ON BIPRESENTABILITY AND POSETS

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The aim of this note originally was to show the ∞-categorical version of the 1-categorical theorem that if $\mathcal{C}$ is a category where $\mathcal{C}$ and $\mathcal{C}^{\mathsf{op}}$ are presentable (these will henceforth be called *bipresentable*), then $\mathcal{C}$ is a small complete lattice (see [GU71, Satz 7.13], [AR94, Theorem 1.64]). This appears only to exist as unverified folklore in the case of ∞-categories (see e.g. [Sch26, Remark 1.25] for an inaccurate statement in this vein). It is worth noting that the classical proof does not generalize to this setting, because it crucially relies on the prevalence of monomorphisms (specifically, that morphisms admitting retracts are monomorphisms) in the case of ordinary (1-)categories, which fails in the case of ∞-categories. This complication has thus far proven itself insurmountable.

Nonetheless, we establish the ∞-categorical analogue in the following cases.

Theorem. *Let $\mathcal{C}$ be a presentable ∞-category satisfying any of the following assumptions:*

(1) $\mathcal{C}$ has disjoint coproducts, in that $X \times_{X \amalg Y} Y$ is initial for all $X, Y \in \mathcal{C}$ (e.g. if $\mathcal{C}$ is semiadditive).
(2) In $\mathcal{C}$ arbitrary products distribute over binary coproducts. In fact, we only need that for $X \in \mathcal{C}$ and I a small set we have $\coprod_{2^I} X \simeq \prod_I (X \amalg X)$.
(3) The cartesian product in $\mathcal{C}$ preserves countable weakly contractible colimits in each variable separately.

Then if $\mathcal{C}^{\mathsf{op}}$ is also presentable, $\mathcal{C}$ must be a poset (in fact a small complete lattice).

Under the first two conditions, we show the following stronger result:

Theorem. *Let $\mathcal{C}$ be a presentable ∞-category and suppose it satisfies either (1) or (2) in the above Proposition. Then for any regular cardinal κ, any κ-cocompact object in $\mathcal{C}$ is (-1)-truncated.*

We expect that this stronger claim holds in general, and formulate this as the following question[1]:

Question. *Let $\mathcal{C}$ be a presentable ∞-category. For any regular cardinal κ, is any κ-cocompact object in $\mathcal{C}$ (-1)-truncated?*

We also comment on a curious relation of this question to the existence of measurable cardinals.

Remark 0.1. From here on out, we will drop the prefix '∞-' and simply write 'category'.

Remark 0.2. We have implicitly fixed an inaccessible cardinal modeling the universe with respect to which we handle size issues. The regular cardinals appearing in the following are all assumed to be small with respect to this universe.

[1] This appears to be open even for ordinary (1-)categories.

1. Some Observations

Proposition 1.1. *A poset is presentable if and only if it is a small complete lattice. These posets are all bipresentable categories.*

Proof. In a poset, the colimit resp. limit of a diagram is equivalently the join resp. meet of its vertices. Thus, a presentable poset is a complete lattice and, furthermore, it is small since there is set (the κ-compacts for some regular cardinal κ) of objects s.t. every object is a small (κ-filtered) colimit of a diagram valued in this set of objects, but this means every object is a meet over subsets of a small set, which still only gives a set of objects. In the converse direction, every small complete lattice admits all (co-)limits and every object is both κ-compact and κ-cocompact for $\kappa \gg 0$ (since every κ-(co-)filtered diagram is eventually constant), hence it is bipresentable. □

Proposition 1.2. *Let $\mathcal{C}$ be a presentable category that satisfies the conclusion of the Conjecture. If $\mathcal{C}$ is copresentable, then it is a poset.*

Proof. For $\mathcal{C}$ to be copresentable means that there is a $\kappa \gg 0$ such that $\mathcal{C}$ is generated under small limits by a set of κ-cocompact objects, but these are (-1)-truncated by the assumption and those objects are stable under small limits, hence $\mathcal{C}$ consists of (-1)-truncated objects, i.e. is a poset. □

Lemma 1.3. *A category with finite coproducts and no nontrivial retracts is a poset.*

Proof. Given any map $f\colon X \to Y$, we get that the inclusion $Y \to X \amalg Y$ admits a retraction $(f, \mathrm{id}_Y)\colon X \amalg Y \to Y$ hence is an equivalence. Thus the projection $\mathrm{map}(X,Y) \times \mathrm{map}(Y,Y) \to \mathrm{map}(Y,Y)$ is an equivalence, and we conclude that $\mathrm{map}(X,Y) = *$, since $\mathrm{map}(Y,Y) \neq \emptyset$. □

Let $\mathsf{biPr}^L \subseteq \mathsf{Pr}^L$ be the full subcategory on bipresentable categories.

Lemma 1.4. *The full subcategory $\mathsf{biPr}^L \subseteq \mathsf{Pr}^L$ is closed under*

1. *small limits,*
2. *small colimits,*
3. *$(-) \otimes \mathcal{C}$ for any presentable category $\mathcal{C}$,*
4. *$\mathsf{Fun}^L(\mathcal{C}, -)$ for any presentable category $\mathcal{C}$,*
5. *taking over- and under-slices.*

Proof. Clearly the (co)limit in Pr^L of a diagram in biPr^L is again presentable, and it remains to verify that its opposite is presentable. Since we already know it is bicomplete, it in fact suffices to show the opposite is accessible. By the way we compute (co)limits in Pr^L as limits in $\widehat{\mathsf{Cat}}$, both cases reduce to showing that the subcategory of accessible categories and accessible functors is closed under limits in the $\widehat{\mathsf{Cat}}$, which is true (see e.g. [Lur18, Tag 06LQ]).

To see (3) and (4), fix a bipresentable category $\mathcal{D}$ and observe that $\mathcal{D} \otimes \mathcal{C} \simeq \mathsf{Fun}^R(\mathcal{D}^{\mathsf{op}}, \mathcal{C}) \simeq \mathsf{Fun}^L(\mathcal{C}, \mathcal{D}^{\mathsf{op}})^{\mathsf{op}}$ is copresentable and $\mathsf{Fun}^L(\mathcal{C}, \mathcal{D})^{\mathsf{op}} \simeq \mathsf{Fun}^R(\mathcal{C}^{\mathsf{op}}, \mathcal{D}^{\mathsf{op}}) \simeq \mathcal{C} \otimes \mathcal{D}^{\mathsf{op}}$ is presentable, since $\mathcal{D}^{\mathsf{op}}$ is presentable. Lastly, (5) is clear. □

2. Special Cases

2.1. Disjoint Coproducts.

Definition 2.1. Let $\mathcal{C}$ be a category with finite coproducts. We say it has disjoint coproducts if for any objects $X, Y \in \mathcal{C}$, the following pushout square is also a pullback square:

$$\begin{array}{ccc} \varnothing & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \amalg Y \end{array}$$

Example 2.2. Any topos has disjoint coproducts. Free finite coproduct completions of categories have disjoint coproducts, e.g. the category of finite sets. Also Cat has disjoint coproducts.

Lemma 2.3. *Semiadditive categories have disjoint coproducts.*

Proof. Let X, Y be objects in a semiadditive category and consider the pushout squares

$$\begin{array}{ccccc} 0 & \longrightarrow & Y & \longrightarrow & 0 \\ \downarrow & \ulcorner & \downarrow & \ulcorner & \downarrow \\ X & \longrightarrow & X \oplus Y & \longrightarrow & X \\ & & \downarrow & \ulcorner & \downarrow \\ & & Y & \longrightarrow & 0 \end{array}$$

The top left one exists by definition of coproducts. Then the top right one exists by pushout pasting, and the bottom one also by pushout pasting. But the bottom one is now also a pullback by semiadditivity, so by pullback pasting the top right one is a pullback, and by another pullback pasting the top left one is too, as desired. $\square$

Example 2.4. Let $\mathcal{C}$ be a pointedly symmetric monoidal category, then $\mathrm{CAlg}(\mathcal{C})^{\mathsf{op}}$ has disjoint coproducts. Indeed, the product projections $A \times B \to A$ and $A \times B \to B$ are localizations at orthogonal idempotents, so the pushout of $A \leftarrow A \times B \to B$ is the terminal commutative algebra.

Lemma 2.5. *Let $\mathcal{C}$ be a bicomplete category with disjoint coproducts and κ some regular cardinal. Suppose that κ-filtered colimits commute with pullbacks. Then for any $X \in \mathcal{C}$ we have*

$$\varnothing \xrightarrow{\simeq} \lim_{\alpha \in \kappa^{\mathsf{op}}} \coprod_{\kappa \setminus \alpha} X.$$

Proof. For $\alpha, \beta < \kappa$ we have the following pullback square:

$$\begin{array}{ccc} P_{\alpha,\beta} & \longrightarrow & \coprod_{\beta} X \\ \downarrow & \lrcorner & \downarrow \\ \coprod_{\kappa \setminus \alpha} X & \longrightarrow & \coprod_{\kappa} X \end{array}$$

Note that for $\alpha > \beta$, we have $P_{\alpha,\beta} = \varnothing$ by disjointness of coproducts in $\mathcal{C}$. In particular, taking the limit over $\alpha \in \kappa^{\mathsf{op}}$, we obtain

$$\begin{array}{ccccc} \varnothing & \longrightarrow & \lim_{\alpha \in \kappa^{\mathsf{op}}} \coprod_{\beta} X & = & \coprod_{\beta} X \\ \downarrow & \lrcorner & \downarrow & & \downarrow \\ \lim_{\alpha \in \kappa^{\mathsf{op}}} \coprod_{\kappa \setminus \alpha} X & \longrightarrow & \lim_{\alpha \in \kappa^{\mathsf{op}}} \coprod_{\kappa} X & = & \coprod_{\kappa} X \end{array}$$

Passing to the colimit over $\beta < \kappa$ and using that κ-filtered colimits commute with pullbacks (note κ is κ-filtered) we obtain the desired pullback square

$$\begin{array}{ccc} \varnothing & \longrightarrow & \operatorname{colim}_{\beta < \kappa} \coprod_{\beta} X \\ \simeq \downarrow & \lrcorner & \downarrow \simeq \\ \lim_{\alpha \in \kappa^{\mathsf{op}}} \coprod_{\kappa \setminus \alpha} X & \longrightarrow & \coprod_{\kappa} X \end{array}$$

$\square$

Example 2.6. The disjoint coproducts assumption is necessary. For example, the category CRing of ordinary commutative rings does not have disjoint coproducts, since the coproduct agrees with the tensor product, and there are many idempotents for it. In particular, if we chose $X = \mathbb{Q}$, then $\lim_{\kappa^{\mathsf{op}}} \coprod_{\kappa \setminus \alpha} \mathbb{Q} = \lim_{\kappa^{\mathsf{op}}} \mathbb{Q} = \mathbb{Q}$ is an inverse limit of a constant diagram.

Proposition 2.7. *Let $\mathcal{C}$ be a presentable category with disjoint coproducts. Then for all regular cardinals κ, the only κ-cocompact object of $\mathcal{C}$ is the terminal one.*

Proof. Suppose that Y is κ-cocompact. After possibly increasing κ, we may assume that $\mathcal{C}$ is κ-compactly generated, and hence that κ-filtered colimits commute with κ-small limits. Now take any $X \in \mathcal{C}$, and note that

$$* = \operatorname{map}(\varnothing, Y) = \operatorname{map}(\lim_{\alpha \in \kappa^{\mathrm{op}}} \coprod_{\kappa \setminus \alpha} X, Y) = \operatorname*{colim}_{\alpha < \kappa} \prod_{\kappa \setminus \alpha} \operatorname{map}(X, Y).$$

Applying π_0 and using that π_0 commutes with filtered colimits and products, one sees that $\pi_0 \operatorname{map}(X, Y)$ not having exactly one element leads to a contradiction. Similarly, applying π_n for $n \geq 1$, one deduces $\pi_n \operatorname{map}(X, Y) = 0$, and hence that $\operatorname{map}(X, Y) = *$. Thus Y is terminal, as desired. □

Corollary 2.8. *A bipresentable category with disjoint coproducts is trivial.*

Proof. By the above, all cocompacts will be terminal, but copresentability also guarantees that the category is generated under limits by cocompacts. Since the full subcategory on the terminal object is closed under limits, we conclude. □

Remark 2.9. This argument, slightly more generally, works as soon as $\mathcal{C}$ admits a jointly conservative collection of objects, each of which has disjoint coproducts with itself. For example, this applies to $\mathcal{C}_*$ for any extensive category $\mathcal{C}$, e.g. $\mathcal{C} = \mathsf{An}, \mathsf{Cat}$, etc.

Remark 2.10. In particular, by Example 2.4, we know that, say, $\mathrm{CAlg}(\mathsf{Sp})$ is not bipresentable. However, it is the perhaps most well-understood category for which we do not currently know whether it satisfies the Conjecture.

2.2. **Distributive Categories.**

Proposition 2.11. *Let κ be a regular cardinal and $\mathcal{C}$ a category in which κ-indexed products distribute over binary coproducts. Then, every κ-cocompact object of $\mathcal{C}$ is (-1)-truncated.*

Proof. Let Y be a κ-cocompact object of $\mathcal{C}$. For an arbitary $X \in \mathcal{C}$, we calculate

$$\begin{aligned} \prod_{2^\kappa} \operatorname{map}(X, Y) &\simeq \operatorname{map}(\coprod_{2^\kappa} X, Y) \simeq \operatorname{map}(\prod_{\kappa} \coprod_{2} X, Y) \\ &\simeq \operatorname*{colim}_{\lambda < \kappa} \operatorname{map}(\prod_{\lambda} \coprod_{2} X, Y) \simeq \operatorname*{colim}_{\lambda < \kappa} \operatorname{map}(\coprod_{2^\lambda} X, Y) \simeq \operatorname*{colim}_{\lambda < \kappa} \prod_{2^\lambda} \operatorname{map}(X, Y). \end{aligned}$$

Now, apply π_0 and then $\pi_n(-, *)$, $n \geq 1$, which commute with $(\kappa-)$filtered colimits and arbitrary products. This yields an analogous equivalence in Set, however surjectivity of the map $\operatorname{colim}_{\lambda<\kappa} \prod_{2^\lambda} A \to \prod_{2^\kappa} A$ implies that every function $2^\kappa \to A$ factors through a projection $2^\kappa \to 2^\lambda$ with $\lambda < \kappa$ and this is absurd if $|A| \geq 2$. Thus, we conclude that $\operatorname{map}(X, Y)$ is (-1)-truncated. Since X was arbitrary, it follows that Y is (-1)-truncated. □

Corollary 2.12. *A bipresentable category where arbitrary products distribute over binary coproducts is a poset.* □

Remark 2.13. This distributivity condition and the disjoint coproduct condition are both weaker conditions implied by $(\kappa-)$extensitivity, which is satisfied in most examples (e.g. topoi, categories of categories, etc.). Nonetheless, there are non-distributive categories with disjoint coproducts (e.g. any non-trivial stable category) and distributive categories with coproducts that aren't disjoint (e.g. any completely distributive lattice).

2.3. LCC Categories.

Corollary 2.14. *Let $\mathcal{C}$ be a bipresentable category. Then $\mathsf{CMon}(\mathcal{C}_{/X}) = *$ for all $X \in \mathcal{C}$.*

Proof. Also $\mathcal{C}_{/X}$ is bipresentable, so consider the case $X = *$. Then $\mathsf{CMon}(\mathcal{C}) = \mathsf{CMon} \otimes \mathcal{C}$ is semiadditive and bipresentable, hence trivial by Corollary 2.8. □

We do not know whether $\mathsf{CMon}(\mathcal{C}_{/X}) = *$ for all $X \in \mathcal{C}$ already implies that $\mathcal{C}$ is a poset. The next example witnesses that at least the case $X = *$ alone is not enough.

Example 2.15. The category $\mathsf{An}_*^{\mathsf{acyclic}}$ of pointed acyclic anima sits in a pullback square in Pr^L

$$\begin{array}{ccc} \mathsf{An}_*^{\mathsf{acyclic}} & \longrightarrow & \mathsf{An}_* \\ \big\downarrow & & \big\downarrow \Sigma \\ \{*\} & \longrightarrow & \mathsf{An}_*, \end{array}$$

hence is a presentable category. Furthermore, the full inclusion $\mathsf{An}_*^{\mathsf{acyclic}} \subseteq \mathsf{An}_*$ preserves finite products by the Künneth theorem. Thus, the Eckmann-Hilton argument implies that a (commutative) monoid X in $\mathsf{An}_*^{\mathsf{acyclic}}$ has $\pi_1(X)$ an abelian group, but this group is perfect by acyclicity, hence trivial. Then, Hurewicz implies that $X \simeq *$, so we have $\mathsf{CMon}(\mathsf{An}_*^{\mathsf{acyclic}}) = \mathsf{Mon}(\mathsf{An}_*^{\mathsf{acyclic}}) = *$.

Lemma 2.16. *Let $\mathcal{C}$ be a bipresentable category and suppose that $\mathcal{C}$ enhances to a symmetric monoidal category $(\mathcal{C}, \otimes)$ whose tensor product $\otimes$ commutes with colimits indexed by countable ordinary groupoids with finite automorphism groups in each variable and furthermore the identity enhances to a lax symmetric monoidal functor $\mathrm{id}_{\mathcal{C}} \colon (\mathcal{C}, \otimes) \to (\mathcal{C}, \times)$. Then $\mathcal{C}_* := \mathcal{C}_{*/} = *$.*

Proof. Under these assumptions, the composite $\mathcal{C} \to \mathrm{CAlg}(\mathcal{C}, \otimes) \to \mathcal{C}$ of the free and forgetful functors takes the familiar form $X \mapsto \coprod_{n \geq 0} (X^{\otimes n})_{h\Sigma_n}$. The latter factors as $\mathrm{CAlg}(\mathcal{C}, \otimes) \to \mathsf{CMon}(\mathcal{C}) \to \mathcal{C}$ via the lax symmetric monoidal functor $(\mathcal{C}, \otimes) \to (\mathcal{C}, \times)$ and $\mathsf{CMon}(\mathcal{C}) = *$ by Corollary 2.14. Since the forgetful functor preserves limits, we obtain a natural equivalence $* \simeq \coprod_{n \geq 0} (X^{\otimes n})_{h\Sigma_n}$ for $X \in \mathcal{C}$. In particular, we see that $* \simeq X \amalg Y$ for $Y = \coprod_{0 \leq n \neq 1} (X^{\otimes n})_{h\Sigma_n}$. Since $\mathbf{1} = (X^{\otimes 0})_{h\Sigma_0}$ occurs as a summand in Y and the lax symmetric monoidal structure supplies a map $* \to \mathbf{1}$, we also always have a map $X \to * \to \mathbf{1} \to Y$, and can exhibit Y as a retract of $X \amalg Y \simeq *$, hence $Y \simeq *$. Thus we get $X \amalg * \simeq *$. By the same argument, if there is a map $* \to X$, then X is a retract of $X \amalg * \simeq *$, and hence $X \simeq *$. This shows $\mathcal{C}_{*/} = *$. □

Corollary 2.17. *Suppose that $\mathcal{C}$ is a bipresentable category in which the cartesian product commutes with countable weakly contractible colimits in each variable (e.g. if $\mathcal{C}$ is locally cartesian closed). Then $\mathcal{C}$ is a poset.*

Proof. For any $X \in \mathcal{C}$, the cartesian symmetric monoidal category $(\mathcal{C}_{/X}, \times_X)$ is compatible with countable, weakly contractible colimits by assumption and this induces a 'smash product' symmetric monoidal structure $(\mathcal{C}_{X//X}, \wedge_X)$ on $(\mathcal{C}_{/X})_* = \mathcal{C}_{X//X}$, which is then compatible with countable colimits and comes equipped with a lax symmetric monoidal functor $(\mathcal{C}_{X//X}, \wedge_X) \to (\mathcal{C}_{X//X}, \times_X)$ enhancing the identity.

The category $\mathcal{C}_{X//X}$ is again bipresentable, so the above applies and we get $\mathcal{C}_{X//X} \simeq *$ for all $X \in \mathcal{C}$. In particular, it follows that $\mathcal{C}$ does not have any nontrivial retracts. But since $\mathcal{C}$ admits finite coproducts, we conclude that $\mathcal{C}$ must be a poset by Lemma 1.3. □

2.4. Truncated Categories.

Proposition 2.18. *Let $\mathcal{C}$ be a bipresentable category. Then every truncated morphism in $\mathcal{C}$ is a monomorphism.*

Proof. The full subcategory $\tau_{\leq 1}\mathcal{C} \subseteq \mathcal{C}$ on the 0-truncated objects is an ordinary (1-)category and bipresentable since it is $\tau_{\leq 1}\mathcal{C} \simeq \mathcal{C} \otimes \mathsf{Set}$, hence it is a poset by the classical Theorem. Thus, every 0-truncated object in a bipresentable category is (-1)-truncated. This passes to over-slices since these are again bipresentable, so an induction over n shows that every n-truncated morphism in a bipresentable category is (-1)-truncated, i.e. a monomorphism. □

Corollary 2.19. *Let $\mathcal{C}$ be a bipresentable category in which every object is truncated. Then $\mathcal{C}$ is a poset.* □

3. The relation to Measurable Cardinals

Recall that a cardinal κ is measurable if it is uncountable and admits a κ-complete non-principal ultrafilter. Measurable cardinals are inaccessible, and thus their existence cannot be proven in ZFC. Nevertheless, assuming their existence is not too crazy (ask your local set theorist), though certainly stronger than assuming that inaccessible cardinals exist. For example, it turns out that if there exists an ω_1-complete non-principal ultrafilter, and if κ is the smallest cardinal which admits one, then that ultrafilter is already κ-complete, so κ is measurable. It also turns out that if κ is the smallest measurable cardinal, then κ is the κ-th inaccessible, so measurable cardinals are huge if they exist.

The proof of the following is surprisingly straightforward, and essentially rests on identifying the set of κ-complete ultrafilters on a given set I with $\lim_{J \in \mathsf{Set}^{\kappa}_{I/}} J$.

Theorem 3.1 ([AR94, Theorem A.5])**.** *The following are equivalent*

(1) Set^{op} is a Bousfield localization of a presentable category.
(2) There do not exist arbitrarily large measurable cardinals.

On the other hand, there is the following argument, which we learned from Maxime Ramzi.

Proposition 3.2. *Let κ, λ be regular cardinals. Suppose that $\mathcal{C} \in \mathsf{Pr}^L_\kappa$ and $X \in \mathcal{C}$ is λ-cocompact. If there exists a measurable cardinal $\mu > \kappa, \lambda$ so that $\operatorname{map}(Y, X) \in \mathsf{An}^\mu$ for all $Y \in \mathcal{C}^\kappa$, then X is (-1)-truncated.*

Corollary 3.3. *If arbitrarily large measurable cardinals exist, then for any regular cardinal λ, the λ-cocompact objects of a presentable category $\mathcal{C}$ form a poset. In particular, bipresentable categories are posets.*

Proof. Say $\mathcal{C} \in \mathsf{Pr}^L_\kappa$ and $X \in \mathcal{C}$ is λ-cocompact for some regular cardinal λ. Since $\mathcal{C}^\kappa$ is small and $\mathcal{C}$ is locally small, we can always find a μ' so that $\operatorname{map}(Y, X) \in \mathsf{An}^{\mu'}$ for all $Y \in \mathcal{C}^\kappa$. Picking a measurable $\mu > \mu'$, we see that X is (-1)-truncated. Thus, $\mathcal{C}$ satisfies the Conjecture and 1.2 applies. □

For the proof, we will first need the following lemma:

Lemma 3.4. *Let κ be a measurable cardinal. If $X \in \mathsf{An}^\kappa$ and $\mathcal{U}$ is a κ-complete non-principal ultrafilter on κ, then the diagonal map $X \to \prod_\kappa X/\mathcal{U} = \operatorname{colim}_{U \in \mathcal{U}^{op}} \prod_U X$ is an equivalence.*

Proof. Since homotopy groups commute with products and filtered colimits, this reduces to the analogous statement for $X \in \mathsf{Set}^\kappa$, which is clear. □

Proof of Proposition 3.2. By testing on $\operatorname{map}(Y, -)$ for all $Y \in \mathcal{C}^\kappa$, we deduce from the lemma that the diagonal map $X \to \prod_\mu X/\mathcal{U}$ is an equivalence, where $\mathcal{U}$ is a μ-complete non-principal ultrafilter on μ. By λ-cocompactness of X, there is a λ-small subset $I \subseteq \mu$ and a factorization

$$\begin{array}{ccc} \prod_\mu X & \xrightarrow{\;p\;} & \prod_\mu X/\mathcal{U} \\ {\scriptstyle \mathrm{pr}}\downarrow & \overset{f}{\searrow} & \downarrow{\scriptstyle \simeq} \\ \prod_I X & \dashrightarrow & X \end{array}$$

where f is simply defined as the composite. But since I is λ- and hence μ-small, the fact that $\mathcal{U}$ is μ-complete shows that $I \notin \mathcal{U}$ and the fact that $\mathcal{U}$ is an ultrafilter then yields that $(\mu \setminus I) \in \mathcal{U}$. Thus p and hence f factor through $\prod_\mu X \to \prod_{\mu \setminus I} X$.

But the map $X \xrightarrow{\Delta} \prod_\mu X \xrightarrow{f} X$ is an equivalence, and hence the identity on X factors through the map f. We are thus in the situation of the lemma below, and conclude that X is (-1)-truncated. $\square$

Lemma 3.5. *Let $\mathcal{C}$ be a category with finite products and $X \in \mathcal{C}$. Suppose that the identity on X factors through a map $f\colon Y \times Z \to X$, where f factors through both projections pr_Y and pr_Z. Then X is (-1)-truncated.*

Proof. We have a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{s} & Y \times Z & \xrightarrow{\mathrm{pr}_Z} & Z \\ & & {\scriptstyle \mathrm{pr}_Y}\downarrow & \searrow^{f} & \downarrow{\scriptstyle \varphi} \\ & & Y & \xrightarrow[\psi]{} & X \end{array}$$

where $fs = \mathrm{id}_X$. We now claim that the composite

$$X \times X \xrightarrow{(\mathrm{pr}_Y s, \mathrm{pr}_Z s)} Y \times Z \xrightarrow{f} X$$

is an inverse to $\Delta\colon X \to X \times X$, which is straightforward from $\Delta f = (\psi, \varphi)\colon Y \times Z \to X \times X$. $\square$

4. Miscellaneous

Proposition 4.1. *Let κ be a regular cardinal, and $\mathcal{C} \in \mathsf{Pr}^L_\kappa$. If $\mathcal{C}^\kappa$ is closed under κ-indexed coproducts in $\mathcal{C}$, then $\mathcal{C}$ is a poset.*

Proof. Let $X \in \mathcal{C}^\kappa$ and $Y \in \mathcal{C}$. Then

$$\mathrm{map}(\coprod_\kappa X, \operatorname*{colim}_{\alpha<\kappa} \prod_\alpha Y) \simeq \prod_\kappa \operatorname*{colim}_{\alpha<\kappa} \prod_\alpha \mathrm{map}(X, Y)$$

If, on the other hand, $\coprod_\kappa X \in \mathcal{C}^\kappa$, we can also calculate

$$\mathrm{map}(\coprod_\kappa X, \operatorname*{colim}_{\alpha<\kappa} \prod_\alpha Y) \simeq \operatorname*{colim}_{\alpha<\kappa} \prod_\kappa \prod_\alpha \mathrm{map}(X, Y)$$

These equivalences agree under the canonical colimit-limit interchange map. Now, apply π_0 and then $\pi_n(-, *)$, $n \geq 1$, which commute with (κ-)filtered colimits and arbitrary products. This yields the analogous interchange map in Set, however we may identify both $\prod_\kappa \mathrm{colim}_{\alpha<\kappa} \prod_\alpha A$ and $\mathrm{colim}_{\alpha<\kappa} \prod_\kappa \prod_\alpha A$ as subsets of $\prod_\kappa \prod_\kappa A$, the '$\kappa \times \kappa$-matrices in A', with the former consisting of those matrices that are column-wise eventually constant (as transfinite sequences) and the latter consisting of those matrices that are *uniformly* column-wise eventually constant. These two subsets are clearly distinct if $|A| \geq 2$. Thus, we conclude that $\mathrm{map}(X, Y)$ is (-1)-truncated. Since Y was arbitrary and every object of $\mathcal{C}$ is a colimit of objects in $\mathcal{C}^\kappa$, it follows that all mapping spaces in $\mathcal{C}$ are (-1)-truncated, i.e. $\mathcal{C}$ is a poset. $\square$

Remark 4.2. The condition that $\mathcal{C}^\kappa$ is closed under κ-small coproducts in $\mathcal{C}$ is stronger than just $\mathcal{C}^\kappa$ admitting κ-indexed coproducts. The latter is easily achievable, e.g. by taking a small category $\mathcal{C}_0$ admitting κ-small colimits as well as κ-indexed coproducts and taking $\mathcal{C} := \mathrm{Ind}_\kappa(\mathcal{C}_0)$.

Lemma 4.3. *Suppose $\mathcal{C}$ is κ-bipresentable and μ is an inaccessible cardinal bigger than $\max(\kappa, |\mathcal{C}^\kappa|, |\mathcal{C}_\kappa|)$ (but still small w.r.t. the universe). Then $\mathcal{C}^\mu = \mathcal{C}_\mu$, i.e. the full subcategories of μ-compact objects and μ-cocompact objects agree.*

Proof. First we observe that if $\lambda < \mu$ is regular, there exists a regular $\lambda < \lambda' < \mu$ so that $\mathcal{C}^\lambda \subseteq \mathcal{C}_{\lambda'}$ and $\mathcal{C}_\lambda \subseteq \mathcal{C}^{\lambda'}$. Indeed, for every $X \in \mathcal{C}^\lambda$ there exists a diagram of size $\rho_X < \mu$ taking values in $\mathcal{C}_\kappa$ with limit X. Hence there exists regular $\mu > \lambda_X > \rho_X$ with $X \in \mathcal{C}_{\lambda_X}$ (as this is closed under λ_X-small limits). Since $\mathcal{C}^\lambda$ is small, we can take the supremum over λ_X and then a regular cardinal $\lambda' < \mu$ which is still bigger, and this yields $\mathcal{C}^\lambda \subseteq \mathcal{C}_{\lambda'}$. The dual inclusion is analogous, and then we can take another maximum between the two λ'. Now we conclude using

$$\mathcal{C}^\mu = \bigcup_{\lambda<\mu \text{ regular}} \mathcal{C}^\lambda = \bigcup_{\lambda<\mu \text{ regular}} \mathcal{C}_\lambda = \mathcal{C}_\mu$$

where we can argue for the middle equalities as follows; given $X \in \mathcal{C}^{\mu}$, there is a μ-small κ-filtered diagram I of κ-compacts with colimit X. But if I is μ-small, then it is also λ-small for some $\lambda < \mu$, and hence $X \in \mathcal{C}^{\lambda}$. The statement for cocompacts is analogous. □

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