a non-equivariant flat/projective cofibration. Every projective cofibration is a flat cofibration.

3. If $H \leq G$, then restriction $\mathrm{res}^G_H : G\mathsf{Sp}^\Sigma \to H\mathsf{Sp}^\Sigma$ preserves the flat/projective cofibrations and sends G-stable equivalences to H-stable equivalences. In particular, restriction along injections is left Quillen for both model structures. Restricting along a non-injective group homomorphism preserves flat/projective cofibrations, but is generally only left Quillen for the projective model structure.

4. Smashing with a flat G-symmetric spectrum preserves G-stable equivalences. Smashing with an arbitrary G-symmetric spectrum preserves G-stable equivalences between flat G-symmetric spectra.

5. For any inclusion of finite groups $H \leq G$, the symmetric monoidal norm $N^G_H : H\mathsf{Sp}^\Sigma \to G\mathsf{Sp}^\Sigma$ preserves projectively cofibrant/flat spectra and sends H-stable equivalences between flat spectra to G-stable equivalences.

Proof. Existence of the model structures with the mentioned weak equivalences and cofibrations, and that they are proper and combinatorial is shown in [Hau17, Theorem 4.7,4.8]. That they are simplicial is proven in [Len21, 3.1.12] and stability is shown in [Len21, 3.1.18]. Compatibility with the monoidal structure and the related points from (4) and (5) are shown in [Hau17, Section 6]. It follows from [Hau17, Remark 2.20] that both the flat and projective cofibrations do not depend on the G-action. Every projective cofibration is also a flat cofibration by definition, cf. [Hau17, 2.18]. That restriction along injections is compatible with weak equivalences and cofibrations also follows directly from the definitions, compare [Hau17, Section 5.2]. Since both types of cofibrations do not depend on the group, restriction along arbitrary group homomorphisms preserves them. That such restrictions are left Quillen for the projective model structure follows directly from this and [Hau17, Section 5.1], compare [CLL23b, Lemma 9.3]. $\square$

Theorem 4.12 (G-Global Model Structures)**.** For a finite group G, category of G-symmetric spectra $G\mathsf{Sp}^\Sigma$ admits a flat G-global model structure with the following properties.

1. It is combinatorial, stable, proper, simplicial and monoidal with cofibrant unit $\mathbb{S}_G$.

2. The weak equivalences are the G-global equivalences, given by those maps $f : X \to Y$ in $G\mathsf{Sp}^\Sigma$ such that $\alpha^* f$ is a K-stable equivalence for every homomorphism of finite groups $\alpha : K \to G$. The underlying presentably symmetric monoidal ∞-category is by definition $\mathsf{Sp}^{G\text{-}\mathsf{gl}}$. The G-global flat cofibrations are simply underlying flat cofibrations of symmetric spectra, so agree with the cofibrations of the flat G-equivariant model structure. For $G = e$ the model structure agrees with the global model structure on symmetric spectra from [Hau19].

3. Restriction along an arbitrary homomorphism of finite groups $\alpha : K \to G$ preserves flat cofibrations and sends G-global equivalences to K-global equivalences, hence is left Quillen for the flat model structure.

4. Smashing with a flat G-symmetric spectrum preserves G-stable equivalences. Smashing with an arbitrary G-symmetric spectrum preserves G-global equivalences between flat G-symmetric spectra.

5. For any inclusion of finite groups $H \leq G$, the symmetric monoidal norm $N_H^G : H\mathsf{Sp}^\Sigma \to G\mathsf{Sp}^\Sigma$ preserves flat spectra and sends H-global equivalences between them to G-global equivalences.

Proof. That the model structure exists with the mentioned weak equivalences and that it is proper, combinatorial simplicial and stable is shown in [Len21, 3.1.40, 3.1.47, 3.1.48]. By [Len21, 3.1.42] the model structure agrees with Hausmann's global stable model structure on symmetric spectra in the case $G = e$. Compatibility with the monoidal structure and the point from (4) are shown in [Len21, 3.1.62,3.1.63,3.1.64], (cf. [LS23, 1.46]. Since G-global flat cofibrations are just flat cofibrations, $\mathbb{S}_G$ is flat. Restrictions along arbitrary group homomorphisms preserves flat cofibrations because they don't depend on the group, as in the equivariant case. It is also immediately clear from the definition of G-global equivalences that such restrictions are strictly homotopical. The final point about norms is [LS23, 5.19]. ☐

Proposition 4.13. We have a diagram of left Quillen functors where the horizontal functor is part of a Quillen equivalence

$$\begin{tikzcd}
G\mathsf{Sp}^\Sigma_{G\text{-equivariant projective}} \arrow[r, "\mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & G\mathsf{Sp}^\Sigma_{G\text{-equivariant flat}} \\
& G\mathsf{Sp}^\Sigma_{G\text{-global flat}} \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

In particular, G-stable equivalences between projectively cofibrant G-symmetric spectra are already G-global equivalences, and the above functors derive to exhibit Sp^G as both a left and right Bousfield-localization of $\mathsf{Sp}^{G\text{-}\mathsf{gl}}$:

$$\begin{tikzcd} \mathsf{Sp}^G \arrow[r, bend left, hook, "i_!"] \arrow[r, bend right, hook, "i_*"'] & \mathsf{Sp}^{G-\mathsf{gl}} \arrow[l, "i^*"'] \end{tikzcd}$$

The functor $i_!$ is strong symmetric monoidal, and the induced lax symmetric monoidal structure on i^* is actually strong symmetric monoidal. Thus also i_* attains a lax symmetric monoidal structure.

Proof. For ease of notation, denote the diagonal, horizontal and vertical identities in the diagram by $\mathrm{id}_d, \mathrm{id}_h, \mathrm{id}_v$. That id_d is left Quillen and induces a right Bousfield localization $i_! := \mathbf{L}\,\mathrm{id}_d \dashv \mathbf{R}\,\mathrm{id}_d^{-1} =: i^*$ is the statement of [Len21, 3.3.1]. Since id_d is strong symmetric monoidal, also $i_!$ is (e.g. by identifying $i_! \simeq \mathsf{DK}(\mathrm{inc})$ for the morphism $\mathrm{inc} : ((G\mathsf{Sp}^\Sigma)^{\mathsf{proj-cof}}, W_{G\text{-stable}}) \subseteq ((G\mathsf{Sp}^\Sigma)^{\mathsf{flat}}, W_{G\text{-global}})$ in $\mathsf{CMon}(\mathsf{Cat}^\dagger)$). In particular, the oplax monoidal structure on $i_!$ induces a unique lax symmetric monoidal structure on i^* by [HHLN23a, Proposition A]. The dual of [HHLN23a, Proposition 3.2.7] shows that these induced structure morphisms are given by $\mathbb{S}_{G\text{-}\mathsf{eqv}} \xrightarrow{\eta_{\mathbb{S}_{G\text{-}\mathsf{eqv}}}} i^* i_! \mathbb{S}_{G\text{-}\mathsf{eqv}} \simeq i^*\mathbb{S}_{G\text{-}\mathsf{gl}}$ and

$$i^*X \otimes i^*Y \xrightarrow{\eta_{i^*X\otimes i^*Y}} i^*i_!(i^*X \otimes i^*Y) \simeq i^*(i_!i^*X \otimes i_!i^*Y) \xrightarrow{i^*(\varepsilon_X\otimes\varepsilon_Y)} i^*(X \otimes Y),$$

where the unlabeled equivalences come from the strong symmetric monoidal structure of $i_!$. Since $i_!$ is fully faithful and hence η an equivalence, it remains to see that $\varepsilon_X \otimes \varepsilon_Y$ is an equivalence. But in the model, the counit is precisely given by the projectively cofibrant replacement $q : Q \Rightarrow \mathrm{id}$ in $G\mathsf{Sp}^\Sigma$. Since we can assume without loss of generality that X and Y are flat, so $q_X \wedge q_Y$ is an equivalence since smashing with flat spectra preserves stable equivalences by Theorem 4.11(4).

Finally, note that since id_v and id_d^{-1} are strictly homotopical, and $\mathbf{L}\,\mathrm{id}_h \dashv \mathbf{R}\,\mathrm{id}_h^{-1}$ is the identity adjunction, we have by commutativity of the square that

$$i^* = \mathbf{R}\,\mathrm{id}_d^{-1} \simeq \mathsf{DK}(\mathrm{id}_d^{-1}) \simeq \mathsf{DK}(\mathrm{id}_h^{-1} \circ \mathrm{id}_v) \simeq \mathsf{DK}(\mathrm{id}_v) \simeq \mathbf{L}\,\mathrm{id}_v$$

In particular, i^* admits a further right adjoint $i_* := \mathbf{R}\,\mathrm{id}_v^{-1}$, which is now fully faithful and lax symmetric monoidal for formal reasons. ☐

Consider the symmetric monoidal 1-category of symmetric spectra $\mathsf{Sp}^\Sigma \in \mathsf{Mack}(\widehat{\mathsf{Cat}})$. In view of Example 3.1, its global Borelification $\mathrm{Bor}_{\mathsf{Glo}}^{\mathsf{Orb}}(\mathsf{Sp}^\Sigma) \in \mathsf{Mack}_{\mathsf{Glo}}^{\mathsf{Orb}}(\widehat{\mathsf{Cat}})$ has the functoriality

$$\left(K \xleftarrow{\alpha} H \hookrightarrow G\right) \mapsto \left(K\mathsf{Sp}^\Sigma \xrightarrow{\alpha^*} H\mathsf{Sp}^\Sigma \xrightarrow{N_H^G} G\mathsf{Sp}^\Sigma\right)$$

and moreover its restriction along $\mathrm{Span}(\mathbb{F}) \to \mathrm{Span}_{\mathsf{all},\mathsf{Orb}}(\mathbb{F}_{\mathsf{Glo}}), * \mapsto G$ encodes the symmetric monoidal structure on G-symmetric spectra $G\mathsf{Sp}^\Sigma$. Let $p : \int \mathrm{Bor}_{\mathsf{Glo}}^{\mathsf{Orb}}(\mathsf{Sp}^\Sigma) \to \mathrm{Span}_{\mathsf{all},\mathsf{Orb}}(\mathbb{F}_{\mathsf{Glo}})$ be the cocartesian unstraightening.

Construction 4.14 (Construction of $\underline{\mathsf{Sp}}^\otimes$)**.** We equip each fiber $p^{-1}(\coprod_{i=1}^n G_i) = \prod_{i=1}^n G_i\mathsf{Sp}^\Sigma$ with weak equivalences $\prod_{i=1}^n W_{G_i-\mathrm{stable}}$ and left-deformations induced by the functorial cofibrant replacements of the projective G_i-equivariant model structures. By Theorem 4.11 this yields a left-deformable cocartesian fibration p_{eqv}. In view of Corollary 4.9, we can take fiberwise full subcategories on the flat and projectively cofibrant spectra and obtain obtain a commutative diagram of cocartesian fibrations over $\mathrm{Span}_{\mathsf{all},\mathsf{Orb}}(\mathbb{F}_{\mathsf{Glo}})$ where the labels denote which cocartesian arrows are preserved (cc means all):

$$\begin{tikzcd}
\int \mathrm{Bor}^{\mathsf{Orb}}_{\mathsf{Glo}}((\mathsf{Sp}^{\Sigma})^{\mathsf{proj-cof}}) \arrow[r, hook, "\mathrm{cc}"] \arrow[dr, "\mathrm{cc}"'] & \int \mathrm{Bor}^{\mathsf{Orb}}_{\mathsf{Glo}}((\mathsf{Sp}^{\Sigma})^{\mathsf{flat}}) \arrow[r, hook, "\mathrm{cc}"] \arrow[d, "\mathrm{Span}(\mathbb{F}_{\mathsf{Orb}})\text{-cc}"] & \int \mathrm{Bor}^{\mathsf{Orb}}_{\mathsf{Glo}}(\mathsf{Sp}^{\Sigma}) \arrow[dl, "\mathbb{F}^{\mathsf{op}}_{\mathsf{Orb}}\text{-cc}"] \\
& \int \underline{\mathsf{Sp}}^{\otimes} &
\end{tikzcd} \tag{13}$$

Here all downwards arrows exhibit $\int \underline{\mathsf{Sp}}^{\otimes}$ as the left-derived cocartesian fibration of their source, i.e. as the Dwyer-Kan localization of its source at the $W_{\bullet\text{-stable}}$ equivalences. The horizontal arrows clearly preserve all cocartesian morphisms, and by Theorem 4.6 the vertical ones preserve cocartesian lifts of those morphisms which induce strictly homotopical pushforward functors. For example, it follows from Theorem 4.11 that when restricting to flat spectra, all relevant functors except inflations are strictly homotopical, and hence the middle vertical localization preserves cocartesian lifts of morphisms in the subcategory $\mathrm{Span}(\mathbb{F}_{\mathsf{Orb}}) \subset \mathrm{Span}_{\mathsf{all},\mathsf{Orb}}(\mathbb{F}_{\mathsf{Glo}})$.

Construction 4.15 (Construction of $\underline{\mathsf{Sp}}^{\otimes}_{\mathsf{Glo}}$)**.** Analogously to the above, by Theorem 4.12 we obtain a left-deformable cocartesian fibration p_{gl} by equipping each fiber of p with the G-global equivalences and left-deformations induced by the functorial cofibrant replacement in the flat G-global model structure. We can also restrict to flat spectra in the global case and obtain a commutative diagram of cocartesian fibrations over $\mathrm{Span}_{\mathsf{all},\mathsf{Orb}}(\mathbb{F}_{\mathsf{Glo}})$

$$\begin{tikzcd}
\int \mathrm{Bor}^{\mathsf{Orb}}_{\mathsf{Glo}}((\mathsf{Sp}^{\Sigma})^{\mathsf{flat}}) \arrow[rr, hook, "\mathrm{cc}"] \arrow[dr, "\mathrm{cc}"'] & & \int \mathrm{Bor}^{\mathsf{Orb}}_{\mathsf{Glo}}(\mathsf{Sp}^{\Sigma}) \arrow[dl, "\mathbb{F}^{\mathsf{op}}_{\mathsf{Glo}}\text{-cc}"] \\
& \int \underline{\mathsf{Sp}}^{\otimes}_{\mathsf{Glo}} &
\end{tikzcd} \tag{14}$$

with the downwards arrows exhibiting $\int \underline{\mathsf{Sp}}^{\otimes}_{\mathsf{Glo}}$ as the left-derived cocartesian fibration of their source, i.e. as Dwyer-Kan localization at the $W_{\bullet\text{-global}}$ equivalences.

Construction 4.16 (Comparing $\underline{\mathsf{Sp}}^{\otimes}$ and $\underline{\mathsf{Sp}}^{\otimes}_{\mathsf{Glo}}$)**.** By Proposition 4.13 and Lemma D.3 we also have natural pointwise fully faithful inclusions

$$\begin{aligned}
(\mathrm{Bor}^{\mathsf{Orb}}_{\mathsf{Glo}}((\mathsf{Sp}^{\Sigma})^{\mathsf{proj-cof}}), W_{\bullet\text{-stable}}) &\subseteq (\mathrm{Bor}^{\mathsf{Orb}}_{\mathsf{Glo}}((\mathsf{Sp}^{\Sigma})^{\mathsf{flat}}), W_{\bullet\text{-global}}) \\
&\subseteq (\mathrm{Bor}^{\mathsf{Orb}}_{\mathsf{Glo}}((\mathsf{Sp}^{\Sigma})^{\mathsf{flat}}), W_{\bullet\text{-stable}})
\end{aligned}$$

which upon cocartesian unstraightening give marked functors of left-deformable cocartesian fibrations preserving all cocartesian morphisms. Taking the left-derivations yields a commutative diagram

$$\begin{tikzcd}
\int \mathrm{Bor}_{\mathsf{Glo}}^{\mathsf{Orb}}((\mathsf{Sp}^{\Sigma})^{\mathsf{proj-cof}}) \arrow[r, hook] \arrow[d, "\gamma_{\mathsf{eqv}}"'] & \int \mathrm{Bor}_{\mathsf{Glo}}^{\mathsf{Orb}}((\mathsf{Sp}^{\Sigma})^{\mathsf{flat}}) \arrow[r, equal] \arrow[d, "\gamma_{\mathsf{gl}}"] & \int \mathrm{Bor}_{\mathsf{Glo}}^{\mathsf{Orb}}((\mathsf{Sp}^{\Sigma})^{\mathsf{flat}}) \arrow[d, "\gamma_{\mathsf{eqv}}"] \\
\int \underline{\mathsf{Sp}}^{\otimes} \arrow[r, hook] & \int \underline{\mathsf{Sp}}_{\mathsf{Glo}}^{\otimes} \arrow[r, "\mathrm{Span}(\mathbb{F}_{\mathsf{Orb}})\text{-cc}"] & \int \underline{\mathsf{Sp}}^{\otimes}
\end{tikzcd} \tag{15}$$

Here all functors except the bottom right horizontal one are morphisms of cocartesian fibrations over $\mathrm{Span}_{\mathsf{all},\mathsf{Orb}}(\mathbb{F}_{\mathsf{Glo}})$, i.e. preserve all cocartesian morphisms. For the bottom left horizontal functor this follows from Proposition 4.7 by noting that the first of the above inclusions is fiberwise compatible with the left deformations. Thus it straightens to an inclusion $\underline{\mathsf{Sp}}^{\otimes} \subseteq \underline{\mathsf{Sp}}_{\mathsf{Glo}}^{\otimes}$ in $\mathsf{Mack}_{\mathsf{Glo}}^{\mathsf{Orb}}(\widehat{\mathsf{Cat}})$, given pointwise by the (symmetric monoidal) left adjoint inclusion $\mathsf{Sp}^{G} \subseteq \mathsf{Sp}^{G\text{-}\mathsf{gl}}$ from Proposition 4.13. However, the second inclusion is not fiberwise compatible with the left-deformations; its image would have to lie inside the projectively cofibrant symmetric spectra. Nevertheless, we can still use the other criterion of Proposition 4.7 to conclude that the bottom right horizontal functor preserves cocartesian lifts of morphisms in $\mathrm{Span}(\mathbb{F}_{\mathsf{Orb}})$. By construction and Proposition 4.13, this is a fiberwise right adjoint to $\underline{\mathsf{Sp}}^{\otimes} \subseteq \underline{\mathsf{Sp}}_{\mathsf{Glo}}^{\otimes}$, and the composite $\int \underline{\mathsf{Sp}}^{\otimes} \to \int \underline{\mathsf{Sp}}_{\mathsf{Glo}}^{\otimes} \to \int \underline{\mathsf{Sp}}^{\otimes}$ is homotopic to the identity.

Proposition 4.17. Let $\underline{\mathsf{Sp}}^{\otimes}, \underline{\mathsf{Sp}}_{\mathsf{Glo}}^{\otimes}$ be as above.

1. The restriction $\underline{\mathsf{Sp}} \coloneqq \underline{\mathsf{Sp}}^{\otimes}|_{\mathsf{Glo}^{\mathsf{op}}}$ is the global category of equivariant spectra constructed in [CLL23b]. In particular $\underline{\mathsf{Sp}}$ is the free equivariantly presentable equivariantly stable global category on one generator, also denoted $\underline{\mathsf{Sp}}_{\mathsf{Orb}\triangleright\mathsf{Glo}}^{\mathsf{Orb}}$ in op. cit.

2. Analogously, $\underline{\mathsf{Sp}}_{\mathsf{Glo}}^{\otimes}|_{\mathsf{Glo}^{\mathsf{op}}}$ is the global category of global spectra $\underline{\mathsf{Sp}}_{\mathsf{Glo}}^{\mathsf{Orb}}$ constructed in [CLL23a], and hence the free globally presentable equivariantly stable global category on one generator.

3. The bottom right horizontal functor in Diagram (15) gives an equivariantly symmetric monoidal structure to the Orb-right adjoint of $\underline{\mathsf{Sp}} \subseteq \underline{\mathsf{Sp}}_{\mathsf{Glo}}^{\mathsf{Orb}}$ constructed in [CLL23b, Lemma 9.12]. Pointwise, this is the symmetric monoidal adjunction $\mathsf{Sp}^{G} \rightleftarrows \mathsf{Sp}^{G\text{-}\mathsf{gl}}$ of Proposition 4.13.

4. For every injection $p\colon H \hookrightarrow G$, the derived multiplicative norms $p_{\otimes}\colon \mathsf{Sp}^{H} \to \mathsf{Sp}^{G}$ of $\underline{\mathsf{Sp}}^{\otimes}$ and $p_{\otimes}\colon \mathsf{Sp}^{H\text{-}\mathsf{gl}} \to \mathsf{Sp}^{G\text{-}\mathsf{gl}}$ of $\underline{\mathsf{Sp}}_{\mathsf{Glo}}^{\otimes}$ preserve sifted colimits.

Proof. 1. Consider the morphism $\gamma_{\mathsf{eqv}}\colon \int \mathrm{Bor}_{\mathsf{Glo}}^{\mathsf{Orb}}((\mathsf{Sp}^{\Sigma})^{\mathsf{proj-cof}}) \to \int \underline{\mathsf{Sp}}^{\otimes}$ of cocartesian fibrations over $\mathrm{Span}_{\mathsf{all},\mathsf{Orb}}(\mathbb{F}_{\mathsf{Glo}})$ from Diagram (13), which exhibits $\int \underline{\mathsf{Sp}}^{\otimes} \to \mathrm{Span}_{\mathsf{all},\mathsf{Orb}}(\mathbb{F}_{\mathsf{Glo}})$ as left-derived cocartesian fibration of its left-deformable source. By Corollary 4.8 this straightens to a morphism $\mathrm{Bor}_{\mathsf{Glo}}^{\mathsf{Orb}}((\mathsf{Sp}^{\Sigma})^{\mathsf{proj-cof}}) \Rightarrow \underline{\mathsf{Sp}}^{\otimes}$ in $\mathsf{Mack}_{\mathsf{Glo}}^{\mathsf{Orb}}(\widehat{\mathsf{Cat}})$ which exhibits $\underline{\mathsf{Sp}}^{\otimes}$ as the functorial DK-localization:

$$\underline{\mathsf{Sp}}^{\otimes} \simeq \mathsf{DK} \circ \ (\mathrm{Bor}_{\mathsf{Glo}}^{\mathsf{Orb}}((\mathsf{Sp}^{\Sigma})^{\mathsf{proj-cof}}), W_{\bullet\text{-stable}}).$$

Clearly all of this can be restricted along $\mathsf{Glo}^{\mathsf{op}} \subseteq \mathbb{F}_{\mathsf{Glo}}^{\mathsf{op}} \to \mathrm{Span}_{\mathsf{all},\mathsf{Orb}}(\mathbb{F}_{\mathsf{Glo}})$, and thus gives

$$\underline{\mathsf{Sp}} \coloneqq \underline{\mathsf{Sp}}^{\otimes}|_{\mathsf{Glo}^{\mathsf{op}}} \simeq \mathsf{DK} \circ \ (\mathrm{Bor}_{\mathsf{Glo}}((\mathsf{Sp}^{\Sigma})^{\mathsf{proj-cof}}), W_{\bullet\text{-stable}}).$$

But this is precisely the definition of $\underline{\mathscr{S}p}$ used in [CLL23b, Section 9.1], and so the claims follow from Theorems 9.4 and 9.5 of op. cit.

2. We consider the localization $\gamma_{\mathsf{gl}} : \int \mathrm{Bor}^{\mathsf{Orb}}_{\mathsf{Glo}}(\mathsf{Sp}^{\Sigma}) \to \int \underline{\mathsf{Sp}}^{\otimes}_{\mathsf{Glo}}$ exhibiting $\int \underline{\mathsf{Sp}}^{\otimes}_{\mathsf{Glo}}$ as the left-derivation of its left-deformable source. By Theorem 4.6(2) we can pull back along $\mathsf{Glo}^{\mathsf{op}} \subseteq \mathbb{F}^{\mathsf{op}}_{\mathsf{Glo}} \to \mathrm{Span}_{\mathsf{all},\mathsf{Orb}}(\mathbb{F}_{\mathsf{Glo}})$ to see that $\gamma_{\mathsf{gl}} : \int \mathrm{Bor}_{\mathsf{Glo}}(\mathsf{Sp}^{\Sigma}) \to \int \underline{\mathsf{Sp}}^{\otimes}_{\mathsf{Glo}}|_{\mathsf{Glo}^{\mathsf{op}}}$ still exhibits the target as the left-derivation of the source. But now the domain is actually the straightening of a strictly homotopical functor, so as above we obtain via Corollary 4.8

$$\underline{\mathsf{Sp}}^{\otimes}_{\mathsf{Glo}}|_{\mathsf{Glo}^{\mathsf{op}}} \simeq \mathsf{DK} \circ \ (\mathrm{Bor}_{\mathsf{Glo}}(\mathsf{Sp}^{\Sigma}), W_{\bullet\text{-global}}).$$

This is precisely the definition of $\underline{\mathscr{S}p^{\mathsf{gl}}}$ from [CLL23a, Section 7.1], and so the claims follow from Theorem 7.3.2 and Corollary 7.3.3 of op. cit.

3. In [CLL23b, Section 9] the so-called injective global model structure on $G\mathsf{Sp}^{\Sigma}$ is used to model $\mathsf{Sp}^{G\text{-}\mathsf{gl}}$, see Proposition 9.10 of op.cit. However, the identity yields a left Quillen equivalence from the flat global model structure to the injective global model structure, and their construction of $\underline{\mathsf{Sp}} \subseteq \underline{\mathsf{Sp}}^{\mathsf{Orb}}_{\mathsf{Glo}}$ and its Orb-right adjoint in [CLL23b, Lemma 9.12] is simply our construction from Construction 4.16 conjugated by (the induced equivalence of ∞-categories of) this Quillen equivalence. In other words, if we pull back along the inclusion $j : \mathrm{Span}(\mathbb{F}_{\mathsf{Orb}}) \to \mathrm{Span}_{\mathsf{all},\mathsf{Orb}}(\mathbb{F}_{\mathsf{Glo}})$ we can straighten the resulting morphism of cocartesian fibrations into $j^*\underline{\mathsf{Sp}}^{\otimes}_{\mathsf{Glo}} \Rightarrow j^*\underline{\mathsf{Sp}}^{\otimes}$ in $\mathsf{Mack}_{\mathsf{Orb}}(\widehat{\mathsf{Cat}})$. Restricting this along $\mathsf{Orb}^{\mathsf{op}} \to \mathrm{Span}(\mathbb{F}_{\mathsf{Orb}})$ then gives the Orb-right adjoint $\underline{\mathsf{Sp}}_{\mathsf{Orb}} = \underline{\mathsf{Sp}}^{\mathsf{Orb}}_{\mathsf{Glo}}|_{\mathsf{Orb}^{\mathsf{op}}} \to \underline{\mathsf{Sp}}|_{\mathsf{Orb}^{\mathsf{op}}}$ constructed in Lemma 9.12 of op.cit.

4. By [Lur09, 5.5.8.17] a functor preserves sifted colimits if and only if it preserves filtered colimits and geometric realizations (i.e. Δ^{op}-indexed colimits). Moreover, by Kerodon 039H a functor preserves filtered colimits if and only if it preserves colimits indexed by directed partially ordered sets, which are filtered 1-categories. Their nerve is also a filtered ∞-category by Kerodon 02PV. Thus it suffices to show that if K is a 1-category such that its nerve NK is a sifted ∞-category (and hence K is sifted in the 1-categorical sense)[18] then each $p_{\otimes}$ preserves NK-indexed colimits. We consider the equivariant case, as the global one is entirely analogous. By definition, $p_{\otimes} : \mathsf{Sp}^H \to \mathsf{Sp}^G$ is the left-derived functor $\mathbf{L}N_H^G$ of the multiplicative norm $N_H^G : H\mathsf{Sp}^{\Sigma} \to G\mathsf{Sp}^{\Sigma}$ with respect to the projective model structures from Theorem 4.11. Since $H\mathsf{Sp}^{\Sigma}$ is combinatorial, by [Lur17, 1.3.4.25] every diagram $NK \to \mathsf{Sp}^H$ is equivalent to one of the form $NK \to H\mathsf{Sp}^{\Sigma} \to \mathsf{Sp}^H$. Moreover, if $NK \to H\mathsf{Sp}^{\Sigma}$ is projectively cofibrant, then its 1-categorical colimit in $H\mathsf{Sp}^{\Sigma}$

[18] Warning: The converse does not hold, see Kerodon 02QF.

also computes the colimit in Sp^H by [BHH17, Remark 2.5.7]. Since N_H^G preserves sifted 1-colimits and cofibrant objects, it remains to show that if $F : K \to H\mathsf{Sp}^\Sigma$ is projectively cofibrant, then $\operatorname{colim} N_H^G F$ is weakly equivalent to $\operatorname{hocolim} N_H^G F$. To this end let $n = [G : H]$ and recall that the norm factors as $H\mathsf{Sp}^\Sigma \to (\Sigma_n \wr H)\mathsf{Sp}^\Sigma \to G\mathsf{Sp}^\Sigma$ where the first functor takes the n-fold smash power and the second one restricts the action. The restriction preserves both colimits and homotopy colimits, so it remains to see that $\operatorname{colim} F^{\wedge n}$ is weakly equivalent to $\operatorname{hocolim} F^{\wedge n}$, where $F^{\wedge n} : K \to (\Sigma_n \wr H)\mathsf{Sp}^\Sigma, k \mapsto F(k)^{\wedge n}$. Without loss of generality, we can take $n = 2$. By assumption on F we have that $\operatorname{colim}_K F$ is cofibrant, represents $\operatorname{hocolim}_K F$, and that $F \wedge F$ and $(\operatorname{colim}_K F) \wedge (\operatorname{colim}_K F)$ also represent the derived smash product $\wedge^L$. Using that K and NK are sifted, we then get isomorphisms in the homotopy category

$$\begin{aligned}\operatorname*{colim}_K(F \wedge F) &\cong \operatorname*{colim}_{K\times K}(F \wedge F) \cong (\operatorname*{colim}_K F) \wedge (\operatorname*{colim}_K F) \cong (\operatorname*{hocolim}_K F) \wedge^L (\operatorname*{hocolim}_K F)\\ &\cong \operatorname*{hocolim}_{K\times K}(F \wedge^L F) \cong \operatorname*{hocolim}_K(F \wedge F).\end{aligned}$$

□

Definition 4.18. For a finite group G, we define the G-symmetric monoidal G-category of G-spectra as the restriction $\underline{\mathsf{Sp}}_G^\otimes \coloneqq \underline{\mathsf{Sp}}^\otimes|_{\mathrm{Span}(G)} \in \mathsf{Mack}_G(\widehat{\mathsf{Cat}}(\mathsf{sift}))$ and $\underline{\mathsf{Sp}}_G \coloneqq \underline{\mathsf{Sp}}_G^\otimes|_{\mathsf{Orb}_G^{\mathrm{op}}}$.

Remark 4.19. We have already discussed some properties of $\underline{\mathsf{Sp}}_G$ in Example 2.26. It was shown in [CLL23b, Theorem 9.13] that $\underline{\mathsf{Sp}}_G \simeq \underline{\mathsf{Sp}}|_{\mathsf{Orb}_G^{\mathrm{op}}}$ is the free G-presentable G-stable G-category on one generator. Moreover, restricting along $\Theta : \mathsf{Orb}_G^{\mathrm{op}} \times \mathsf{Fin}_* \to \mathrm{Span}_{\mathsf{all,fold}}(\mathbb{F}_G) \to \mathrm{Span}(\mathbb{F}_G)$ adjoins over to $\mathsf{Orb}_G^{\mathrm{op}} \to \mathsf{CMon}(\widehat{\mathsf{Cat}})$, $G \mapsto \mathsf{Sp}^G$ by [BH17, Lemma C.4], analogously to what we did in Section 3.2. This actually factors through $\mathrm{CAlg}(\mathsf{Pr}^L)$, and is the initial presentably symmetric monoidal G-stable G-category of [Cno23a, Theorem 4.10].

Corollary 4.20. $\underline{\mathsf{Sp}}_G^\otimes, \underline{\mathsf{Sp}}^\otimes$ and $\underline{\mathsf{Sp}}_{\mathsf{Glo}}^\otimes$ are compatible with sifted colimits in the sense of Definition 2.32, hence factor through the subcategory $\widehat{\mathsf{Cat}}(\mathsf{sift}) \subset \widehat{\mathsf{Cat}}$ on large categories admitting sifted colimits and functors preserving them.

Construction 4.21. By [Hau19, Theorem 3.5] there is a model structure on $\mathrm{CAlg}(\mathsf{Sp}^\Sigma)$ representing the homotopy theory of ultracommutative global ring spectra, so that the weak equivalences are underlying global equivalences and the cofibrations between underlying flat symmetric spectra are underlying flat cofibrations. Hence we define

$$\mathsf{UCom} \coloneqq \mathrm{CAlg}(\mathsf{Sp}^\Sigma)[W_{\mathrm{global}}^{-1}] \simeq \mathrm{CAlg}((\mathsf{Sp}^\Sigma)^{\mathsf{flat}})[W_{\mathrm{global}}^{-1}].$$

Let $\gamma_{\mathsf{gl}}, \gamma_{\mathsf{eqv}}$ and Ξ denote the left, right and bottom functors from the right square in Diagram (15).

Via postcomposition, these functors yield the commutative diagram

$$\begin{tikzcd}
 & \mathrm{CAlg}^{\mathsf{Orb}}_{\mathsf{Glo}}(\mathrm{Bor}^{\mathsf{Orb}}_{\mathsf{Glo}}((\mathsf{Sp}^\Sigma)^{\mathsf{flat}})) \arrow[dl, "\gamma_{\mathsf{gl}}"'] \arrow[d, "\mathrm{res},\simeq"] \arrow[dr, "\gamma_{\mathsf{eqv}}"] & \\
\mathrm{CAlg}^{\mathsf{Orb}}_{\mathsf{Glo}}(\underline{\mathsf{Sp}}^{\otimes}_{\mathsf{Glo}}) \arrow[rr, bend right, "\Xi"] & \mathrm{CAlg}((\mathsf{Sp}^\Sigma)^{\mathsf{flat}}) \arrow[d, "\gamma_{\mathsf{gl}}"] & \mathrm{Sect}^{\mathbb{F}^{\mathrm{op}}_{\mathsf{Orb}}-\mathsf{cc}}(\int \underline{\mathsf{Sp}}^{\otimes}) \\
 & \mathsf{UCom} \arrow[ul, dashed, "\Phi_{\mathsf{gl}}"] \arrow[ur, dashed, "\Phi_{\mathsf{eqv}}"'] &
\end{tikzcd}$$

Here res is the restriction along $\mathrm{Span}(\mathbb{F}) \to \mathrm{Span}_{\mathsf{all},\mathsf{Orb}}(\mathbb{F}_{\mathsf{Glo}})$ induced by the Borel-inclusion $(*,*) \subseteq (\mathsf{Glo},\mathsf{Orb})$. It is an equivalence by Proposition 3.7. We check that the composites $(\gamma_{\mathsf{eqv}})_* \circ \mathrm{res}^{-1}$ and $(\gamma_{\mathsf{gl}})_* \circ \mathrm{res}^{-1}$ invert underlying global equivalences, so that the functors Φ_{gl} and Φ_{eqv} exist and make the diagram commute by the universal property of Dwyer-Kan localization. So let $f : R \to S$ be a morphism in $\mathrm{CAlg}^{\mathsf{Orb}}_{\mathsf{Glo}}(\mathrm{Bor}^{\mathsf{Orb}}_{\mathsf{Glo}}(\mathsf{Sp}^\Sigma))$ so that $\mathrm{res}\, f$ is a weak equivalence in $\mathrm{CAlg}(\mathsf{Sp}^\Sigma)$, i.e. such that $f(e) : R(e) \to S(e)$ is a global equivalence of symmetric spectra. Then since R and S are cocartesian on backwards morphisms, we see that $f(G)$ is actually $\mathrm{infl}_G\, f(e) : \mathrm{infl}_G\, R(e) \to \mathrm{infl}_G\, S(e)$ and hence a G-global and in particular G-stable equivalence. Such morphisms are inverted by γ_{gl} respectively γ_{eqv}.

Remark 4.22. We really want to land in $\mathrm{Sect}^{\mathbb{F}^{\mathrm{op}}_{\mathsf{Orb}}-\mathsf{cc}}(\int \underline{\mathsf{Sp}}^{\otimes})$ in the equivariant case. This is inspired by the main result of [LNP22], which states $\mathrm{laxlim}^{\dagger}{}_{\mathsf{Glo}^{\mathrm{op}}}\, \underline{\mathsf{Sp}} = \mathrm{Sect}^{\mathsf{Orb}^{\mathrm{op}}-\mathsf{cc}}(\int \underline{\mathsf{Sp}}) \simeq \mathsf{Sp}^{\mathsf{gl}}$, whereas $\mathrm{Sect}^{\mathsf{Glo}^{\mathrm{op}}-\mathsf{cc}}(\int \underline{\mathsf{Sp}}) \simeq \lim_{\mathsf{Glo}^{\mathrm{op}}} \underline{\mathsf{Sp}} \simeq \underline{\mathsf{Sp}}(e) = \mathsf{Sp}$ since e is initial in $\mathsf{Glo}^{\mathrm{op}}$.

Construction 4.23. Analogously to the above, by [Hau17, Corollary 6.5] we have a model structure on $\mathrm{CAlg}(G\mathsf{Sp}^\Sigma)$ representing the homotopy theory of strictly commutative G-ring spectra with weak equivalences the underlying G-stable equivalences. We set $\mathsf{UCom}_G \coloneqq \mathrm{CAlg}(G\mathsf{Sp}^\Sigma)[W^{-1}_{G\text{-stable}}]$. In view of Theorem 3.15 and Remark 3.20 we likewise get a commutative square

$$\begin{tikzcd}
\mathrm{CAlg}_G(\mathrm{Bor}_G(\mathsf{Sp}^\Sigma)) \arrow[r, "(\gamma_G)_*"] \arrow[d, "\mathrm{ev}_{G/G},\simeq"'] & \mathrm{CAlg}_G(\underline{\mathsf{Sp}}^{\otimes}_G) \\
\mathrm{CAlg}(G\mathsf{Sp}^\Sigma) \arrow[r, "\gamma_{G-\mathrm{stable}}"] & \mathsf{UCom}_G \arrow[u, dashed, "\Phi_G"']
\end{tikzcd}$$

where γ_G is the restriction of γ_{eqv} along $\mathrm{Span}(G) \to \mathrm{Span}_{\mathsf{all},\mathsf{Orb}}(\mathbb{F}_{\mathsf{Glo}})$. Indeed, if $f : R \to S$ is a morphism of G-commutative algebras in $\mathrm{Bor}_G(\mathsf{Sp}^\Sigma)$ such that $f(G/G) : R(G/G) \to S(G/G)$ is a G-stable equivalence, then $f(G/H) = \mathrm{res}^G_H\, f(G/G)$ is an H-stable equivalence, hence f is inverted by γ_G, and Φ_G exists by the universal property of Dwyer-Kan localizations.

Conjecture 4.24. The functors $\Xi, \Phi_{\mathsf{gl}}, \Phi_{\mathsf{eqv}}$ and Φ_G are equivalences. It is expected that this can be shown using the Barr-Beck-Lurie theorem [Lur17, 4.7.3.5] by comparing the induced monads, however we did not have the time left to seriously attempt this. For Φ_G, this is claimed without proof at the end of [BH17, Section 9].

Warning 4.25. Suppose we define Φ'_{eqv} as above, but restricting to projectively cofibrant symmetric spectra $\gamma'_{\mathsf{eqv}} : \int \mathrm{Bor}^{\mathsf{Orb}}_{\mathsf{Glo}}((\mathsf{Sp}^{\Sigma})^{\mathsf{proj-cof}}) \to \int \underline{\mathsf{Sp}}^{\otimes}$ instead of only flat symmetric spectra. This will give the "wrong" functor. Indeed, note that γ'_{eqv} preserves all cocartesian morphisms, and hence we would land in the full subcategory $\mathrm{CAlg}^{\mathsf{Orb}}_{\mathsf{Glo}}(\underline{\mathsf{Sp}}^{\otimes}) \subseteq \mathrm{Sect}^{\mathbb{F}^{\mathsf{op}}_{\mathsf{Orb}}}(\int \underline{\mathsf{Sp}}^{\otimes})$. The restriction along $\mathsf{Glo}^{\mathsf{op}} \to \mathrm{Span}_{\mathsf{all},\mathsf{Orb}}(\mathbb{F}_{\mathsf{Glo}})$ would then likewise factor through $\mathrm{Sect}^{\mathsf{Glo}^{\mathsf{op}}}(\int \underline{\mathsf{Sp}}) \simeq \mathsf{Sp} \subseteq \mathsf{Sp}^{\mathsf{gl}}$, which shows that Φ_{eqv} would have image in algebras whose underlying global spectrum is left-induced.[19] In some sense this is also to be expected, because stable equivalences between projectively cofibrant symmetric spectra are already global equivalences by Proposition 4.13, and so inverting global equivalences in $(\mathsf{Sp}^{\Sigma})^{\mathsf{proj-cof}}$ already gives Sp instead of $\mathsf{Sp}^{\mathsf{gl}}$.

5 Parametrized Picard Spectra

In this section we address the original thesis problem; to construct equivariant and global Picard spectra. We begin in Section 5.1 by giving an introduction to the classical notion of Picard spectra. We finally consider parametrized Picard spectra in Section 5.2.

5.1 Picard Groups, Spaces, Spectra

Let us begin by recalling the classical notion of Picard group, space and spectrum. Given a symmetric monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$, we call an object $x \in \mathcal{C}$ invertible if there exists $x^{-1} \in \mathcal{C}$ such that $x \otimes x^{-1} \simeq \mathbb{1}$. Note that such a x^{-1}, if it exists, is unique; the space of inverses for x is (-1)-truncated. Equivalently, x is invertible if $x \otimes - : \mathcal{C} \to \mathcal{C}$ is an equivalence. The Picard group of $\mathcal{C}$ is then defined as the group of isomorphism classes of invertible elements in $\mathcal{C}$, with group operation given by tensor product again. In other words, we are considering $\pi_0(\mathcal{C}^{\simeq}, \otimes, \mathbb{1})^{\times}$. While this group is all the information we care about, it is more convenient to consider the whole commutative group of units in the underlying commutative monoid $\mathcal{C}^{\simeq}$ of $\mathcal{C}$, as it is categorically more well behaved. Since commutative groups are equivalently connective spectra $\mathsf{CGrp} \simeq \mathsf{Sp}_{\geq 0}$, we may as well deloop the resulting group and obtain what is called the Picard spectrum of $\mathcal{C}$. Note that taking units of commutative monoids assembles into a functor $(-)^{\times} : \mathsf{CMon} \to \mathsf{CGrp}$, right adjoint to the fully faithful inclusion $\mathsf{CGrp} \subseteq \mathsf{CMon}$.

Definition 5.1. The Picard space / Picard spectrum functor $\mathsf{pic} : \mathsf{CMon}(\mathsf{Cat}) \to \mathsf{Sp}_{\geq 0}$ is defined as the composite

$$\mathsf{CMon}(\mathsf{Cat}) \xrightarrow{(-)^{\simeq}_{*}} \mathsf{CMon} \xrightarrow{(-)^{\times}} \mathsf{CGrp} \simeq \mathsf{Sp}_{\geq 0}.$$

The Picard group of a symmetric monoidal category $\mathcal{C}$ is given by $\pi_0 \,\mathsf{pic}\, \mathcal{C}$.

[19] Using [Lin24, Proposition 3.16] one checks that $\mathsf{Sp} \simeq \mathrm{Sect}^{\mathsf{Glo}^{\mathsf{op}}-\mathsf{cc}}(\underline{\mathsf{Sp}}) \subseteq \mathrm{Sect}^{\mathsf{Orb}^{\mathsf{op}}-\mathsf{cc}}(\underline{\mathsf{Sp}}) \simeq \mathsf{Sp}^{\mathsf{gl}}$, preserves colimits, hence this inclusion really corresponds to left-induction in the sense of [Sch18, Section 4.5].

Remark 5.2. The Picard space functor can also be extended to monoidal categories, however we will only land in ($\mathbb{E}_1$-)groups, which do not deloop to connective spectra:

$$\mathsf{pic} : \mathsf{Mon}(\mathsf{Cat}) \xrightarrow{(-)^{\simeq}_{*}} \mathsf{Mon} \xrightarrow{(-)^{\times}} \mathsf{Grp}.$$

However, all of our examples will be symmetric monoidal, so we will rarely consider this extension.

As composite of two right adjoints pic automatically preserves limits. Note that for this it is important to set the codomain of pic to be $\mathsf{CGrp} \simeq \mathsf{Sp}_{\geq 0}$, and not all of spectra, as connective spectra are not closed under limits in Sp. Moreover, both right adjoints also preserve filtered colimits, hence so does pic. These are the categorical advantages to considering the whole Picard spectrum as opposed to only the Picard group.

Remark 5.3. We can also apply pic to presentably symmetric monoidal categories (which are *not* small) and still end up in the same universe. The reason for this is that any object in a presentable category is κ-compact for some regular cardinal κ, and in particular $\mathbb{1}$ will be. But then if x is an invertible object, then $x \otimes -$ is an equivalence sending $\mathbb{1}$ to x, so that x must also be κ-compact. In other words, the full subcategory on invertible objects will be a full subcategory of the κ-compact objects, which is small.

We will thus also consider

$$\mathsf{pic} : \mathrm{CAlg}(\mathsf{Pr}^L) \xrightarrow{((-)^{\simeq})^{\times}} \mathsf{CGrp} \simeq \mathsf{Sp}_{\geq 0}.$$

Since limits in commutative algebras are computed in the underlying category, and limits in Pr^L are computed in $\widehat{\mathsf{Cat}}$, it follows that also this version of the Picard spectrum functor preserves limits (however, while filtered colimits of algebras are again computed in the underlying category, the inclusion $\mathsf{Pr}^L \subset \widehat{\mathsf{Cat}}$ generally does not preserve filtered colimits).

The main example for us will be Picard groups of commutative ring spectra. More generally, if $\mathcal{C} \in \mathrm{CAlg}(\mathsf{Pr}^L)$ is presentably symmetric monoidal and $R \in \mathrm{CAlg}(\mathcal{C})$, we will consider the Picard spectrum of R, defined as $\mathsf{pic}\, R \coloneqq \mathsf{pic}\, \mathsf{Mod}_R(\mathcal{C})$. This association can be made functorial by [Lur17, 4.8.5.21], so overall we obtain another version of the Picard functor

$$\mathsf{pic}_{\mathcal{C}} : \mathrm{CAlg}(\mathcal{C}) \to \mathsf{Sp},\ R \mapsto \mathsf{pic}(\mathsf{Mod}_R(\mathcal{C})).$$

When it is clear from context, we will leave out the subscript and simply write $\mathsf{pic}\, R \coloneqq \mathsf{pic}_{\mathcal{C}}\, R$. In the case of $\mathcal{C} = \mathsf{Sp}$ essentially all interesting information of $\mathsf{pic}\, R \in \mathsf{Sp}$ lies in the Picard group.

Lemma 5.4. Let $R \in \mathrm{CAlg}(\mathsf{Sp})$ be a commutative ring spectrum. Then

$$\pi_n \,\mathsf{pic}\, R \cong \begin{cases} 0, & n < 0 \\ (\pi_0 R)^{\times}, & n = 1 \\ \pi_{n-1} R, & n \geq 2 \end{cases}.$$

Proof. By definition $\mathsf{pic}\, R$ is connective. Now note that for a symmetric monoidal category $\mathcal{C}$, we have equivalences of spaces

$$\Omega\, \mathsf{pic}(\mathcal{C}) \simeq \Omega_{\mathbb{1}}(\mathcal{C}^{\simeq})^{\times} \simeq (\Omega_{\mathbb{1}}\mathcal{C}^{\simeq})^{\times} \simeq \Omega_{\mathbb{1}}\mathcal{C}^{\simeq} \simeq \mathcal{C}^{\simeq}(\mathbb{1}, \mathbb{1}),$$

where we use that $(-)^{\times}$ preserves limits and hence loops, that $\Omega : \mathsf{CMon} \to \mathsf{CMon}$ actually already lands in CGrp (see [HW, II.21b]). We now claim that this last space is equivalent to the units of the endomorphism monoid $\mathcal{C}(\mathbb{1}, \mathbb{1})$. Indeed, we can compute the units of this monoid as the pullback

$$\begin{tikzcd} \mathcal{C}(\mathbb{1},\mathbb{1})^{\times} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}(\mathbb{1},\mathbb{1}) \arrow[d] \\ (\pi_0\mathcal{C}(\mathbb{1},\mathbb{1}))^{\times} \arrow[r] & \pi_0\mathcal{C}(\mathbb{1},\mathbb{1}) \end{tikzcd}$$

Now $\pi_0\mathcal{C}(\mathbb{1}, \mathbb{1}) = (h\mathcal{C})(\mathbb{1}, \mathbb{1})$ is also the hom-set in the homotopy category of $\mathcal{C}$, and the inherited monoid structure from functoriality of π_0 corresponds again to the endomorphism monoid structure of this hom-set. But for the 1-category $h\mathcal{C}$ it is clear that $(h\mathcal{C})(\mathbb{1}, \mathbb{1})^{\times} = (h\mathcal{C})^{\cong}(\mathbb{1}, \mathbb{1}) = \pi_0\mathcal{C}^{\simeq}(\mathbb{1}, \mathbb{1})$, and hence we obtain the desired equivalence $\mathcal{C}^{\simeq}(\mathbb{1}, \mathbb{1}) \simeq \mathcal{C}(\mathbb{1}, \mathbb{1})^{\times}$.

Now the unit of Mod_R is R, and the forgetful functor $\mathsf{Mod}_R \to \mathsf{Sp}$ has left adjoint given by $R \otimes -$, so that $\mathsf{Mod}_R(R, R) \simeq \mathsf{Sp}(\mathbb{S}, R) \simeq \Omega^{\infty} R$. Overall, we obtain the claimed formula:

$$\pi_{n+1}\, \mathsf{pic}\, R \cong \pi_n \Omega_{\mathbb{1}}\, \mathsf{pic}\, R \cong (\pi_n \mathsf{Mod}_R(R, R))^{\times} \cong (\pi_n \Omega^{\infty} R)^{\times} = \pi_n(R)^{\times}.$$

□

The following example is well known.

Example 5.5. We have $\pi_0\, \mathsf{pic}\, \mathbb{S} = \pi_0\, \mathsf{pic}\, \mathsf{Sp} = \{\Sigma^n \mathbb{S} \mid n \in \mathbb{Z}\}$.

Proof. Let X be an invertible spectrum. Since $X \otimes - : \mathsf{Sp} \to \mathsf{Sp}$ is an equivalence that sends the compact object $\mathbb{S}$ to X, we see that also X is compact, hence finite (i.e. a finite colimit of $\mathbb{S}$, or equivalently shift of suspension spectrum of finite CW complex). In particular X is bounded below. Moreover, the Künneth theorem tells us that for any field K the functor $HK_* = H_*(-; K) : \mathsf{Sp} \to \mathsf{GrVect}_K$ is symmetric monoidal, hence preserves invertible objects. An easy dimension argument then shows that for every field K there is an integer $n_K \in \mathbb{Z}$ such that $HK_*(X) \cong \Sigma^{n_K} K$ is a single copy of K in degree n_K. By shifting X we can assume $n_{\mathbb{Q}} = 0$. Since X is has finitely generated integral homology, we can write

$$H_0(X; \mathbb{Z}) \cong \mathbb{Z}^{\ell} \oplus \bigoplus_{i=1}^{s} \mathbb{Z}/p_i^{t_i}.$$

Now $\mathbb{Q} \cong H\mathbb{Q}_0 X \cong H\mathbb{Z}_0 X \otimes \mathbb{Q} \cong \mathbb{Q}^\ell$, so $\ell = 1$. But then the universal coefficient theorem tells us

$$H_0(X; \mathbb{F}_p) \cong H_0(X; \mathbb{Z}) \otimes \mathbb{F}_p \oplus A \cong \mathbb{F}_p \oplus B \oplus A$$

for some abelian groups A, B. By the above, we must have $A = B = 0$ and $n_{\mathbb{F}_p} = 1$. Doing this for all primes p, we arrive at $H_0(X; \mathbb{Z}) \cong \mathbb{Z}$. Moreover, since by the UCT $H_n(X; \mathbb{Z}) \otimes K = 0$ for all $n \neq 0$, we finally see that $H\mathbb{Z}_* X = \mathbb{Z}$ is a single copy of $\mathbb{Z}$ concentrated in degree 0. Since X is bounded below, the Hurewicz theorem yields $\pi_0 X \cong H_0 X \cong \mathbb{Z}$, and so there exists a map $f : \mathbb{S} \to X$ inducing an isomorphism on π_0 and thus on H_0. But then it already induces an isomorphism on $H\mathbb{Z}_*$, so by the Whitehead theorem f is an equivalence, as desired. $\square$

In general, Picard groups are quite hard to compute, even for classical and well-understood spectra such as real and complex K-theory, whose Picard groups reflect their Bott-periodicity.

Example 5.6. $\pi_0 \operatorname{\mathsf{pic}}(\mathrm{KU}) = \{\mathrm{KU}, \Sigma\mathrm{KU}\} \cong \mathbb{Z}/2$ and analogously $\pi_0 \operatorname{\mathsf{pic}}(\mathrm{KO}) \cong \mathbb{Z}/8$.

Proof. I am not aware of where this was first shown, but a proof is given in [MS16, Section 7]. $\square$

Let G be a compact Lie group. Since the geometric fixed point functors are all symmetric monoidal and hence preserve invertible objects, it follows from Example 5.5 that if $X \in \mathsf{Sp}^G$ is an invertible genuine G-spectrum, then $\Phi^H X$ is a shifted sphere for every closed subgroup $H \leq G$. Despite this, the Picard group of the category of genuine G-spectra Sp^G is generally quite complicated. The representation sphere S^V associated to a finite-dimensional orthogonal G-representations V becomes invertible in Sp^G.[20] In this way one obtains a map $\mathbf{RO}(G) \to \pi_0 \operatorname{\mathsf{pic}} \mathsf{Sp}^G, V \mapsto S^V$, but this is in general neither injective nor surjective[21] Because of this map, invertible G-spectra are often called (stable) homotopy representations. There is much more to be said about invertible genuine G-spectra, but this will take us too far afield. We refer the interested reader to [FLM01] and [Kra20]. Surprisingly, the global situation is much simpler.

Example 5.7. $\pi_0 \operatorname{\mathsf{pic}}(\mathbb{S}_{\mathsf{gl}}) = \pi_0 \operatorname{\mathsf{pic}}(\mathsf{Sp}^{\mathsf{gl}}) = \{\Sigma^n \mathbb{S}_{\mathsf{gl}} \mid n \in \mathbb{Z}\}$.

The proof of this is a direct consequence of the following general lemma applied to $\varepsilon : LU \Rightarrow \mathrm{id}$ for $L : \mathsf{Sp} \to \mathsf{Sp}^{\mathsf{gl}}$ the unique symmetric monoidal left adjoint, which also admits a symmetric monoidal right adjoint U (e.g. take $G = e$ in Proposition 4.13) and the fact that $L\mathbb{S} = \mathbb{S}_{\mathsf{gl}}$, compare [Sch18, 4.5.5].

[20] In fact, it was shown in [GM23, Corollary C.7] that this is essentially the defining property of genuine G-spectra; one can define Sp^G as the symmetric monoidal localization $\mathsf{Spc}_*^G[\{S^V\}^{-1}]$ of pointed G-spaces at all representation spheres, and $\Sigma^\infty : \mathsf{Spc}_*^G \to \mathsf{Sp}^G$ is the unique symmetric monoidal left adjoint which inverts representation spheres.

[21] I do not know of a reference in the literature for this, but two counterexamples are given in this lecture by Stefan Schwede https://www.youtube.com/watch?v=4CdVetQn2vg&t=2180s.

Lemma 5.8 ([Sch18, 4.5.4])**.** Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a symmetric monoidal category, $P : \mathcal{C} \to \mathcal{C}$ a strong symmetric monoidal functor, and $\varepsilon : P \Rightarrow \mathrm{id}_{\mathcal{C}}$ a symmetric monoidal transformation. Then ε induces a homotopy of the map of Picard spectra

$$\mathsf{pic}(P) : \mathsf{pic}\,\mathcal{C} \to \mathsf{pic}\,\mathcal{C}$$

to the identity. The dual statement with a symmetric monoidal transformation $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow P$ in place of ε holds as well.

Proof. For the convenience of the reader, we adapt the referenced proof to the ∞-categorical context. Note that the cited statement contains the superfluous hypothesis that also $\varepsilon_{\mathbb{1}} : P\mathbb{1} \to \mathbb{1}$ is an equivalence, which automatically follows from P and id being strong monoidal and ε being a monoidal transformation.

The data of such a symmetric monoidal functor P and natural transformation ε can be formalized as a natural transformation $\varepsilon^{\otimes} : P^{\otimes} \Rightarrow \mathrm{id}_{\mathcal{C}^{\otimes}}$ of the morphism of cocartesian fibrations $P^{\otimes} : \mathcal{C}^{\otimes} \to \mathcal{C}^{\otimes}$ where $\mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ encodes the symmetric monoidal structure on $\mathcal{C}$. Let $X \in \mathcal{C}$ be invertible with inverse Y, and let $\alpha : X \otimes Y \to \mathbb{1}$ be an equivalence. Since $P^{\otimes}$ preserves inert morphisms we can identify $P^{\otimes}(X, Y) = (PX, PY)$. Under this identification $\varepsilon^{\otimes}_{P^{\otimes}(X,Y)} : P^{\otimes}(X, Y) \to (X, Y)$ corresponds to $(\varepsilon_X, \varepsilon_Y)$ by naturality of $\varepsilon^{\otimes}$. Now let $f : (X, Y) \to X \otimes Y$ and $g : (PX, PY) \to PX \otimes PY$ denote the cocartesian lifts of the unique active map $2_+ \to 1_+$. We can factor $(PX, PY) = P^{\otimes}(X, Y) \xrightarrow{P^{\otimes}(f)} P(X \otimes Y)$ into $\mu \circ g$ for a unique $\mu : PX \otimes PY \to P(X \otimes Y)$ which is an equivalence as $P^{\otimes}$ preserves cocartesian morphisms. By naturality of $\varepsilon^{\otimes}$ we obtain a commutative diagram in $\mathcal{C}^{\otimes}$ as on the left, which then induces the commutative diagram in $\mathcal{C}$ on the right:

$$\begin{tikzcd}
(PX, PY) \arrow[r, equal] \arrow[d, "g"'] \arrow[rr, bend left, "(\varepsilon_X, \varepsilon_Y)"] & P^{\otimes}(X, Y) \arrow[r, "\varepsilon^{\otimes}_{(X,Y)}"'] \arrow[d, "P^{\otimes}(f)"'] & (X, Y) \arrow[d, "f"] \\
PX \otimes PY \arrow[r, "\simeq"] & P(X \otimes Y) \arrow[r, "\varepsilon_{X \otimes Y}"] & X \otimes Y
\end{tikzcd}
\qquad
\begin{tikzcd}
 & P(X \otimes Y) \arrow[r, "P(\alpha)", "\simeq"'] \arrow[d, "\varepsilon_{X \otimes Y}"] & P(\mathbb{1}) \arrow[d, "\simeq"', "\varepsilon_{\mathbb{1}}"] \\
PX \otimes PY \arrow[ur, "\simeq"] \arrow[r, "\varepsilon_X \otimes \varepsilon_Y"'] & X \otimes Y \arrow[r, "\simeq", "\alpha"'] & \mathbb{1}
\end{tikzcd}$$

Indeed, since g and f are cocartesian, $\varepsilon_X \otimes \varepsilon_Y$ is homotopic to the bottom horizontal composite in the left diagram. By 2-out-of-3 we see that

$$\varepsilon_X \otimes \varepsilon_Y \simeq (X \otimes \varepsilon_Y) \circ (\varepsilon_X \otimes PY) \simeq (\varepsilon_X \,\mathrm{id} \otimes Y) \circ (PX \otimes \varepsilon_Y)$$

is an equivalence. Since strong symmetric monoidal functors preserve invertible objects, we see that $- \otimes PY$ and $- \otimes Y$ are equivalences, hence the above tells us that ε_X has both a left and right inverse, and is therefore an equivalence. Thus ε is an equivalence on all invertible objects, and hence (since $\varepsilon^{\otimes}$

is fiberwise given by ε) also $\varepsilon^{\otimes}_{(X_1,\ldots,X_n)}$ is an equivalence for all invertible objects $X_1,\ldots,X_n$.
Let $\mathsf{pic}(\mathcal{C})^{\otimes} \subset \mathcal{C}^{\otimes}$ be the maximal subcategory which is fiberwise $\mathsf{pic}(\mathcal{C})^n \subset \mathcal{C}^n$. Then $\mathsf{pic}(\mathcal{C})^{\otimes} \to \mathsf{Fin}_*$ is still a cocartesian (even left) fibration, encoding the induced monoidal structure on the subcategory $\mathsf{pic}(\mathcal{C}) \subset \mathcal{C}$, i.e. the group structure. Since P preserves invertible objects, $P^{\otimes}$ restricts to $\mathsf{pic}(\mathcal{C})^{\otimes} \to \mathsf{pic}(\mathcal{C})^{\otimes}$, and the restriction of $\varepsilon^{\otimes}$ then yields the natural equivalence $P^{\otimes}|_{\mathsf{pic}(\mathcal{C})^{\otimes}} \simeq \mathrm{id}_{\mathsf{pic}(\mathcal{C})^{\otimes}}$ which upon cocartesian straightening yields the homotopy $\mathsf{pic}(P) = P|_{\mathsf{pic}(\mathcal{C})} \simeq \mathrm{id}_{\mathsf{pic}(\mathcal{C})}$ of morphisms in Grp. $\square$

Corollary 5.9. Let $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ be an adjunction with L strong symmetric monoidal, and suppose the induced lax symmetric monoidal structure on R is actually strong symmetric monoidal[22]. Then L and R induce mutually inverse equivalences $\mathsf{pic}(\mathcal{C}) \simeq \mathsf{pic}(\mathcal{D})$.

Proof. By [Lur17, 7.3.2.7] the fact that L is strong symmetric monoidal implies that we have a relative adjunction i.e. we have $L^{\otimes} : \mathcal{C}^{\otimes} \rightleftarrows \mathcal{D}^{\otimes} : R^{\otimes}$ over Fin_*, where $L^{\otimes}$ preserves all cocartesian edges and $R^{\otimes}$ the inert ones, giving the lax symmetric monoidal structure on R. Now by hypothesis $R^{\otimes}$ actually preserves all cocartesian morphisms, and so we can apply Lemma 5.8 to both $\varepsilon : LR \Rightarrow \mathrm{id}$ and $\eta : \mathrm{id} \Rightarrow RL$, which gives the claim. $\square$

Example 5.10. Let G be a finite groups and $R \in \mathrm{CAlg}(\mathsf{Sp}^G)$. The adjunction $i_! \dashv i^*$ from Proposition 4.13 induces an equivalence of Picard spectra $\mathsf{pic}(\mathsf{Mod}_R(\mathsf{Sp}^G)) \simeq \mathsf{pic}(\mathsf{Mod}_{i_!R}(\mathsf{Sp}^{G\text{-}\mathsf{gl}}))$. In particular, $\mathsf{pic}(\mathsf{Sp}^G) \simeq \mathsf{pic}(\mathsf{Sp}^{G\text{-}\mathsf{gl}})$.

Proof. Generally, given a symmetric monoidal adjunction $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ and an algebra object $A \in \mathrm{CAlg}(\mathcal{D})$, then the induced functor $M_R : \mathsf{Mod}_A(\mathcal{D}) \to \mathsf{Mod}_{RA}(\mathcal{C})$ admits a left adjoint given by the composite $M_L : \mathsf{Mod}_{RA}(\mathcal{C}) \xrightarrow{L} \mathsf{Mod}_{LRA}(\mathcal{D}) \xrightarrow{A \otimes_{LRA} -} \mathsf{Mod}_A(\mathcal{D})$, see e.g. [Lur17, 7.3.2.8] where one uses that $\mathcal{D}$ becomes left-tensored over $\mathcal{C}$ via L. Alternatively, this can also be deduced from [Lin24, Proposition 3.25] applied to the cocartesian fibrations $\mathsf{Mod}(\mathcal{C}) \to \mathrm{CAlg}(\mathcal{C})$ and $\mathsf{Mod}(\mathcal{D}) \to \mathrm{CAlg}(\mathcal{D})$ of [Lur17, 4.5.3.6]. Moreover, if the symmetric monoidal structures on $\mathcal{C}, \mathcal{D}$ are compatible with geometric realizations and $F : \mathcal{C} \to \mathcal{D}$ is a lax symmetric monoidal functor preserving geometric realizations, then this induces a lax symmetric monoidal functor $M_F : \mathsf{Mod}_A(\mathcal{C}) \to \mathsf{Mod}_{FA}(\mathcal{D})$ via structure morphisms

$$FM \otimes_{FA} FN \simeq \operatorname*{colim}_{\Delta^{\mathsf{op}}} \mathsf{Bar}(FM, FA, FN) \to \operatorname*{colim}_{\Delta^{\mathsf{op}}} F\,\mathsf{Bar}(M, A, N) \simeq F(M \otimes_A N),$$

where $\mathsf{Bar}(M, A, N) \in \mathcal{C}^{\Delta^{\mathsf{op}}}$ denotes the Bar-construction sending $[n] \mapsto M \otimes A^{\otimes n} \otimes N$, and the morphism $\mathsf{Bar}(FM, FA, FN) \to F\,\mathsf{Bar}(M, A, N)$ is induced by the lax symmetric monoidal structure of F. If F is even strong symmetric monoidal, then so is M_F, which also follows from Construction/Theorem 2.34.

[22]The lax monoidality is of R is derived in [Lur17, 7.3.2.7]. More generally, lax symmetric monoidal structures on right adjoints canonically give rise to oplax symmetric monoidal structures on the corresponding left adjoints and vice versa, by [HHLN23a, Theorem A].

Now we specialize to the case at hand; we consider the adjunction $i_! : \mathsf{Sp}^G \rightleftarrows \mathsf{Sp}^{G\text{-gl}} : i^*$. Recall that $i_!$ is fully faithful, so the unit is an equivalence. We apply the above for $A = i_! R$, and after composing with equivalences induced by $i^* i_! \simeq \mathrm{id}$ obtain an induced adjunction $M_{i_!} : \mathsf{Mod}_R(\mathsf{Sp}^G) \rightleftarrows \mathsf{Mod}_{i_! R}(\mathsf{Sp}^{G\text{-gl}}) : M_{i^*}$. Now recall from Proposition 4.13 that *both* $i_!$ and i^* are symmetric monoidal left adjoints, and so by the above we obtain symmetric monoidal structures on $M_{i_!}$ and M_{i^*}. Using [HHLN23a, Proposition 3.2.7] as in Proposition 4.13 to identify the lax symmetric monoidal structure on M_{i^*} induced from the oplax symmetric monoidal structure on $M_{i_!}$ via [HHLN23b, Proposition A], we see that it agrees with the one induced from the lax symmetric monoidal structure on i^* as above (which is itself induced from the oplax symmetric monoidal structure of $i_!$). Hence $M_{i_!} \dashv M_{i^*}$ is a symmetric monoidal adjunction where both functors are strong symmetric monoidal, and applying Corollary 5.9 yields the claim. The addendum about $\mathsf{pic}(\mathsf{Sp}^G) \simeq \mathsf{pic}(\mathsf{Sp}^{G\text{-gl}})$ either follows by applying Corollary 5.9 directly to $i_! \dashv i^*$ or from the above since $\mathsf{Mod}_{\mathbb{S}_G}(\mathsf{Sp}^G) \simeq \mathsf{Sp}^G$ and likewise for $\mathsf{Sp}^{G\text{-gl}}$ by [Lur17, 4.2.4.9]. $\square$

5.2 Parametrized Picard Spectra

In this subsection we finally construct the equivariant and global Picard spectra which were originally conjectured to exist in [Sch18, Remark 5.1.18].

Definition 5.11. Let $P \subset T$ be orbital. Since $\mathsf{pic} : \mathsf{CMon}(\mathsf{Cat}) \to \mathsf{Sp}_{\geq 0}$ preserves limits and $\mathsf{Sp}_{\geq 0} \subseteq \mathsf{Sp}$ finite products, we can define the parametrized Picard spectrum functor

$$\mathsf{pic}_* : \underline{\mathsf{Mack}}_T^P(\mathsf{Cat}) \simeq \underline{\mathsf{Mack}}_T^P(\mathsf{CMon}(\mathsf{Cat})) \to \underline{\mathsf{Mack}}_T^P(\mathsf{Sp}).$$

Analogous to the non-parametrized version, given a fixed $\mathcal{C} \in \mathsf{Mack}_T^P(\mathsf{Cat}; \{\Delta^{\mathsf{op}}\})$ we will also want to consider the Picard spectrum of module categories, and also define

$$\mathsf{pic}_T^P : \mathrm{CAlg}_T^P(\mathcal{C}) \xrightarrow{\mathsf{Mod}_{(-)}(\mathcal{C})} \mathsf{Mack}_T^P(\mathsf{Cat}) \xrightarrow{\mathsf{pic}_*} \mathsf{Mack}_T^P(\mathsf{Sp}).$$

Note that if P is even atomic, then by Corollary 2.56 we have $\underline{\mathsf{Mack}}_T^P(\mathsf{Sp}) \simeq \underline{\mathsf{Sp}}_T^P$.

Remark 5.12. In the case of $(T, P) = (\mathsf{Orb}_G, \mathsf{Orb}_G)$, such a G-Picard spectrum of a G-symmetric monoidal category has already been considered in [HHK+24, Section 5.1]. There it was used to construct a G-symmetric monoidal Thom spectrum functor $(\underline{\mathsf{Spc}}_G)_{/\underline{\mathsf{pic}}_G(\underline{\mathsf{Sp}}_G^{\otimes})} \to \underline{\mathsf{Sp}}_G$. The work is done in the setting of [NS22], and their Picard G-space of a G-symmetric monoidal G-category $\underline{\mathcal{C}}^{\otimes}$ is defined as the maximal G-subgroupoid on the invertible objects, with G-commutative monoid structure inherited from the G-symmetric monoidal structure. Here an object in $\underline{\mathcal{C}}$, i.e. a G-functor const $* \to \underline{\mathcal{C}}$ adjoint to $* \to \Gamma\underline{\mathcal{C}} = \underline{\mathcal{C}}(G/e)$ is equivalently a section $x : \mathsf{Orb}_G^{\mathsf{op}} \to \int \underline{\mathcal{C}}$, and is called invertible if it is so fiberwise; for each $H \leq G$, the object $x(G/H) \in \underline{\mathcal{C}}(G/H)$ is invertible in the symmetric monoidal structure on $\underline{\mathcal{C}}(G/H)$ defined by $\underline{\mathcal{C}}^{\otimes}$. This precisely agrees with our definition of postcomposing

$\mathcal{C}^{\otimes} \in \mathsf{Mack}_G(\mathsf{CMon}(\mathsf{Cat}))$ with $\mathsf{pic} : \mathsf{CMon}(\mathsf{Cat}) \xrightarrow{(-)^{\simeq}_{*}} \mathsf{CMon} \xrightarrow{(-)^{\times}} \mathsf{CGrp}$.

Construction 5.13. Consider $(T, P) = (\mathsf{Glo}, \mathsf{Orb})$ and $\mathcal{C} = \underline{\mathsf{Sp}}^{\otimes}_{\mathsf{Glo}}$, which is compatible with sifted colimits by Corollary 4.20. Then we can build the composite

$$\mathsf{pic}_{\mathsf{gl}} : \mathsf{UCom} \xrightarrow{\Phi_{\mathsf{gl}}} \mathrm{CAlg}^{\mathsf{Orb}}_{\mathsf{Glo}}(\underline{\mathsf{Sp}}^{\otimes}_{\mathsf{Glo}}) \xrightarrow{\mathsf{pic}^{\mathsf{Orb}}_{\mathsf{Glo}}} \mathsf{Mack}^{\mathsf{Orb}}_{\mathsf{Glo}}(\mathsf{Sp}) \simeq \mathsf{Sp}^{\mathsf{gl}}$$

where the first functor was defined in Construction 4.21 and the last equivalence comes from Remark 2.57.[23] This is the functor sending an ultracommutative ring spectrum R to its global Picard spectrum $\mathsf{pic}_{\mathsf{gl}}(R)$ with $\mathsf{pic}_{\mathsf{gl}}(R)^G \simeq \mathsf{pic}(\mathsf{Mod}_{R_G}(\mathsf{Sp}^{G\text{-}\mathsf{gl}}))$ for $G \in \mathsf{Glo}^{\mathsf{op}}$.

Construction 5.14. Now let $\mathcal{C} = \underline{\mathsf{Sp}}^{\otimes}$. Note that the definition of the $\underline{\mathsf{Mod}}_{(-)}(\underline{\mathsf{Sp}}^{\otimes})$ from Theorem 2.37 does not actually require the input section to be cocartesian on all backwards morphisms; it suffices to be cocartesian on projections, i.e. backwards summand inclusions, giving analogously

$$\mathsf{pic}^{\mathsf{Orb}}_{\mathsf{Glo}} := \mathsf{pic}_{*} \circ \underline{\mathsf{Mod}}_{(-)}(\underline{\mathsf{Sp}}^{\otimes}) : \mathrm{Sect}^{\mathbb{F}^{\mathrm{op}}_{\mathsf{Orb}}-\mathsf{cc}}(\textstyle\int \underline{\mathsf{Sp}}^{\otimes}) \to \mathsf{Mack}^{\mathsf{Orb}}_{\mathsf{Glo}}(\mathsf{Sp}) \simeq \mathsf{Sp}^{\mathsf{Orb}}_{\mathsf{Glo}} \simeq \mathsf{Sp}^{\mathsf{gl}}.$$

Hence as above we can build

$$\mathsf{pic}_{\mathsf{eqv}} : \mathsf{UCom} \xrightarrow{\Phi_{\mathsf{eqv}}} \mathrm{Sect}^{\mathbb{F}^{\mathrm{op}}_{\mathsf{Orb}}-\mathsf{cc}}\left(\textstyle\int \underline{\mathsf{Sp}}^{\otimes}\right) \xrightarrow{\mathsf{pic}^{\mathsf{Orb}}_{\mathsf{Glo}}} \mathsf{Sp}^{\mathsf{gl}}.$$

This functor sends an ultracommutative ring spectrum R to the equivariant Picard spectrum $\mathsf{pic}_{\mathsf{eqv}}(R)$ with $\mathsf{pic}_{\mathsf{eqv}}(R)^G \simeq \mathsf{pic}(\mathsf{Mod}_{R_G}(\mathsf{Sp}^G))$ for $G \in \mathsf{Glo}^{\mathsf{op}}$.

Example 5.15. Consider the global sphere spectrum $\mathbb{S}_{\mathsf{gl}} \in \mathsf{UCom}$ coming from the symmetric sphere spectrum $\mathbb{S} \in \mathsf{Sp}^{\Sigma}$. Chasing the definitions of Φ_{gl} and Φ_{eqv} from Construction 4.21, this is the image of $(G \mapsto \mathbb{S}_G = \mathrm{infl}_G\, \mathbb{S} \in G\mathsf{Sp}^{\Sigma}) \in \mathrm{CAlg}^{\mathsf{Orb}}_{\mathsf{Glo}}(\mathrm{Bor}^{\mathsf{Orb}}_{\mathsf{Glo}}(\mathsf{Sp}^{\Sigma}))$ under the equivalence $\mathrm{res} : \mathrm{CAlg}^{\mathsf{Orb}}_{\mathsf{Glo}}(\mathrm{Bor}^{\mathsf{Orb}}_{\mathsf{Glo}}(\mathsf{Sp}^{\Sigma})) \simeq \mathrm{CAlg}(\mathsf{Sp}^{\Sigma})$ from Proposition 3.7. Note that each $\mathbb{S}_G \in G\mathsf{Sp}^{\Sigma}$ is the symmetric monoidal unit, and both localizations $G\mathsf{Sp}^{\Sigma} \to \mathsf{Sp}^{G\text{-}\mathsf{gl}}$ and $G\mathsf{Sp}^{\Sigma} \to \mathsf{Sp}^G$ are symmetric monoidal. In particular, since there is an equivalence $\mathcal{C} \simeq \mathsf{Mod}_{\mathbb{1}}(\mathcal{C})$ for any symmetric monoidal $\mathcal{C}$ with unit $\mathbb{1}$ (c.f. [Lur17, 4.2.4.9]), the free module functor from Theorem 2.37 induces equivalences $\underline{\mathsf{Sp}}^{\otimes}_{\mathsf{Glo}} \simeq \underline{\mathsf{Mod}}_{\Phi_{\mathsf{gl}}\mathbb{S}_{\mathsf{gl}}}(\underline{\mathsf{Sp}}^{\otimes}_{\mathsf{Glo}})$ and $\underline{\mathsf{Sp}}^{\otimes} \simeq \underline{\mathsf{Mod}}_{\Phi_{\mathsf{eqv}}\mathbb{S}_{\mathsf{gl}}}(\underline{\mathsf{Sp}}^{\otimes})$. So by Example 5.10 the inclusion $i : \underline{\mathsf{Sp}}^{\otimes} \subseteq \underline{\mathsf{Sp}}^{\otimes}_{\mathsf{Glo}}$ induces an equivalence of global spectra

$$\mathsf{pic}_{\mathsf{eqv}}(\mathbb{S}_{\mathsf{gl}}) \simeq \mathsf{pic}^{\mathsf{Orb}}_{\mathsf{Glo}}(\underline{\mathsf{Sp}}^{\otimes}) \xrightarrow{\simeq} \mathsf{pic}^{\mathsf{Orb}}_{\mathsf{Glo}}(\underline{\mathsf{Sp}}^{\otimes}_{\mathsf{Glo}}) \simeq \mathsf{pic}_{\mathsf{gl}}(\mathbb{S}_{\mathsf{gl}}).$$

Construction 5.16. Finally, in the equivariant case one considers the composite

$$\mathsf{pic}_G : \mathsf{UCom}_G \xrightarrow{\Phi_G} \mathrm{CAlg}_G(\underline{\mathsf{Sp}}^{\otimes}_G) \xrightarrow{\mathsf{pic}_{\mathsf{Orb}_G}} \mathsf{Mack}_G(\mathsf{Sp}) \simeq \mathsf{Sp}^G.$$

This functor sends a strictly commutative G-ring spectrum R to the G-spectrum $\mathsf{pic}_G(R) \in \mathsf{Sp}^G$ with

[23] Secretly, since we are dealing with large categories here, we also use presentability of $\mathsf{Sp}^{G\text{-}\mathsf{gl}}$ here in conjunction with Remark 5.3.

$\mathsf{pic}_G(R)^H \simeq \mathsf{pic}(\mathsf{Mod}_{\mathrm{res}^G_H R}(\mathsf{Sp}^H))$.

Lemma 5.17. Let $R \in \mathrm{Sect}^{\mathbb{F}^{\mathrm{op}}_{\mathsf{Orb}}}(\underline{\mathsf{Sp}}^{\otimes})$. For every finite group G, there is an equivalence $\mathrm{res}_G\, \mathsf{pic}^{\mathsf{Orb}}_{\mathsf{Glo}}(R) \simeq \mathsf{pic}_{\mathsf{Orb}_G}(\mathrm{res}_G R)$. In particular, given $R \in \mathsf{UCom}$ with $\mathrm{res}_G R \in \mathsf{UCom}_G$, there is an equivalence $\mathrm{res}_G\, \mathsf{pic}_{\mathsf{eqv}}(R) \simeq \mathsf{pic}_G(\mathrm{res}_G R)$ in Sp^G.

Proof. We want to apply Lemma 2.39 to the inclusion $\mathrm{Span}(G) \to \mathrm{Span}_{\mathsf{all},\mathsf{Orb}}(\mathbb{F}_{\mathsf{Glo}})$ induced by $\mathsf{Orb}_G \simeq \mathsf{Orb}_{/G} \to \mathsf{Orb} \subset \mathsf{Glo}$. Analogously to our construction of $\mathsf{pic}_{\mathsf{eqv}}$ above, this is possible as we do not actually need our algebras to be cocartesian on $\mathbb{F}^{\mathrm{op}}_{\mathsf{Glo}}$, and can instead consider $\mathrm{Sect}^{\mathbb{F}^{\mathrm{op}}_{\mathsf{Orb}}}(\underline{\mathsf{Sp}}^{\otimes})$ which restricts via i to $\mathrm{CAlg}_G(\underline{\mathsf{Sp}}^{\otimes}_G)$. The same proof as in Lemma 2.39 then gives a natural equivalence $i^* \circ \underline{\mathsf{Mod}}_{(-)}(\underline{\mathsf{Sp}}^{\otimes}) \simeq \underline{\mathsf{Mod}}_{(-)}(\underline{\mathsf{Sp}}^{\otimes}_G) \circ i^*$ of functors $\mathrm{Sect}^{\mathbb{F}^{\mathrm{op}}_{\mathsf{Orb}}}(\underline{\mathsf{Sp}}^{\otimes}) \to \mathsf{Mack}_G(\widehat{\mathsf{Cat}}(\mathsf{sift}))$. Postcomposing with pic_* yields the claim. $\square$

6 Outlook

Let us mention some more statements which would be interesting to investigate; we plan to come back to some of them in future work. The main problem of interest was already stated in Conjecture 4.24, which would give a purely higher categorical way to reason about ultracommutative ring spectra. Moreover, the same methods we used to construct $\underline{\mathsf{Sp}}^{\otimes}$ can be used to construct equivariantly symmetric monoidal functors $\underline{\mathsf{Spc}}^{\times} \xrightarrow{(-)_+} \underline{\mathsf{Spc}_*}^{\wedge} \xrightarrow{\Sigma^\infty} \underline{\mathsf{Sp}}^{\otimes}$ in $\mathsf{Mack}^{\mathsf{Orb}}_{\mathsf{Glo}}(\widehat{\mathsf{Cat}}(\mathsf{sift}))$. One should then be able to argue that the latter two are essentially uniquely determined by the $\underline{\mathsf{Spc}}^{\times}$, which allows for an easy comparison with the restriction of the globally symmetric monoidal global category of equivariant spectra $\mathcal{SH}^{\otimes} \in \mathsf{Mack}_{\mathsf{Glo}}(\widehat{\mathsf{Cat}}(\mathsf{sift}))$ constructed in [BH17, Section 9]. In fact, it is like that one can carry out their construction analogously to also define a globally symmetric monoidal enhancement of $\underline{\mathsf{Sp}}^{\otimes}_{\mathsf{Glo}}$ with forwards surjections encoding geometric fixed points. Notice that since Glo is only orbital and not atomic orbital, this does not fit into the framework of P-commutative T-monoids in $\underline{\mathsf{Cat}}_T$ from [CLL23a] or the parametrized higher algebra developed by Nardin-Shah, which is one advantage of the added generality of our framework.

Next, it would be interesting to investigate distributivity for P-symmetric monoidal T-categories using bispans as in [EH21]. Ideally, this would allow one to formulate universal properties for the equivariantly (and possibly globally) symmetric monoidal structures on $\underline{\mathsf{Sp}}$ and $\underline{\mathsf{Sp}}_{\mathsf{Glo}}$ similar to Nardin's characterization of $\underline{\mathsf{Sp}}^{\otimes}_G$ as the unique G-distributive G-symmetric monoidal structure with unit $\mathbb{S}$ on $\underline{\mathsf{Sp}}_G$ in [Nar17]. It seems feasible to adapt Nardin's strategy and show that for $P \subset T$ *both* orbital the T-category of P-presentable T-categories $\underline{\mathsf{Pr}}^P_T$ from [Lin24, Definition 2.11] admits a suitable P-symmetric monoidal structure, and then consider idempotent algebras. Relatedly, one should be able to show that the P-relative cocompletion adjunction $\mathcal{P}^P_T : \mathsf{Pr}^P_T \to \mathsf{Pr}^L_T : \mathrm{fgt}$ enhances to a P-symmetric monoidal adjunction, which in the case $(T,P) = (\mathsf{Glo},\mathsf{Orb})$ would give us a way to construct $\underline{\mathsf{Sp}}^{\otimes}_{\mathsf{Glo}}$ as the “globalization” of $\underline{\mathsf{Sp}}^{\otimes}$.

A Extensive and Atomic Orbital Categories

Definition A.1 (Summand Inclusion, [CLL23a, 4.1.3])**.** A map $f : A \to B$ in a category $\mathcal{C}$ is called a (disjoint) *summand inclusion* if there is another morphism $g : C \to B$ such that f and g exhibit B as a coproduct of A and C. Generally $\mathcal{C}^{\mathsf{si}} \subset \mathcal{C}$ denotes the wide subcategory of summand inclusions.

Definition A.2. Let $\mathcal{C}$ be a category with finite coproducts. We call $\mathcal{C}$ extensive if in every commutative diagram of the form

$$\begin{tikzcd} X_Y \arrow[r] \arrow[d] & X \arrow[d] & X_Z \arrow[l] \arrow[d] \\ Y \arrow[r] & Y \sqcup Z & Z \arrow[l] \end{tikzcd} \tag{16}$$

the two squares are pullbacks if and only if the top row is a coproduct diagram. In particular, pullbacks along summand inclusions exist.

Notation A.3. For a category T we denote by $\mathbb{F}_T$ the free finite-coproduct completion of T, the "category of finite T-sets". Formally, this can be defined as the full subcategory of $\mathrm{PSh}(T)$ on the objects which are finite coproducts of objects in the image of $よ : T^{\mathsf{op}} \subseteq \mathrm{PSh}(T)$, which then restricts to $T \subseteq \mathbb{F}_T$. For a wide subcategory $P \subset T$ we write $\mathbb{F}_T^P \subset \mathbb{F}_T$ for the wide subcategory on those morphisms that are finite coproducts of morphisms of the form $(p_i) : \coprod_{i=1}^n A_i \to B$ where each p_i lies in P. Note that $\mathbb{F}_P = \mathbb{F}_P^P = \mathbb{F}_T^P$.

Lemma A.4. Let $\mathcal{C}$ be an extensive category.

1. Coproducts in $\mathcal{C}$ are disjoint, every morphism $X \to \varnothing$ is an equivalence, every summand inclusion in $\mathcal{C}$ is a monomorphism, and $\mathcal{C}^{\mathsf{si}} \subset \mathcal{C}$ is left-cancellable.

2. $\mathcal{C}_{/X}$ is extensive for every $X \in \mathcal{C}$. If $\mathcal{C}^0 \subseteq \mathcal{C}$ is a full subcategory such that $X \sqcup Y \in \mathcal{C}^0$ if and only if $X, Y \in \mathcal{C}^0$, then also $\mathcal{C}^0$ is extensive.

3. For $Y, Z \in \mathcal{C}$ taking coproducts induces an equivalence $\mathcal{C}_{/Y} \times \mathcal{C}_{/Z} \xrightarrow{\simeq} \mathcal{C}_{/(Y \sqcup Z)}$.[24] The inverse sends takes pullbacks along $Y \to Y \sqcup Z$ respectively $Z \to Y \sqcup Z$, sending X to (X_Y, X_Z) in the notation of Definition A.2.

4. $\mathbb{F}_T$ is extensive for any category T. The connected objects, i.e. those $X \in \mathbb{F}_T$ where $\mathbb{F}_T(X, -)$ preserves finite coproducts, are exactly those in the image of $T \subseteq \mathbb{F}_T$. For $X \simeq \coprod_{i=1}^n X_i$ with $X_i \in T$, the summand inclusions induce equivalences

$$\coprod_{i=1}^n T_{/X_i} \xrightarrow{\simeq} T_{/(\coprod_{i=1}^n X_i)} \qquad \text{respectively} \qquad \prod_{i=1}^n (\mathbb{F}_T)_{/X_i} \xrightarrow{\simeq} (\mathbb{F}_T)_{/(\coprod_{i=1}^n X_i)}.$$

5. The free finite-coproduct completion $\mathsf{Cat} \to \mathsf{Cat}_{\sqcup}, T \mapsto \mathbb{F}_T$ preserves fully faithful functors.

[24] In fact, given that $\mathcal{C}$ admits finite coproducts, this is equivalent to $\mathcal{C}$ being extensive, see [CLW93, Proposition 2.2].

Proof. 1. Disjointness of coproducts follows from the definition by taking the coproduct-decomposition $Y = Y \sqcup \varnothing$ in the top row of Diagram (16). Similarly, given $X \to \varnothing$, we put the identities on X in the top row of Diagram (16) and the identities on $\varnothing$ in the bottom row. Then both squares are clearly pullbacks, and hence $X \sqcup X \simeq X$. It follows that $\mathcal{C}(X,X) \simeq \mathcal{C}(X \sqcup X, X) \simeq \mathcal{C}(X,X)^2$ and hence $\mathcal{C}(X,X)$ is contractible. Thus $\varnothing \to X$ and $X \to \varnothing$ are mutual inverses. Next, if $i : c \to c \sqcup d$ is a summand inclusion, then we have pullback squares

$$\begin{tikzcd} c \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, very near start, "\lrcorner"] & c \arrow[d, "i"] \arrow[dr, phantom, very near end, "\llcorner"] & \varnothing \arrow[l] \arrow[d] \\ c \arrow[r, "i"] & c \sqcup d & d \arrow[l] \end{tikzcd}$$

the left of which implies that $\Delta_i : c \to c \times_{c \sqcup d} c$ is an equivalence, hence i is a monomorphism, i.e. (-1)-truncated, cf. [Lur09, 5.5.6.15]. Finally, let $a \xrightarrow{f} b \xrightarrow{g} c$ be morphisms in $\mathcal{C}$ such that gf and g are summand inclusions. Since summand inclusions are stable under pullback, $f : a \to b$ factors as $(\mathrm{id}_a, f) : a \to a \times_c b$ followed by the summand inclusion $a \times_c b \to b$. But (id_a, f) is a section to the summand inclusion $a \times_c b \to a$ and hence an equivalence. Hence f is a summand inclusion as desired.

2. Coproducts and pullbacks in $\mathcal{C}_{/X}$ are computed in $\mathcal{C}$, so this is clear. Analogously

3. Since $X \to Y \sqcup Z$ is the image of $(X_Y \to Y, X_Z \to Z)$ in the notation of Diagram (16), the functor is essentially surjective. For $A, A' \to Y$ in $\mathcal{C}_{/Y}$ and $B, B' \to Z$ in $\mathcal{C}_{/Z}$ the induced map on mapping spaces is a product of the map

$$\begin{aligned} \mathcal{C}_{/Y}(A, A') &\simeq \mathcal{C}(A, A') \times_{\mathcal{C}(A,Y)} \{A \to Y\} \\ &\to \mathcal{C}(A, A' \sqcup B') \times_{\mathcal{C}(A, Y \sqcup Z)} \{A \to Y \to Y \sqcup Z\} \\ &\simeq \mathcal{C}_{/Y \sqcup Z}(A, A' \sqcup B') \end{aligned}$$

with the analogous map $\mathcal{C}_{/Y}(B, B') \to \mathcal{C}_{/Y \sqcup Z}(B, A' \sqcup B')$. The middle morphism is a monomorphism since the summand inclusions $A \to A' \sqcup B'$ and $Y \to Y \sqcup Z$ are. Moreover, it is surjective on π_0 since $Y \times_{Y \sqcup Z} (A' \sqcup B') \simeq A'$, so every morphism in the codomain factors through $A' \to A' \sqcup B'$. Hence it is an equivalence, also showing that $\mathcal{C}_{/Y} \times \mathcal{C}_{/Z} \to \mathcal{C}_{/(Y \sqcup Z)}$ is fully faithful and thus an equivalence.

4. The connectivity of objects in T follows from the Yoneda lemma, since

$$\mathrm{map}(よ(A), よ(B) \sqcup よ(C)) \simeq よ(B)(A) \sqcup よ(C)(A) \simeq \mathrm{map}(A, B) \sqcup \mathrm{map}(A, C).$$

Clearly any nontrivial coproduct is not connected. This connectivity statements tell us the canonical map $\coprod_{i=1}^n T_{/X_i} \to T_{/X}$ is essentially surjective and that the inclusions $X_i \to X$ induce an equivalence $\coprod_i T(A, X_i) \simeq \mathbb{F}_T(A, X)$. So given $A, B \in T_{/X_i}$, the bottom face of the following cube is a pullback

$$\begin{tikzcd}[column sep=small]
 & T_{/X_i}(A,B) \arrow[rr] \arrow[dl] \arrow[dd] \arrow[dr, phantom, very near start, "\lrcorner"] & & T(A,B) \arrow[dl, equal] \arrow[dd] \\
T_{/X}(A,B) \arrow[rr, crossing over] \arrow[dd] \arrow[dr, phantom, very near start, "\lrcorner"] & & T(A,B) & \\
 & \{A \to X_i\} \arrow[rr] \arrow[dl] & & T(A, X_i) \arrow[dl] \\
\{A \to X_i \to X\} \arrow[rr] & & \mathbb{F}_T(A,X) \arrow[from=uu, crossing over] &
\end{tikzcd}$$

By pullback pasting we then get that the top face is also a pullback square, hence the functor $T_{/X_i} \to T_{/X}$ is fully faithful. Now given $A \in T_{/X_i}$ and $B \in T_{/X_j}$ for $i \neq j$, then a map $A \to B$ in $T_{/X}$ would induce $A \to X_i \times_X X_j \simeq \varnothing$, and thus $A \simeq \varnothing$. This shows that the $\coprod_{i=1}^n T_{/X_i} \to T_{/X}$ is fully faithful, giving the first equivalence. The second is obtained by induction from point (3).

5. Recall that we have the covariant presheaf functor $\mathcal{P} : \mathsf{Cat} \to \mathsf{Pr}^L$ for which Yoneda becomes a natural transformation よ $: \mathrm{id} \Rightarrow \mathrm{forget} \circ \mathcal{P}$ by the main result of [Ram23]. Now the Yoneda embedding factors through the full subcategories $\mathbb{F}_T \subseteq \mathcal{P}(T)$, and using Lemma D.3 we obtain unique pointwise fully faithful natural transformations $\mathrm{id} \Rightarrow \mathbb{F}_{(-)} \Rightarrow \mathcal{P}$ which compose to よ $: \mathrm{id} \Rightarrow \mathcal{P}$. Finally if $f : S \hookrightarrow T$ is fully faithful, then we have a commutative diagram

$$\begin{tikzcd}
S \arrow[r, hook] \arrow[d, hook, "f"'] & \mathbb{F}_S \arrow[r, hook] \arrow[d] & \mathcal{P}(S) \arrow[d, hook, "f_!"] \\
T \arrow[r, hook] & \mathbb{F}_T \arrow[r, hook] & \mathcal{P}(T)
\end{tikzcd}$$

and so the middle vertical map is also fully faithful.

□

Definition A.5 ([CLL23a, 4.2.2, 4.3.1])**.** Let T be a category and $P \subset T$ a wide subcategory.

1. We say P is *orbital in* T if the base change of a morphism in $\mathbb{F}_T^P$ along an arbitrary morphism in $\mathbb{F}_T$ exists and is again in $\mathbb{F}_T^P$.

2. We say P is *atomic orbital in* T if furthermore for every morphism $p : A \to B$ in P the diagonal $\Delta_p : A \to A \times_B A$ is a summand inclusion.

In the case that $P = T$ we simply say that T is (atomic) orbital. Note that if $P \subset T$ is (atomic) orbital in T, then P is (atomic) orbital in P itself.

Lemma A.6 ([CLL23a, 4.3.2])**.** An orbital subcategory $P \subset T$ is atomic if and only if every morphism $p : A \to B$ in P which admits a section in T is an equivalence.

Proposition A.7 ([NS22, 2.5.1])**.** Let T be an atomic orbital category that admits a final object. Then T is equivalent to the nerve of a 1-category.

Example A.8.

- The minimal wide subcategory $T^{\simeq} \subset T$ is atomic orbital.
- For a finite group G, the category of finite G-sets $\mathbb{F}_G \simeq \mathbb{F}_{\mathsf{Orb}_G}$ is atomic orbital.
- The global indexing category for finite groups Glo is defined as the Duskin Nerve of the $(2,1)$-category of finite groups, group homomorphisms and conjugations. Equivalently, it is the full subcategory $\mathsf{Glo} \subseteq \mathsf{Spc}$ on the finite connected 1-groupoids, and hence $\mathbb{F}_{\mathsf{Glo}} \subseteq \mathsf{Spc}$ is the $(2,1)$-category of finite 1-groupoids, which admits pullbacks, cf. [CLL23a, 4.2.5]. So Glo is orbital, however the maximal *atomic* orbital subcategory is given by $\mathsf{Orb} \subset \mathsf{Glo}$, the wide subcategory on the injective group homomorphisms / faithful functors of groupoids, cf. [CLL23a, 4.3.3]
- Many other examples are listed in [Nar16, 4.2].

For $X \in \mathbb{F}_T$ we let π_X denote the forgetful maps $(\mathbb{F}_T)_{/X} \to \mathbb{F}_T$ and $T_{/X} \coloneqq T \times_{\mathbb{F}_T} (\mathbb{F}_T)_{/X} \to T$ For $P \subset T$ we will denote by $\pi_X^{-1}(P) \subset T_{/X}$ the wide subcategory of those morphisms sent into P by π_X. Note that $P_{/X} \subseteq \pi_X^{-1}(P)$ is generally a strict full subcategory. For $\in \mathbb{F}_T$, we also denote by $\underline{\mathbb{F}}_T^P(X) \subseteq \underline{\mathbb{F}}_T(X) = (\mathbb{F}_T)_{/X}$ the full subcategory on the morphisms in $\mathbb{F}_T^P$. Note that we have inclusions

$$\mathbb{F}_T^{\mathsf{si}} \subset \mathbb{F}_T^P \qquad \text{and} \qquad (\mathbb{F}_T)_{/X} = \underline{\mathbb{F}}_T(X) \supseteq \underline{\mathbb{F}}_T^P(X) \supset (\mathbb{F}_T^P)_{/X} \subseteq \mathbb{F}_{T_{/X}}^{\pi_X^{-1}(P)}.$$

Lemma A.9. Let $P \subset T$ be a wide subcategory.

1. If $X \in \mathbb{F}_T$, then the projection $T_{/X} \coloneqq T \times_{\mathbb{F}_T} (\mathbb{F}_T)_{/X} \to (\mathbb{F}_T)_{/X}$ exhibits $(\mathbb{F}_T)_{/X}$ as the free finite-coproduct completion of $T_{/X}$. In particular, we obtain an equivalence $\mathbb{F}_{T_{/X}} \xrightarrow{\simeq} (\mathbb{F}_T)_{/X}$ This restricts to a fully faithful inclusion $(\mathbb{F}_T^P)_{/X} \subseteq \mathbb{F}_{T_{/X}}^{\pi_X^{-1}(P)}$.
2. If $P \subset T$ is [atomic] orbital, then $\pi_X^{-1}(P) \subset T_{/X}$ and $P_{/X}$ (in itself) are [atomic] orbital for $X \in \mathbb{F}_T$.
3. Let P be atomic orbital and $X \in \mathbb{F}_T$.
   (a) $P \subset T$ is left-cancellable, meaning that for composable f in T and g in P, we have gf in P if and only if f in P.
   (b) $\mathbb{F}_T^P \subset \mathbb{F}_T$ is left-cancellable.
   (c) We have fully faithful inclusions $P_{/X} \subseteq T_{/X}$ and $(\mathbb{F}_T^P)_{/X} = \underline{\mathbb{F}}_T^P(X) \subseteq (\mathbb{F}_T)_{/X}$.
   (d) $(\mathbb{F}_T^P)_{/X} = \underline{\mathbb{F}}_T^P(X)$ is a 1-category for any $X \in \mathbb{F}_T$.

4. Let $P \subset T$ be atomic orbital and $(X \to Y)$, $(U \to X)$, $(V \to X)$ in $\mathbb{F}_T^P$. Then

$$X \to X \times_Y X \qquad \text{and} \qquad U \times_X V \to U \times_Y V$$

are summand inclusions.

Proof.

1. Clearly $(\mathbb{F}_T)_{/X}$ admits finite coproducts computed in $\mathbb{F}_T$, and so $T_{/X} \to (\mathbb{F}_T)_{/X}$ factors uniquely through a finite-coproduct preserving functor $\varphi : \mathbb{F}_{T_{/X}} \to (\mathbb{F}_T)_{/X}$. Clearly φ is essentially surjective, and it is fully faithful by the commutative diagram

$$\begin{tikzcd}
\mathbb{F}_{T_{/X}}(\coprod_i (A_i \to X), \coprod_j (B_j \to X)) \arrow[r, "\varphi"] \arrow[d, "\simeq"] & (\mathbb{F}_T)_{/X}(\coprod_i (A_i \to X), \coprod_j (B_j \to X)) \arrow[d, "\simeq"] \\
\prod_i \mathbb{F}_{T_{/X}}((A_i \to X), \coprod_j (B_j \to X)) \arrow[r, "\varphi"] & \prod_i (\mathbb{F}_T)_{/X}((A_i \to X), \coprod_j (B_j \to X)) \\
\prod_i \coprod_j T_{/X}((A_i \to X), (B_j \to X)) \arrow[r, "\simeq"] \arrow[u, "\simeq"] & \prod_i \coprod_j (\mathbb{F}_T)_{/X}((A_i \to X), (B_j \to X)) \arrow[u, "\simeq"]
\end{tikzcd}$$

2. For orbitality, note that pullbacks and coproducts in $\mathbb{F}_{T_{/X}} \simeq (\mathbb{F}_T)_{/X}$ are computed in $\mathbb{F}_T$, and this forgetful map sends $\mathbb{F}_{T_{/X}}^{\pi_X^{-1}(P)}$ to $\mathbb{F}_T^P$ so it follows from orbitality of P. Analogously, $\mathbb{F}_{P_{/X}} \simeq (\mathbb{F}_P)_{/X}$ has pullbacks. For atomicity, we use Lemma A.6. So if $p : (A \to X) \to (C \to X)$ is a morphism in $\pi_X^{-1}(P)$ which admits a section in $T_{/X}$, then applying π_X we get that $A \to C$ is a morphism in P admitting a section in T, and is hence an equivalence. But $\pi_{/X}$ is conservative, so p is an equivalence. Of course this also verifies atomicity of $P_{/X}$.

3. Part (a) is shown in [CLL23a, 4.3.5]. For part (b), by definition of maps in $\mathbb{F}_T^P$, it suffices to restrict to one summand in the domain. Let $f : A \to \coprod_i B_i$ and $g : \coprod_i B_i \to \coprod_j C_j$ so $A \in T$ and that gf and g are in $\mathbb{F}_T^P$. Since A is connected by Lemma A.4, f factors through precisely one of the B_i, and analogously $g|_{B_i}$ then lands in precisely one C_j, which reduces us to to (a). Part (c) is then clear. Finally, note that for $X = \coprod_{i=1}^n X_i$ we have

$$\underline{\mathbb{F}}_T^P(X) = (\mathbb{F}_T^P)_{/X} = (\mathbb{F}_P)_{/X} \simeq \prod_{i=1}^n (\mathbb{F}_P)_{/X_i} \simeq \prod_{i=1}^n \mathbb{F}_{P_{/X_i}}$$

where each $P_{/X_i}$ is atomic orbital by (2) and has a final object id_{X_i}, hence is a 1-category by Proposition A.7.

4. This is [CLL23a, 4.3.7, 4.9.1].

$\square$

Lemma A.10. Let $P \subset T$ be orbital.

1. We have a parametrized adequate triple $(\underline{\mathbb{F}}_T^P, (\underline{\mathbb{F}}_T^P)^{\mathsf{si}}, \underline{\mathbb{F}}_T^P) : T^{\mathsf{op}} \to \mathsf{AdTrip}$.
2. If P is furthermore atomic, then there is an equivalence of T-categories

$$\mathrm{Span}_{\mathsf{si},\mathsf{all}}(\underline{\mathbb{F}}_T^P) \simeq \underline{\mathbb{F}}_{T,*}^P$$

which at $B \in T$ sends a span $X \overset{\mathsf{si}}{\leftarrow} Z \xrightarrow{f} Y$ over B to the morphism $X \sqcup B = Z \sqcup Z' \sqcup B \to Y \sqcup B$ which on Z is given by f and on $Z' \sqcup B$ by the structure map to B.

Proof. Let $X \in \mathbb{F}_T$ and consider a cospan $V \hookrightarrow U \leftarrow W$ in $\underline{\mathbb{F}}_T^P(X)$ where $V \hookrightarrow U$ is a summand inclusion. Then the pullback $V \times_U W$ is computed in $\mathbb{F}_T$, and since summand inclusions are stable under basechange in $\mathbb{F}_T$ the map $V \times_U W \to W \to X$ is again in $\mathbb{F}_T^P$ (which contains summand inclusions). Thus each $(\underline{\mathbb{F}}_T^P(X), \underline{\mathbb{F}}_T^P(X)^{\mathsf{si}}, \underline{\mathbb{F}}_T^P(X))$ is an adequate triple. Moreover, for $f : X \to X'$ the basechange $f^* : \underline{\mathbb{F}}_T^P(X') \to \underline{\mathbb{F}}_T^P(X)$ clearly preserves pullbacks, coproducts and summand inclusions since those are computed in $\mathbb{F}_T$.

For the second point, recall from Lemma A.4, Proposition A.7 and Lemma A.9 that since P is atomic orbital, $\underline{\mathbb{F}}_T^P(X) = (\mathbb{F}_T^P)_{/X}$ is an extensive 1-category admitting a final object. More generally, let $\mathcal{C}$ be any extensive 1-category with final object $*$. Note that under these hypotheses also $\mathrm{Span}_{\mathsf{si},\mathsf{all}}(\mathcal{C})$ is a 1-category. Indeed, since summand inclusions are monomorphisms by Lemma A.4 they do not have any nontrivial automorphisms as objects in $\mathcal{C}_{/X}$, so $(\mathcal{C}^{\mathsf{si}}_{/X})^{\simeq} \simeq \{X_i \to X\}$ is discrete on the summand inclusions $X_i \to \coprod_{i=1}^n X_i = X$. The mapping space formula Lemma B.1 then gives

$$\mathrm{Span}_{\mathsf{si},\mathsf{all}}(\mathcal{C})(X,Y) \simeq (\mathcal{C}^{\mathsf{si}}_{/X})^{\simeq} \times_{\mathcal{C}^{\simeq}} (\mathcal{C}_{/Y})^{\simeq}$$

which is also a set because the fibers of $\mathcal{C}_{/Y} \to \mathcal{C}$ are sets, as $\mathcal{C}$ is a 1-category. In other words, $\mathrm{Span}_{\mathsf{si},\mathsf{all}}(\mathcal{C})$ is a 1-category with objects those of $\mathcal{C}$ and morphisms given by isomorphisms classes of spans $X \overset{\mathsf{si}}{\leftarrow} Z \to Y$. We define $\Phi_{\mathcal{C}} : \mathrm{Span}_{\mathsf{si},\mathsf{all}}(\mathcal{C}) \to \mathcal{C}_*$ by sending a span $X \overset{\mathsf{si}}{\leftarrow} Z \xrightarrow{f} Y$ to the unique morphism of pointed objects $X_+ \to Y_+$ induced by the universal property of $X_+ = X \sqcup *$ as a coproduct of $*, Z$ and some $Z^{\perp}$, which is determined by $Z \xrightarrow{f} Y \to Y_+$ and $Z^{\perp} \sqcup * \to * \to Y_+$.

Indeed, suppose we are given an isomorphism $\varphi : Z \xrightarrow{\cong} V$ from the above span to a span $X \overset{\mathsf{si}}{\leftarrow} V \xrightarrow{g} Y$. In this case there also exists $\psi : Z^{\perp} \xrightarrow{\cong} V^{\perp}$ commuting with the inclusions into X. One constructs this using Diagram (16) with bottom row $V^{\perp} \to X \leftarrow V$ then pulling back along $Z^{\perp} \to X$. The right square will have empty pullback since $V \to X$ factors through $Z \to X$. But the top row needs to be a coproduct decomposition, hence the top left arrow will be an isomorphism, and so we get a map $Z^{\perp} \to V^{\perp}$ commuting with inclusion into X. Dually we get $V^{\perp} \to Z^{\perp}$, and then we use that summand inclusions are monomorphisms to see the composites are identities. Given this it follows that the induced morphisms $X_+ \to Y_+$ agree. For composition one does a similar argument.

We now show that $\Phi_{\mathcal{C}}$ is fully faithful. Given $X, Y \in \mathcal{C}$ and a morphism $f : X_+ \to Y_+$, taking pullbacks yields by extensivity of $\mathcal{C}$ a commutative diagram where all faces are cartesian

$$\begin{tikzcd}
 & \varnothing \arrow[rr, equals] \arrow[dl] \arrow[dd] & & \varnothing \arrow[rr, equals] \arrow[dl] \arrow[dd] & & \varnothing \arrow[dl] \arrow[dd] \\
Z \arrow[rr, equals] \arrow[dd, "\mathsf{si}"] & & Z \arrow[rr, "f"] \arrow[dd, "\mathsf{si}"] & & Y \arrow[dd, "\mathsf{si}"] & \\
 & Z^{\perp} \arrow[rr] \arrow[dl] & & Z^{\perp} \sqcup * \arrow[rr] \arrow[dl] & & * \arrow[dl] \\
X \arrow[rr] & & X_+ \arrow[rr, "f"'] & & Y_+ &
\end{tikzcd}$$

In particular we see that the span $X \overset{\mathsf{si}}{\leftarrow} Z \xrightarrow{f} Y$ gets sent to f_+ by $\Phi_{\mathcal{C}}$, proving surjectivity on hom-sets. But conversely, if we start with the span $X \overset{\mathsf{si}}{\leftarrow} Z \to Y$ and the coproduct decomposition $Z \sqcup Z^{\perp} = X$, then by extensivity we see that all the squares in the above diagram are cartesian, which shows that the span is unique up to isomorphism, proving injectivity.

Now suppose that $\mathcal{D}$ is another extensive 1-category with final object, and let $F : \mathcal{C} \to \mathcal{D}$ be a functor preserving this structure. Analogous to the above one checks that $F_* \circ \Phi_{\mathcal{C}} = \Phi_{\mathcal{D}} \circ \mathrm{Span}_{\mathsf{si},\mathsf{all}}(F)$ and thus we obtain a pointwise fully faithful natural transformation $\Phi : \mathrm{Span}_{\mathsf{si},\mathsf{all}}(-) \Rightarrow (-)_*$ of functors $\mathsf{Cat}^{*}_{\mathsf{Ext},1} \to \mathsf{Cat}^{\mathrm{pt}}_{\sqcup,1}$ out of the category of extensive 1-categories admitting final objects into the category of pointed 1-categories admitting finite coproducts. (Note that in nontrivial pointed categories the initial object is not strict, hence they are not extensive anymore).

Coming back to our original example, We saw in the first point that that all structure morphisms of $\underline{\mathbb{F}}^P_T$ preserve finite coproducts and clearly they also preserve the final object. Furthermore, it was shown in [CLL23a, 4.7.3] that

$$(-)_+ : \underline{\mathbb{F}}^P_T(X) \to \underline{\mathbb{F}}^P_{T,*}(X), Y \mapsto Y \sqcup X$$

is essentially surjective. It follows that Φ induces a natural equivalence $\mathrm{Span}_{\mathsf{si},\mathsf{all}}(\underline{\mathbb{F}}^P_T) \simeq \underline{\mathbb{F}}^P_{T,*}$ as desired. $\square$

B Span Categories

Lemma B.1 (Mapping space formula)**.** There is an equivalence $\mathrm{Span}_{b,f}(\mathcal{X})^{\simeq} \simeq \mathcal{X}^{\simeq}$ and a commutative diagram

$$\begin{tikzcd}
\mathsf{Cat}(\Delta^1, \mathrm{Span}_{b,f}(\mathcal{X})) \arrow[rr, "\simeq"] \arrow[dr, "(\mathrm{ev}_0,\mathrm{ev}_1)"'] & & \mathsf{Cat}(\Delta^1, \mathcal{B}) \times_{\mathrm{ev}_0, \mathcal{X}^{\simeq}, \mathrm{ev}_0} \mathsf{Cat}(\Delta^1, \mathcal{F}) \arrow[dl, "(\mathrm{ev}_1,\mathrm{ev}_1)"] \\
& \mathcal{X}^{\simeq} \times \mathcal{X}^{\simeq} &
\end{tikzcd}$$

both of which are natural in $(\mathcal{X}, \mathcal{B}, \mathcal{F}) \in \mathsf{AdTrip}$. In particular, given a morphism of adequate triples $F : (\mathcal{X}, \mathcal{B}, \mathcal{F}) \to (\mathcal{Y}, \mathcal{Y}^b, \mathcal{Y}^f)$ we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{map}_{\mathrm{Span}_{b,f}(\mathcal{X})}(x, y) \arrow[r, "\mathrm{Span}(F)"] \arrow[d, "\simeq"'] & \mathrm{map}_{\mathrm{Span}_{\ell,r}(\mathcal{Y})}(Fx, Fy) \arrow[d, "\simeq"] \\
(\mathcal{B}_{/x} \times_{\mathcal{X}} \mathcal{F}_{/y})^{\simeq} \arrow[r, "(F,F)^{\simeq}"'] & ((\mathcal{Y}^b)_{/Fx} \times_{\mathcal{Y}} (\mathcal{Y}^f)_{/Fy})^{\simeq}
\end{tikzcd}$$

Moreover, composition is given by pullback, i.e. is identified with the map on groupoids induced by

$$\begin{aligned}
\mathcal{B}_{/x} \times_{\mathcal{X}} \mathcal{F}_{/y} \times \mathcal{B}_{/y} \times_{\mathcal{X}} \mathcal{F}_{/z} &\to \mathcal{B}_{/x} \times_{\mathcal{B},t} \mathrm{Ar}(\mathcal{B}) \times_{s,\mathcal{X},s} \mathrm{Ar}(\mathcal{F}) \times_{t,\mathcal{F}} \mathcal{F}_{/z} \\
&\to \mathcal{B}_{/x} \times_{\mathcal{X}} \mathcal{F}_{/z}
\end{aligned}$$

where the first functor takes the pullback over y and the second composes the morphisms in $\mathcal{B}$ respectively $\mathcal{F}$, i.e. we map $(x \leftarrow a \to y \leftarrow b \to z)$ to $x \leftarrow a \times_y b \to z$.

Proof. We can view $\mathrm{Span}_{b,f}(\mathcal{X})$ as the complete Segal space

$$\Delta^{\mathsf{op}} \to \mathsf{Spc}, \ \ [n] \mapsto \mathsf{Cat}([n], \mathrm{Span}_{b,f}(\mathcal{X})) \simeq \mathsf{AdTrip}(\mathrm{Tw}^r[n], (\mathcal{X}, \mathcal{B}, \mathcal{F}))$$

using the adjunction $\mathrm{Tw}^r \dashv \mathrm{Span}$ from Theorem 2.11 (In fact Span was defined this way in [HHLN23b]). Following [HHLN23b, Example 2.9] we depict $\mathrm{Tw}^r[2]$ as

$$\begin{tikzcd}[column sep=small]
& & & 0 \leq 2 \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee" very near start] & & & \\
& & 0 \leq 1 \arrow[dl] \arrow[dr] & & 1 \leq 2 \arrow[dl] \arrow[dr] & & \\
& 0 \leq 0 & & 1 \leq 1 & & 2 \leq 2 &
\end{tikzcd}$$

The backwards morphisms are the left-pointing ones, and the forwards morphisms are the right-pointing ones. The square is the only nontrivial pullback of a forwards along a backwards morphism. We can identify $\mathrm{Tw}^r[1]$ with the full subcategory on the objects $0 \leq 1,\ 0 \leq 2$ and $1 \leq 1$. Note that $\mathrm{Tw}^r[1]$ has no nontrivial pullbacks of forwards morphisms along backwards morphisms, so that any functor of triples $\mathrm{Tw}^r[1] \to (\mathcal{X}, \mathcal{B}, \mathcal{F})$ is automatically a map of adequate triples. We obtain natural equivalences

$$\begin{aligned}
\mathsf{Cat}([1], \mathrm{Span}_{b,f}(\mathcal{X})) &\simeq \mathsf{AdTrip}(\mathrm{Tw}^r[1], (\mathcal{X}, \mathcal{B}, \mathcal{F})) \\
&\simeq \mathsf{Cat}^{\Lambda^2_2}(\mathrm{Tw}^r[1], (\mathcal{X}, \mathcal{B}, \mathcal{F})) \\
&\simeq \mathsf{Cat}([1], \mathcal{B}) \times_{\mathsf{Cat}([1],\mathcal{X})} \mathsf{Cat}(\Lambda^2_0, \mathcal{X}) \times_{\mathsf{Cat}([1],\mathcal{X})} \mathsf{Cat}([1], \mathcal{F}) \\
&\simeq \mathsf{Cat}([1], \mathcal{B}) \times_{\mathrm{ev}_0, \mathcal{X}^{\simeq}, \mathrm{ev}_0} \mathsf{Cat}([1], \mathcal{F}).
\end{aligned}$$

The remaining claims about the mapping spaces are then clear. Now for any category $\mathcal{C}$ general composition is given by

$$\mathsf{Cat}([1], \mathcal{C}) \times_{\mathcal{C}^{\simeq}} \mathsf{Cat}([1], \mathcal{C}) \xleftarrow[\simeq]{(\mathrm{ev}_{0\to 1}, \mathrm{ev}_{1\to 2})} \mathsf{Cat}([2], \mathcal{C}) \xrightarrow{\mathrm{ev}_{0\to 2}} \mathsf{Cat}([1], \mathcal{C}).$$

Applying the adjunction $\mathrm{Tw}^r \dashv \mathrm{Span}$ in our case shows that this is given by extending the horizontal zig-zag of the above diagram in $\mathcal{X}$ to a diagram defined on all of $\mathrm{Tw}^r[2]$, and then again restricting along $\mathrm{Tw}^r[1] \to \mathrm{Tw}^r[2]$ induced by $d_1 : [1] \to [2]$. Clearly this latter restriction is given by composing the long diagonal edges of such a diagram $\mathrm{Tw}^r[2] \to (\mathcal{X}, \mathcal{B}, \mathcal{F})$, and since maps of adequate triples preserve ambigressive pullbacks, the unique extension is simply given by taking pullbacks. It follows that under the whole mapping space equivalence from above the composition map agrees with the claimed one. □

Lemma B.2. Let $\mathcal{C}$ be an extensive category and $m \subset \mathcal{C}$ a wide subcategory closed under finite coproducts, base change and containing all fold maps.

1. The inclusion $\mathcal{C}^{\mathsf{op}} \to \mathrm{Span}_{\mathsf{all},m}(\mathcal{C})$ preserves and creates products, in the sense that a functor $F : \mathrm{Span}_{\mathsf{all},m}(\mathcal{C}) \to \mathcal{D}$ preserves finite products if and only if its restriction to $\mathcal{C}^{\mathsf{op}}$ does. In particular, the backwards summand inclusions

   $$\rho_X = (X \sqcup Y \leftarrow X = X) : (X \sqcup Y) \to X \qquad \text{and} \qquad \rho_Y = (X \sqcup Y \leftarrow Y = Y) : (X \sqcup Y) \to Y$$

   exhibit $X \sqcup Y$ as product of X and Y in $\mathrm{Span}_{\mathsf{all},m}$.

2. The inclusion $m \to \mathrm{Span}_{\mathsf{all},m}(\mathcal{C})$ preserves and creates coproducts.

3. $\mathrm{Span}_{\mathsf{all},m}(\mathcal{C})$ is semiadditive.

4. If $(\mathcal{D}, n)$ is another such pair with the same properties as $(\mathcal{C}, m)$, and $F : \mathcal{C} \to \mathcal{D}$ is a functor sending m into n which preserves finite coproducts and basechanges along m, then the induced

functor $\mathrm{Span}(F) : \mathrm{Span}_{\mathsf{all},m}(\mathcal{C}) \to \mathrm{Span}_{\mathsf{all},n}(\mathcal{D})$ preserves finite biproducts.

Proof. The second point follows immediately from the proof [BH17, Lemma C.3]. Under the equivalence $\mathrm{Span}_{\mathsf{all},m}(\mathcal{C})^{\mathsf{op}} \simeq \mathrm{Span}_{m,\mathsf{all}}(\mathcal{C})$ from [HHLN23b, Lemma 2.14], the first and third point also follow from said lemma. The fourth point is an immediate consequence, since we can now check whether $\mathrm{Span}(F)$ preserves finite products on $F^{\mathsf{op}} : \mathcal{C}^{\mathsf{op}} \to \mathcal{D}^{\mathsf{op}}$. $\square$

Lemma B.3. Let $(\mathcal{X}, \mathcal{B}, \mathcal{F})$ be an adequate triple, and $X \in \mathcal{X}$. The inclusion $\mathcal{B}^{\mathsf{op}} \to \mathrm{Span}_{b,f}(\mathcal{X})$ induces a morphism

$$L : (\mathcal{B}^{\mathsf{op}})_{X/} \to \mathrm{Span}_{b,f}(\mathcal{X})_{X/}$$

which admits a right adjoint R sending a span $X \leftarrow Z \to Y$ to $X \leftarrow Z$ in $\mathcal{B}^{\mathsf{op}}_{X/}$. In particular, L is left cofinal.

Proof. The existence of the adjoint follows from choosing $\mathcal{C} = \mathcal{D} = \mathrm{Span}_{b,f}(\mathcal{X})$ in [Bar23a, A.8], using the canonical backwards/forwards factorization system on $\mathrm{Span}_{b,f}(\mathcal{X})$, which exists by [HHLN23b, 4.9]. However, this does not give the specific description claimed in the lemma, so we give an explicit argument.

Given a span $\varphi = (X \leftarrow B \to A)$ in $\mathrm{Span}_{b,f}(\mathcal{X})_{X/}$, we define the counit-morphism $\varepsilon_\varphi : (X \leftarrow B = B) \to \varphi$ in $\mathrm{Span}_{b,f}(\mathcal{X})_{X/}$ as $B = B \to A$. Indeed, note that the composite $(B = B \to A) \circ (X \leftarrow B = B)$ is simply $(X \leftarrow B \to A)$, and so this trivially gives a commuting triangle in the category $\mathrm{Span}_{b,f}(\mathcal{X})$. By the local existence criterion for adjunctions, it now suffices to show that the following composite is an equivalence:

$$(\mathcal{B}^{\mathsf{op}})_{X/}\left(\begin{smallmatrix} X \\ \uparrow \\ Y \end{smallmatrix}, \begin{smallmatrix} X \\ \uparrow \\ B \end{smallmatrix}\right) \xrightarrow{L} \mathrm{Span}_{b,f}(\mathcal{X})_{X/}\left(\begin{smallmatrix} X \\ \uparrow \\ Y \\ \| \\ Y \end{smallmatrix}, \begin{smallmatrix} X \\ \uparrow \\ B \\ \| \\ B \end{smallmatrix}\right) \xrightarrow{(\varepsilon_\varphi)_*} \mathrm{Span}_{b,f}(\mathcal{X})_{X/}\left(\begin{smallmatrix} X \\ \uparrow \\ Y \\ \| \\ Y \end{smallmatrix}, \begin{smallmatrix} X \\ \uparrow \\ B \\ \downarrow \\ A \end{smallmatrix}\right) \qquad (*)$$

Using the formula for mapping spaces in slice categories as well as Lemma B.1, we see that the middle space in the above composite is computed by first taking horizontal and then vertical pullbacks in the following diagram

$$\begin{tikzcd}
\{B \to X\} \arrow[r] \arrow[d] & \{B\} \arrow[d] & \{\mathrm{id}_B\} \arrow[l] \arrow[d] \\
(\mathcal{B}_{/X})^{\simeq} \arrow[r] & \mathcal{C}^{\simeq} & (\mathcal{F}_{/B})^{\simeq} \arrow[l] \\
(\mathcal{B}_{/Y})^{\simeq} \arrow[r] \arrow[u] & \mathcal{C}^{\simeq} \arrow[u, equal] & (\mathcal{F}_{/B})^{\simeq} \arrow[l] \arrow[u, equal]
\end{tikzcd}$$

If we instead first take the pullback vertically and then horizontally, this proves

$$\mathrm{Span}_{b,f}(\mathcal{X})_{X/}\left(\begin{array}{c} X \\ \uparrow \\ Y \\ \| \\ Y \end{array} , \begin{array}{c} X \\ \uparrow \\ B \\ \| \\ B \end{array} \right) \simeq \mathcal{B}_{/X}\left(\begin{array}{c} B \\ \downarrow \\ X \end{array} , \begin{array}{c} Y \\ \downarrow \\ X \end{array} \right) \times_{\{B\}} \{\mathrm{id}_B\} \simeq \mathcal{B}_{/X}\left(\begin{array}{c} B \\ \downarrow \\ X \end{array} , \begin{array}{c} Y \\ \downarrow \\ X \end{array} \right).$$

Analogously one shows

$$\mathrm{Span}_{b,f}(\mathcal{X})_{X/}\left(\begin{array}{c} X \\ \uparrow \\ Y \\ \| \\ Y \end{array} , \begin{array}{c} X \\ \uparrow \\ B \\ \downarrow \\ A \end{array} \right) \simeq \mathcal{B}_{/X}\left(\begin{array}{c} B \\ \downarrow \\ X \end{array} , \begin{array}{c} Y \\ \downarrow \\ X \end{array} \right) \times_{\{B\}} \{B \to A\} \simeq \mathcal{B}_{/X}\left(\begin{array}{c} B \\ \downarrow \\ X \end{array} , \begin{array}{c} Y \\ \downarrow \\ X \end{array} \right).$$

Clearly the composite $(*)$ is then an equivalence. □

The following lemma will appear in [CHLL].

Lemma B.4. Let $f : (\mathcal{X}, \mathcal{X}^b, \mathcal{X}^f) \to (\mathcal{Y}, \mathcal{Y}^b, \mathcal{Y}^f)$ be a map of adequate triples. Assume $\mathcal{X}^f \to \mathcal{Y}^f$ is a right fibration and $\mathcal{X}^b \to \mathcal{Y}^b$ is a localization at some class $W \subset \mathcal{X}^b$. Then $\mathrm{Span}(f) : \mathrm{Span}_{b,f}(\mathcal{X}) \to \mathrm{Span}_{b,f}(\mathcal{Y})$ is a localization at the backwards morphisms which lie in W.

C Astrology[25]

In this appendix we briefly recall some basic results from the calculus of mates (theory of Beck-Chevalley transformations) for the specific case of adjunctions arising from Kan extensions. While this theory can be developed much more generally, internal to any $(\infty, 2)$-category, we will only need this special case throughout the text. For a more extensive overview, we refer the reader to [Cno23b, Appendix F], which can also be found at https://sites.google.com/view/bastiaan-cnossen.

Our setting is as follows. We are given a natural transformation $\alpha : jq \Rightarrow kp$ giving rise to the following lax commutative square in the homotopy 2-category of $(\infty, 1)$-categories on the left, which via precomposition induces the lax commutative square on the right:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "k"] \arrow[d, "q"'] & \mathcal{E} \arrow[d, "p"] \\ \mathcal{D} \arrow[r, "j"'] \arrow[ur, phantom, "\Nearrow \alpha"] & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{X}^{\mathcal{F}} & \mathcal{X}^{\mathcal{E}} \arrow[l, "k^*"'] \\ \mathcal{X}^{\mathcal{D}} \arrow[u, "q^*"] \arrow[ur, phantom, "\Nearrow \alpha"] & \mathcal{X}^{\mathcal{C}} \arrow[l, "j^*"] \arrow[u, "p^*"'] \end{tikzcd} \tag{17}$$

Suppose that $\mathcal{X}$ admits enough limits, so that $p^* : \mathcal{X}^{\mathcal{C}} \to \mathcal{X}^{\mathcal{E}}$ and $q^* : \mathcal{X}^{\mathcal{D}} \to \mathcal{X}^{\mathcal{F}}$ admit right adjoints p_* respectively q_*. We will denote their (co)units by η^p, ε^p respectively η^q, ε^q. This induces

[25] The title is inspired by the syntactic presentation of the content; we will encounter many stars $q^*, q_*, \ldots$

the Beck-Chevalley transformation $BC(\alpha) : j^*p_* \xRightarrow{\eta^q j^* p_*} q_*q^*j^*p_* \xRightarrow{q_*\alpha p_*} q_*k^*p^*p_* \xRightarrow{q_*k^*\varepsilon^p} q_*k^*$ by pasting:

$$\begin{tikzcd}
\mathcal{X}^{\mathcal{F}} \arrow[d, "q_*"'] & \mathcal{X}^{\mathcal{E}} \arrow[l, "k^*"'] \arrow[d, "p_*"] \arrow[dl, phantom, "\Leftarrow BC(\alpha)"] \\
\mathcal{X}^{\mathcal{D}} & \mathcal{X}^{\mathcal{C}} \arrow[l, "j^*"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{X}^{\mathcal{F}} \arrow[dr, bend right, equal] \arrow[dr, phantom, "\Rightarrow \eta^q"] & \mathcal{X}^{\mathcal{F}} \arrow[l, "q_*"'] \arrow[dr, phantom, "\Rightarrow \alpha"] & \mathcal{X}^{\mathcal{E}} \arrow[l, "k^*"'] \arrow[dr, bend left, equal] \arrow[dr, phantom, "\Rightarrow \varepsilon^p"] & \\
& \mathcal{X}^{\mathcal{D}} \arrow[u, "q^*"] & \mathcal{X}^{\mathcal{C}} \arrow[l, "j^*"] \arrow[u, "p^*"'] & \mathcal{X}^{\mathcal{E}} \arrow[l, "p_*"]
\end{tikzcd}$$

Of course there is a dual version in the case where p and q admit left adjoints. A reference for the following lemma is [CSY20, 2.2.4] or also [Cno23b, Lemma F.6], but it essentially follows straight from the definitions.

Lemma C.1. Beck-Chevalley transformations compose horizontally: Given the diagram

$$\begin{tikzcd}
\mathcal{X}^{\mathcal{F}} & \mathcal{X}^{\mathcal{E}} \arrow[l, "k^*"'] & \mathcal{X}^{\mathcal{A}} \arrow[l, "\ell^*"'] \\
\mathcal{X}^{\mathcal{D}} \arrow[u, "q^*"] \arrow[ur, phantom, "\Rightarrow \alpha"] & \mathcal{X}^{\mathcal{C}} \arrow[l, "j^*"] \arrow[u, "p^*"] \arrow[ur, phantom, "\Rightarrow \beta"] & \mathcal{X}^{\mathcal{B}} \arrow[l, "i^*"] \arrow[u, "r^*"']
\end{tikzcd}$$

the transformation $BC(\alpha\beta) : j^*i^*r_* \Rightarrow q_*k^*\ell^*$ is homotopic to the composite

$$j^*i^*r_* \xRightarrow{j^*BC(\beta)} j^*p_*\ell^* \xRightarrow{BC(\alpha)\ell^*} q_*k^*\ell^*.$$

The following lemma gives a more convenient description of the Beck-Chevalley map for adjunctions arising from Kan extensions. The special case for pullbacks of anima is implicitly used in [Lur17, 6.1.6.3].

Lemma C.2. Suppose we have a commutative square of categories as on the left, which induces for a category $\mathcal{X}$ a commutative square as on the right

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "k"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{D} \arrow[r, "j"'] & \mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{Fun}(\mathcal{F}, \mathcal{X}) & \mathsf{Fun}(\mathcal{E}, \mathcal{X}) \arrow[l, "k^*"'] \\
\mathsf{Fun}(\mathcal{D}, \mathcal{X}) \arrow[u, "q^*"] & \mathsf{Fun}(\mathcal{C}, \mathcal{X}) \arrow[l, "j^*"] \arrow[u, "p^*"']
\end{tikzcd}$$

Suppose $\mathcal{X}$ has enough limits so that p^*, q^* admit right adjoints p_*, q_*, and let $F : \mathcal{E} \to \mathcal{X}$ and $d \in \mathcal{D}$. Under the equivalences coming from the pointwise formula for right Kan extensions

$$(p_*F)(jd) \simeq \lim(\mathcal{C}_{jd/} \times_{\mathcal{C}} \mathcal{E} \to \mathcal{E} \xrightarrow{F} \mathcal{X}) \quad \text{and} \quad (q_*k^*F)(d) \simeq \lim(\mathcal{D}_{d/} \times_{\mathcal{D}} \mathcal{F} \to \mathcal{F} \to \mathcal{E} \xrightarrow{F} \mathcal{X})$$

we can identify the Beck-Chevalley map $BC_F(d) : (j^*p_*F)(d) \to (q_*k^*F)(d)$ with the map on limits induced by restricting along $\mathcal{D}_{d/} \times_{\mathcal{D}} \mathcal{F} \to \mathcal{C}_{jd/} \times_{\mathcal{C}} \mathcal{E}$. In particular, if this map is left cofinal for every $d \in \mathcal{D}$, then the existence of p_* already follows from that of q_* and $BC : j^*p_* \Rightarrow q_*k^*$ is an equivalence.

Proof. This proof is an elaboration on Rune Haugseng's answer to the math overflow question https://mathoverflow.net/questions/286886. Define the lax pullback diagram as on the left via the pullbacks on the right, and the canonical natural transformation $\alpha : s \Rightarrow t$ from source to target functor $\mathrm{Ar}(\mathcal{D}) \to \mathcal{D}$:

$$\begin{tikzcd}
\mathcal{D}_{d/} \times_{\mathcal{D}} \mathcal{F} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & ? \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{F} \arrow[d, "q"] \\
\mathcal{D}_{d/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Ar}(\mathcal{D}) \arrow[r, "t"] \arrow[d, "s"'] & \mathcal{D} \\
* \arrow[r, "d"] & \mathcal{D} \arrow[ur, phantom, "\overset{\alpha}{\Nearrow}"] \arrow[ur, bend right, equal] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{D}_{d/} \times_{\mathcal{D}} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d, "q"] \\
* \arrow[r, "d"'] \arrow[ur, Rightarrow, "\alpha"] & \mathcal{D}
\end{tikzcd}$$

The associated Beck-Chevalley transformation for this lax square is given at $F : \mathcal{F} \to \mathcal{X}$ by

$$\begin{aligned}
(q_*F)(d) &\xrightarrow{\eta} \lim(\mathcal{D}_{d/} \times_{\mathcal{D}} \mathcal{F} \to * \xrightarrow{d} \mathcal{D} \xrightarrow{q_*F} \mathcal{X}) \\
&\xrightarrow{\lim(\alpha)} \lim(\mathcal{D}_{d/} \times_{\mathcal{D}} \mathcal{F} \xrightarrow{q^*q_*F} \mathcal{X}) \\
&\xrightarrow{\lim(\varepsilon^q)} \lim(\mathcal{D}_{d/} \times_{\mathcal{D}} \mathcal{F} \xrightarrow{F} \mathcal{X}).
\end{aligned}$$

This is an equivalence, as it precisely encodes the pointwise formula for right Kan extensions. Analogously we can construct the lax pullback for the cospan $\mathcal{C}_{jd/} \to \mathcal{C} \leftarrow \mathcal{E}$, which admits a map from $\mathcal{D}_{d/} \to \mathcal{D} \leftarrow \mathcal{F}$ to give the following cube:

$$\begin{tikzcd}
& & \mathcal{D}_{d/} \times_{\mathcal{D}} \mathcal{F} \arrow[rr] \arrow[dl] \arrow[dd] & & \mathcal{F} \arrow[dl] \arrow[dd, "q"] \\
& \mathcal{C}_{jd/} \times_{\mathcal{C}} \mathcal{E} \arrow[rr] \arrow[dd] & & \mathcal{E} \arrow[dd] & \\
& & * \arrow[rr, "d"] \arrow[uurr, Rightarrow, "\alpha"] & & \mathcal{D} \arrow[dl] \\
& * \arrow[rr] \arrow[ur] \arrow[uurr, Rightarrow, "\beta"] & & \mathcal{C} &
\end{tikzcd}$$

Here the left, right, top and bottom faces commute strictly, whereas the back and front faces commute laxly via the above natural transformations, induced from $s \Rightarrow t$ on $\mathrm{Ar}(\mathcal{D})$ for α respectively $\mathrm{Ar}(\mathcal{C})$ for β. As one readily checks on the arrow categories, this gives a homotopy of the composite 2-morphisms rightface $\circ\, \alpha \simeq \beta \circ$ leftface. Now Beck-Chevalley transformations compose by Lemma C.1, and $BC(\alpha)$ and $BC(\beta)$ are equivalences, so it remains to identify the Beck-Chevalley map induced by the left face. But right Kan extension along $\mathcal{D}_{d/} \times_{\mathcal{D}} \mathcal{F} \to *$ precisely computes the limit, and the Beck-Chevalley map is then clearly given by restricting said limit along $\mathcal{D}_{d/} \times_{\mathcal{D}} \mathcal{F} \to \mathcal{C}_{jd/} \times_{\mathcal{C}} \mathcal{E}$ as claimed. $\square$

Lemma C.3 ([Cno23b, Lemma F.5])**.** Consider a lax-commutative square as in Diagram (17). Then the following diagram commutes:

$$\begin{tikzcd}
q_*k^*p^* & j^*p_*p^* \arrow[l, "BC(\alpha)p^*"'] \\
q_*q^*j^* \arrow[u, "q_*\alpha"] & j^* \arrow[l, "\eta^q j^*"] \arrow[u, "j^*\eta^p"']
\end{tikzcd}$$

Proof. This follows from the triangle identity via the following diagram; the top horizontal is $BC(\alpha)p^*$:

$$\begin{tikzcd}
q_*k^*p^* & q_*k^*p^*p_*p^* \arrow[l, "q_*k^*\varepsilon^p p^*"'] & q_*q^*j^*p_*p^* \arrow[l, "q_*\alpha p_*p^*"'] & j^*p_*p^* \arrow[l, "\eta^q j^*p_*p^*"'] \\
& q_*k^*p^* \arrow[ul, equal] \arrow[u, "q_*k^*p^*\eta^p"'] & q_*q^*j^* \arrow[l, "q_*\alpha"] \arrow[u, "q_*q^*j^*\eta^p"'] & j^* \arrow[l, "\eta^q j^*"] \arrow[u, "j^*\eta^p"']
\end{tikzcd}$$

□

Lemma C.4. Let $I : \mathcal{C} \hookrightarrow \mathcal{D}$ and $J : \mathcal{D} \hookrightarrow \mathcal{E}$ be fully faithful functors. Denote $K = JI$. If $\mathcal{X}$ is a category with enough limits, then the following square commutes via the Beck-Chevalley transformation

$$\begin{tikzcd}
\mathcal{X}^{\mathcal{C}} \arrow[r, equal] \arrow[d, hook, "I_*"'] & \mathcal{X}^{\mathcal{C}} \arrow[d, hook, "K_*"] \\
\mathcal{X}^{\mathcal{D}} & \mathcal{X}^{\mathcal{E}} \arrow[l, "J^*"]
\end{tikzcd}$$

Proof. Note that $\varepsilon^J : J^*J_* \xRightarrow{\simeq} \mathrm{id}$ is an equivalence as J is fully faithful. Using the natural equivalence $\alpha : K \simeq JI$, we obtain a commutative diagram of natural transformations

$$\begin{tikzcd}
J^*K_* \arrow[r, "J^*\alpha"] \arrow[d, "\eta^I J^*K_*"'] & J^*J_*I_* \arrow[r, "\varepsilon^J I_*"] \arrow[d, "\eta^I J^*J_*I_*"'] & I_* \arrow[d, "\eta I_*"'] \arrow[dd, equal, bend left] \\
I_*I^*J^*K_* \arrow[r, "I_*I^*J^*\alpha"'] \arrow[d, "I_*\alpha K_*"'] & I_*I^*J^*J_*I_* \arrow[r, "I_*I^*\varepsilon^J I_*"] & I_*I^*I_* \arrow[d, "I_*\varepsilon"'] \\
I_*K^*K_* \arrow[rr, "I_*\varepsilon^K"'] & & I_*
\end{tikzcd}$$

Finally note that the left-bottom composite is by definition the Beck-Chevalley map we are interested in, and the top-right composite is an equivalence. □

Definition/Lemma C.5 ([Cno23b, Lemma F.10])**.** Consider again the situation of Diagram (17), with BC-transformation $BC_*(\alpha) : j^*p_* \Rightarrow q_*k^*$. Suppose that α is an equivalence and k^*, j^* admit left adjoints $k_!, j_!$. Then we also have a Beck-Chevalley transformation $BC_!(\alpha^{-1}) : k_!q^* \Rightarrow p^*j_!$. This is called the total mate, and is also obtained by taking left adjoints of $BC_*(\alpha)$. In particular $BC_*(\alpha)$ is an equivalence if and only if $BC_!(\alpha^{-1})$ is one.

D Miscellaneous Lemmas

Lemma D.1. Let $\mathcal{E} \to \mathcal{C}$ be a cocartesian fibration and $c \in \mathcal{C}$. Suppose we have a collection of morphisms $f^i : c \to c_i$ in $\mathcal{C}$ so that the induced $f^i_! : \mathcal{E}_c \to \mathcal{E}_{c_i}$ are jointly conservative. Then a morphism $e' \to e$ in $\mathcal{E}$ lying over $f : c' \to c$ is cocartesian if and only if the composites $e \to e' \to f^i_! e'$ are cocartesian for all i.

In particular, if $s : \mathcal{C} \to \mathcal{E}$ is a section which is cocartesian on all the f^i, then it is cocartesian on f if and only if it is cocartesian on all composites $f^i f$.

Proof. For each i we obtain a commutative diagram

$$\begin{tikzcd}
e' \arrow[r, "cc"] \arrow[dr] & f_! e' \arrow[r, "cc"] \arrow[d, "\varphi"] & (f^i f)_! e' \arrow[d, "f^i_! \varphi"] \\
& e \arrow[r, "cc"'] & f^i_! e
\end{tikzcd}$$

where cocartesian morphisms are labeled by cc. By assumption the whole composite from e' to $f^i_! e$ is cocartesian. But then by right-cancellability of cocartesian morphisms also $f^i_! \varphi$ is cocartesian, and thus an equivalence as it lives in the fiber over c_i. Since the $f^i_!$ are jointly conservative, it follows that φ is an equivalence, and thus $e' \to e$ is cocartesian as desired. $\square$

Corollary D.2. Let $\mathcal{C}$ be a category with finite products, $F : \mathcal{C} \xrightarrow{\times} \mathsf{Cat}$ a functor preserving finite products and $s : \mathcal{C} \to \int F$ a section of its cocartesian unstraightening. Suppose that s is cocartesian on all projections in $\mathcal{C}$. Then:

1. s is cocartesian on morphisms $f : a \to b$ and $g : a \to c$ if and only if it is cocartesian on $(f, g) : a \to b \times c$.

2. If s is cocartesian on $f : a \to b$ and $g : a' \to b'$, then it is cocartesian on $f \times g : a \times a' \to b \times b'$.

Lemma D.3. Suppose we are given a functor $F : \mathcal{C} \to \mathsf{Cat}$ and for each $c \in \mathcal{C}$ a functor $\alpha_c : G(c) \to F(c)$ which is conservative and faithful (e.g. the inclusion of a wide or full subcategory). If for every morphism $f : c \to c'$ in $\mathcal{C}$, the composition $G(c) \to F(c) \xrightarrow{Ff} F(c')$ factors[26] through $G(c') \to F(c')$ then the $G(c)$ assemble into a functor $G : \mathcal{C} \to \mathsf{Cat}$ and there exists a unique natural transformation $\alpha : G \Rightarrow F$ which pointwise agrees with the α_c above.

Proof. Using [Lur09, 5.5.6.15] one deduces that the monomorphisms in Cat are precisely the conservative faithful functors. Now the conclusion follows from the more general statement [Ram23, Theorem A.1]. $\square$

[26] This really is a property, as the space of possible factorizations through a subcategory is (-1)-truncated.

Lemma D.4. The following diagram commutes and is natural in $A, B \in \mathsf{Cat}^{\mathsf{op}}$:

$$\begin{tikzcd}
(\mathsf{Cat}_{/A})_{/\mathrm{pr}_A} & \mathrm{Cocart}(A)_{/\mathrm{pr}_A} \arrow[l, hook'] \arrow[r, "\simeq", "\mathrm{St}_A"'] & \mathsf{Fun}(A, \mathsf{Cat}_{/B}) \\
\mathsf{Cat}_{/A\times B} \arrow[u, hook, "\simeq"'] & \mathrm{Cocart}(A \times B) \arrow[l, hook'] \arrow[u, hook] \arrow[r, "\simeq", "\mathrm{St}_A"'] \arrow[d, "\mathrm{St}_{A\times B}"', "\simeq"] & \mathsf{Fun}(A, \mathrm{Cocart}(B)) \arrow[u, hook] \arrow[d, "\simeq"', "\mathrm{St}_B"] \\
& \mathsf{Fun}(A \times B, \mathsf{Cat}) \arrow[r, "\simeq", "\mathrm{curry}"'] & \mathsf{Fun}(A, \mathsf{Fun}(B, \mathsf{Cat}))
\end{tikzcd}$$

Here all the hooked arrows are inclusions of subcategories, pr_A denotes the projection $A \times B \to A$, and $\mathrm{St}_A : \mathrm{Cocart}(A \times B) \simeq \mathsf{Fun}(A, \mathrm{Cocart}(B))$ is defined via restriction of $\mathrm{St}_A : \mathrm{Cocart}(A)_{/\mathrm{pr}_A} \simeq \mathsf{Fun}(A, \mathsf{Cat}_{/B})$.

Proof. The fact that $\mathrm{St}_A : \mathrm{Cocart}(A)_{/\mathrm{pr}_A} \simeq \mathsf{Fun}(A, \mathsf{Cat}_{/B})$ restricts to an equivalence $\mathrm{Cocart}(A\times B) \simeq \mathsf{Fun}(A, \mathrm{Cocart}(B))$ follows by combining [HHLN23b, 2.2.4, 2.4.3, 2.4.9]. So all functors in the diagram exist as claimed and are natural in $A, B \in \mathsf{Cat}^{\mathsf{op}}$ (for naturality of straightening see [HHLN23b, p.9]). Finally, commutativity of the bottom right square was shown directly from the definitions in [HHLN23b, 6.20], but also follows from the rigidity theorem [HHLN23b, A.1], by which the space of natural self-equivalences of $\mathsf{Fun}(-\times-, \mathsf{Cat}) : \mathsf{Cat}^{\mathsf{op}} \times \mathsf{Cat}^{\mathsf{op}} \to \widehat{\mathsf{Cat}}$ is discrete on $\{\mathrm{id}, ((-)^{\mathsf{op}})_*\}$. Plugging in $(A, B) = (\Delta^0, \Delta^0)$, it is clear that going once around the square corresponds to the identity. $\square$

Remark D.5. As we use some of the $(\infty, 2)$-categorical of the main theorems from [BHS22] which are discussed in Section 5.3 of op. cit., we briefly explain their usage here. Essentially, we do not need the full power of $(\infty, 2)$-categories, but only that they admit mapping category functors which lift the mapping space functor along $(-)^{\simeq} : \mathsf{Cat} \to \mathsf{Spc}$, and that some adjunctions already induce natural equivalences of these mapping categories. We will mostly consider functor categories $\mathsf{Fun}(\mathcal{C}, \mathsf{Cat})$, which inherit their $(\infty, 2)$-categorical structure from Cat. The model used in op. cit is by considering natural Cat-(co)module structures which give rise to a Cat-enrichment and hence $(\infty, 2)$-categorical structures by results of [Hin20] and [Hei23].

Specifically, since Cat is cartesian closed, then for any (small) category $\mathcal{C}$ also $\mathsf{Fun}(\mathcal{C}, \mathsf{Cat})$ is cartesian closed[27]. We denote the internal hom by $[-,-] : (\mathsf{Cat}^{\mathcal{C}})^{\mathsf{op}} \times \mathsf{Cat}^{\mathcal{C}} \to \mathsf{Cat}^{\mathcal{C}}$ with underlying mapping category $\mathrm{Nat}_{\mathcal{C}}(F, G) \simeq \lim_{\mathcal{C}}[F, G]$. By [GHN17, 6.4, 6.7, 6.8] this is computed as

$$\mathrm{Nat}_{\mathcal{C}}(F, G) \simeq \int_{\mathcal{C}} \mathsf{Fun}(F(-), G(-)) \coloneqq \lim_{(c\to d)\in \mathrm{Tw}^{\ell}(\mathcal{C})} \mathsf{Fun}(F(c), G(d)) \tag{18}$$

where $(s, t) : \mathrm{Tw}^{\ell}(\mathcal{C}) \to \mathcal{C}^{\mathsf{op}} \times \mathcal{C}$ is the twisted arrow left fibration classified by $\mathrm{map}_{\mathcal{C}}$. This is also used extensively in parametrized category theory; with base category T, one writes Fun_T instead of $\mathrm{Nat}_{T^{\mathsf{op}}}$ and $\underline{\mathsf{Fun}}_T$ for the internal mapping object, c.f. [CLL23a, Section 2.2].

[27] See e.g. https://mathoverflow.net/questions/104152.

The cartesian closedness of $\mathsf{Cat}^{\mathcal{C}}$ endows it with a (co)tensoring over Cat via const : $\mathsf{Cat} \to \mathsf{Cat}^{\mathcal{C}}$, given by $\mathcal{E} \odot F := (\mathrm{const}\,\mathcal{E}) \times F$ and

$$F^{\mathcal{E}} := [\mathrm{const}\,\mathcal{E}, F] \simeq \mathsf{Fun}(\mathcal{E}, F)$$

where the equivalence follows from the formula for mapping spaces in functor categories (take groupoid cores in Eq. (18) or see [GHN17, Proposition 5.1]) and the fact that $\mathsf{Fun}(\mathcal{E}, -)$ preserves limits. Ultimately we have natural equivalences

$$\mathsf{Cat}(\mathcal{E}, \mathrm{Nat}_{\mathcal{C}}(F, G)) \simeq \mathsf{Cat}^{\mathcal{C}}(\mathcal{E} \odot F, G) \simeq \mathsf{Cat}^{\mathcal{C}}(F, G^{\mathcal{E}}).$$

Analogously, we consider $\mathsf{Fun}^{\times}(\mathcal{C}, \mathsf{Cat}) \subseteq \mathsf{Fun}(\mathcal{C}, \mathsf{Cat})$ as a full sub $(\infty, 2)$-category of $\mathsf{Fun}(\mathcal{C}, \mathsf{Cat})$, so that the mapping category between $F, G \in \mathsf{Fun}^{\times}(\mathcal{C}, \mathsf{Cat})$ is also given by $\mathrm{Nat}_{\mathcal{C}}(F, G)$.

Lemma D.6. Let $i : \mathcal{C} \to \mathcal{D}$ be a functor. Then $i_* : \mathsf{Cat}^{\mathcal{C}} \to \mathsf{Cat}^{\mathcal{D}}$ and $i^* : \mathsf{Cat}^{\mathcal{D}} \to \mathsf{Cat}^{\mathcal{C}}$ are compatible with the Cat-cotensoring. In particular, the adjunction lifts, i.e. the composite

$$\mathrm{Nat}_{\mathcal{D}}(F, i_*G) \xrightarrow{i^*} \mathrm{Nat}_{\mathcal{C}}(i^*F, i^*i_*G) \xrightarrow{(\varepsilon_G)_*} \mathrm{Nat}_{\mathcal{C}}(i^*F, G)$$

is a natural equivalence of categories and analogously for the composite induced by i_* and η.

Proof. It is clear that i^* commutes with the cotensoring given $F^{\mathcal{E}} := \mathsf{Fun}(\mathcal{E}, -) \circ F$. For i_*, this follows from the pointwise formula for right Kan extensions and the fact that $\mathsf{Fun}(\mathcal{E}, -)$ preserves limits:

$$((i_*F)^{\mathcal{E}})(d) = \mathsf{Fun}(\mathcal{E}, (i_*F)(d)) \simeq \mathsf{Fun}(\mathcal{E}, \lim_{d \to Fc} F(c)) \simeq \lim_{d \to Fc} F^{\mathcal{E}}(c) \simeq (i_*F^{\mathcal{E}})(d).$$

Now formally one defines i^* on the mapping categories via this cotensoring adjunction; we have a natural map

$$\mathsf{Cat}(\mathcal{E}, \mathrm{Nat}_{\mathcal{D}}(F, H)) \simeq \mathsf{Cat}^{\mathcal{D}}(F, H^{\mathcal{E}}) \xrightarrow{i^*} \mathsf{Cat}^{\mathcal{C}}(i^*F, (i^*H)^{\mathcal{E}}) \simeq \mathsf{Cat}(\mathcal{E}, \mathrm{Nat}_{\mathcal{C}}(i^*F, i^*H))$$

so by Yoneda's Lemma we get the functor $i^* : \mathrm{Nat}_{\mathcal{D}}(F, H) \to \mathrm{Nat}_{\mathcal{C}}(i^*F, i^*H)$. Analogously $\mathrm{Nat}_{\mathcal{D}}(F, i_*G) \to \mathrm{Nat}_{\mathcal{C}}(i^*F, i^*i_G) \to \mathrm{Nat}_{\mathcal{C}}(i^*F, G)$ is then induced by the natural equivalence

$$\mathsf{Cat}(\mathcal{E}, \mathrm{Nat}(F, i_*G)) \simeq \mathsf{Cat}^{\mathcal{D}}(F, i_*G^{\mathcal{E}}) \simeq \mathsf{Cat}^{\mathcal{C}}(i^*F, G^{\mathcal{E}}) \simeq \mathsf{Cat}(\mathcal{E}, \mathrm{Nat}(i^*F, G))$$

where in the middle we use the usual adjunction equivalence. $\square$

Index of Notation

Note our conventions in Section 1.3, and in particular that Cat is the ∞-category of ∞-categories, usually denoted Cat_∞. Moreover, for every notation with super-/subscripts orbital $P \subset T$, we will leave out superscript for P in the case $P = T$, e.g. we write Mack_T instead of Mack_T^T. We often write G instead of $T = \mathsf{Orb}_G$.

$\mathcal{A}_T^P$	P-symmetric monoidal envelope of $\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T; T)$	Theorem 2.28
AdTrip	category of adequate triples	Definition/Lemma 2.10
$\mathsf{AlgPatt}$	category of algebraic patterns	Definition 2.1
Bor_G	G-symmetric monoidal Borellification $\mathsf{Mack}(\mathcal{E})^{BG} \to \mathsf{Mack}_G(\mathcal{E})$	Definition 3.12
$\mathrm{Bor}_{\mathsf{Glo}}$	global Borellification $\mathsf{Cat} \to \mathsf{Cat}_{\mathsf{Glo}}$	Section 3.3
$\mathrm{Bor}_{\mathsf{Glo}}^{\mathsf{Orb}}$	equivariantly symmetric monoidal global Borellification $\mathsf{Mack}(\mathcal{E}) \to \mathsf{Mack}_{\mathsf{Glo}}^{\mathsf{Orb}}(\mathcal{E})$	Section 3.3
Cat	(large) category of small categories	
$\mathsf{Cat}(\mathcal{K})$	subcategory of Cat on categories admitting $\mathcal{K}$-indexed colimits and functors preserving these	
$\widehat{\mathsf{Cat}}$	(very large) category of large categories	
$\mathsf{Cat}^\times$	subcategory of Cat on categories admitting finite products and functors preserving these	
$\mathsf{Cat}^\dagger$	category of marked categories	Definition 4.1
Cat_T	category of T-categories $\mathsf{Fun}(T^{\mathsf{op}}, \mathsf{Cat})$	Introduction of 2.4
$\mathrm{CAlg}_T^P(\mathcal{C})$	category of P-commutative T-algebras in $\mathcal{C}$	Definition 2.20
$\mathcal{C}(c,d)$	mapping space $\mathrm{map}_{\mathcal{C}}(c,d)$	
$\mathcal{C}^\simeq$	groupoid core of the category $\mathcal{C}$	
$\mathcal{C}^{\mathsf{si}}$	wide subcategory of summand inclusions in $\mathcal{C}$	Definition A.1
CGrp	short for $\mathsf{CGrp}(\mathsf{Spc})$, the category of commutative groups in spaces	
CMon	short for $\mathsf{CMon}(\mathsf{Spc})$, the category of commutative monoids in spaces	
DK	functorial Dwyer-Kan localization $DK : \mathsf{Cat}^\dagger \to \mathsf{Cat}$	Proposition 4.2
$\underline{\mathcal{E}}_T$	T-category of T-objects in $\mathcal{E}$ / cofree T-category on $\mathcal{E}$	Definition/Lemma 2.42
$\mathbb{F}$	category of finite sets	
$\mathbb{F}_*$	category of finite pointed sets	
$\mathbb{F}_T, \mathbb{F}_T^P$	category of finite T-sets / subcategory of finite P-sets	Notation A.3
$\underline{\mathbb{F}}_T^P$	T-category of finite P-sets	Definition 2.24
$\mathbb{F}_{T,*}$	category of finite pointed T-sets	
$\underline{\mathbb{F}}_{T,*}^P$	T-category of finite pointed P-sets	Example 2.47
$\mathsf{Fbrs}(\mathcal{O})$	category of fibrous $\mathcal{O}$-patterns	Definition 2.4
Fbrs_T^P	short for $\mathsf{Fbrs}(\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T; T))$	Definition 2.20
$\mathsf{Fun}^\times(\mathcal{C},\mathcal{D})$	full subcategory of $\mathsf{Fun}(\mathcal{C},\mathcal{D})$ on functors preserving finite products	

$\mathsf{Fun}^{\dagger}(\mathcal{C},\mathcal{D})$	full subcategory of $\mathsf{Fun}(\mathcal{C},\mathcal{D})$ on functors preserving the marked edges	
$\mathsf{Fun}_T(\mathcal{C},\mathcal{D})$	category of T-functors. Equivalently $\mathrm{Nat}_{T^{\mathsf{op}}}(\mathcal{C},\mathcal{D})$	Remark 2.41
$\mathsf{Fun}_T^{P-\otimes}(\mathcal{C},\mathcal{D})$	category of P-symmetric monoidal T-functors. Equivalently $\mathrm{Nat}_{\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)}(\mathcal{C},\mathcal{D})$	Definition 2.20
Glo	global indexing category of finite groups	Example A.8
$\mathsf{Mack}(\mathcal{E})$	$\mathcal{E}$-valued $\mathrm{Span}(\mathbb{F})$-Mackey functors, i.e. $\mathsf{Fun}^{\times}(\mathrm{Span}(\mathbb{F}),\mathcal{E})$	Example 2.8
$\mathsf{Mack}_T^P(\mathcal{E})$	$\mathcal{E}$-valued $\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)$-Mackey functors, i.e. $\mathsf{Fun}^{\times}(\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T),\mathcal{E})$	Example 2.15
$\underline{\mathsf{Mack}}_T^P(\mathcal{E})$	T-category of $\mathcal{E}$-valued $\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)$-Mackey functors	Definition 2.18
$\underline{\mathsf{Mod}}_R(\mathcal{C})$	parametrized symmetric monoidal module category	Theorem 2.37
$\mathrm{Nat}_{\mathcal{C}}(F,G)$	Mapping category of $F,G\in\mathsf{Fun}(\mathcal{C},\mathsf{Cat})$	Remark D.5
Orb	wide subcategory of Glo on the injective group homomorphisms / faithful functors	Example A.8
$\mathsf{pic}(\mathcal{C})$	Picard spectrum of a symmetric monoidal category $\mathcal{C}$	Definition 5.1
$\mathrm{Seg}(\mathcal{O},\mathcal{C})$	Segal-$\mathcal{O}$ objects in $\mathcal{C}$	Definition 2.3
Sp^{Σ}	1-category of symmetric spectra	
$\underline{\mathsf{Sp}}^{\otimes}$	equivariantly symmetric monoidal global category of equivariant spectra	Construction 4.14
$\underline{\mathsf{Sp}}_G^{\otimes}$	G-symmetric monoidal G-category category of equivariant spectra	Definition 4.18
$\underline{\mathsf{Sp}}_{\mathsf{Glo}}^{\otimes}$	equivariantly symmetric monoidal global category of global spectra	Construction 4.15
$\mathrm{Span}(G)$	spans of finite G-sets, i.e. $\mathrm{Span}(\mathbb{F}_G)$	
$\mathrm{Span}_{\mathsf{all},P}(\mathbb{F}_T)$	spans in $\mathbb{F}_T$ with forwards morphisms in $\mathbb{F}_T^P$	Example 2.15
$\mathrm{St}^{\mathsf{cc}}(p)$	cocartesian straightening of p	
$\mathrm{St}^{\mathsf{ct}}(p)$	cartesian straightening of p	
$\mathrm{Tw}^r(\mathcal{C})$	right twisted arrow category of $\mathcal{C}$	above Theorem 2.11
UCom	underlying ∞-category of ultracommutative spectra	Construction 4.21
$\mathcal{C}\xrightarrow{\times}\mathcal{D}$	functor preserving finite products	
$\int F$	(source of the) cocartesian unstraightening of F	

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