Sheaves on Manifolds

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Contents

Literature

These are lecture notes for a course given in the winter semester 2023/24 and summer semester 2024 at Münster university. They are currently work in progress and will be updated continually. Below we list some relevant literature for the course. This will also be expanded over time.

1. [Lan21] for a lot of the ∞-categorical basics we will rely on.

2. [Lur18, Appendix D.7 and Section 21.1.2] for details on dualizable and compactly assembled ∞-categories.

3. [Cla14] for more details on dualizable and compactly assembled categories, including the intrinsic characterisation and applications.

4. [MFO22, K-theory of inverse limits, by A. Efimov] for K-theory of dualizable categories.

5. [Efi24] for the general theory of localizing invariants on large categories.

6. [Lur17a, Appendix A.1] for the shape of a topos.

7. [Vol23] for the six-functor formalism of sheaves on topological spaces.

8. [Sch23] for a general discussion of six-functor formalisms.

Chapter 1

Overview of the course

The main goal of this course is to describe the six-functor formalism and Verdier duality for topological spaces using newly introduced concepts from ∞-category theory. We will also try to shed now light on some classical aspects, like shape theory and Kashiwara-Shapira's Ind-sheaves from this perspective. If time permits we will also discuss the theory of microsupport. On a technical level the course will mostly deal with these new ∞-categorical concepts. We will also explain how to apply these concepts to algebraic K-theory and explain recent results of Efimov and Bartels–Nikolaus.

The course is aimed at graduate students and postdocs and we will require a solid knowledge of ∞-category theory. While we will recall some concepts that we need (such as presentable ∞-categories) we assume that the reader is familiar with the basic concepts, such as limits and colimits, adjunctions and the Yoneda lemma. We also assume that the reader is familiar with the ∞-category of spectra, that will be crucial later in the course. Let us start by giving an overview some results and topics covered in the course.

1.1 The six functors on spaces

Let us first describe the six-functor formalism that we are after. Let X be a locally compact Hausdorff space. Then we can consider the ∞-category

$$\mathrm{Shv}(X; \mathcal{D}\mathbb{Z})$$

of sheaves on X with values in the ∞-categorical derived category of $\mathbb{Z}$. Concretely such a sheaf is given by a functor

$$\mathcal{F} : \mathrm{Open}(X)^{\mathrm{op}} \to \mathcal{D}\mathbb{Z}$$

which satisfies descent, i.e. $\mathcal{F}(\emptyset) = 0$, for two open sets $U, V \subseteq X$ we have that

$$\begin{tikzcd} \mathcal{F}(U \cup V) \arrow[r] \arrow[d] & \mathcal{F}(U) \arrow[d] \\ \mathcal{F}(V) \arrow[r] & \mathcal{F}(U \cap V) \end{tikzcd}$$

is a pullback and that for an increasing union of open subsets $\{U_i\}_{i\in I}$ indexed by a filtered poset I the map

$$\mathcal{F}\left(\bigcup_i U_i\right) \to \lim_I \mathcal{F}(U_i)$$

is an equivalence in $\mathcal{D}\mathbb{Z}$. Note that everything we say in this chapter will be more generally true for sheaves with values in any presentable, stable ∞-category in place of $\mathcal{D}\mathbb{Z}$, but for concreteness we stick with $\mathcal{D}\mathbb{Z}$ here.

Remark 1.1.1. The ∞-category $\mathrm{Shv}(X;\mathcal{D}\mathbb{Z})$ is closely related to the derived category $\mathcal{D}(\mathrm{Shv}(X,\mathrm{Ab}))$ of sheaves of abelian groups on X, but generally not equivalent. The latter is a Bousfield localization of $\mathrm{Shv}(X;\mathcal{D}\mathbb{Z})$, more precisely it is equivalent to the ∞-category of hypersheaves on X. If X is paracompact and has finite covering dimension, then the two ∞-categories are equivalent though.

The category $\mathcal{D}\mathbb{Z}$ has some extra structure, namely it has a symmetric monoidal structure $\otimes$ given by the tensor product of sheaves. This is defines as the sheafification of the pointwise tensor product of functors. It turns out that this is a closed symmetric monoidal structure, that is for any pair of sheaves $\mathcal{F},\mathcal{G}$ on X there exists another sheaf $\underline{\mathrm{Hom}}(\mathcal{F},\mathcal{G}) \in \mathrm{Shv}(X;\mathcal{D}\mathbb{Z})$ with the universal property that maps $\mathcal{H} \to \underline{\mathrm{Hom}}(\mathcal{F},\mathcal{G})$ in $\mathrm{Shv}(X;\mathcal{D}\mathbb{Z})$ are naturally the same as maps $\mathcal{H}\otimes\mathcal{F} \to \mathcal{G}$. The functors $\otimes$ and $\underline{\mathrm{Hom}}$ are functors number 1 and 2 in our six-functor formalism. Now for any continuous map $f : Y \to X$ we have the pushforward functor

$$f_* : \mathrm{Shv}(Y;\mathcal{D}\mathbb{Z}) \to \mathrm{Shv}(X,\mathcal{D}\mathbb{Z}) \qquad (f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U)) \ .$$

For example for $f : Y \to \mathrm{pt}$ we have that $\mathrm{Shv}(\mathrm{pt},\mathcal{D}\mathbb{Z}) = \mathcal{D}\mathbb{Z}$ and $f_*\mathcal{F} = \mathcal{F}(Y)$ is given by global section and thus also written as $\Gamma(\mathcal{F})$. This functor has a left adjoint

$$f^* : \mathrm{Shv}(X,\mathcal{D}\mathbb{Z}) \to \mathrm{Shv}(Y;\mathcal{D}\mathbb{Z})$$

given by pullback of sheaves. Concretely $(f^*\mathcal{F})$ is given by the sheafification of the presheaf $U \mapsto \mathrm{colim}_{V\supseteq f(U) \text{ open}}\, \mathcal{F}(V)$. For example if $f : U \to X$ is the inclusion of an open set, then $f^*\mathcal{F}$ is simply the restriction of $\mathcal{F}$ to opens in U and thus sometimes written as $\mathcal{F}|_U$. For the inclusion $f : \{x\} \to X$ of a point the pullback $f^*\mathcal{F}$ is the stalk and written as $\mathcal{F}_x$. For the projection $f : X \to \mathrm{pt}$ the pullback f^*C for $C \in \mathcal{D}\mathbb{Z}$ is given by the constant sheaf with value C, that is the sheafification of the presheaf that is constant with value C. We shall also write this as $\underline{C}$.

The functors f^* and f_* are functors 3 and 4 of the six functors. Finally for a map $f : Y \to X$ there is also the functor

$$f_! : \mathrm{Shv}(Y,\mathcal{D}\mathbb{Z}) \to \mathrm{Shv}(X,\mathcal{D}\mathbb{Z})$$

of proper pushforward defined as

$$(f_!\mathcal{F})(U) = \mathrm{colim}_{K\subseteq f^{-1}U \text{ s.t. } K\to U \text{ proper}}\, \mathrm{fib}\left(\mathcal{F}(f^{-1}U) \to \mathcal{F}(f^{-1}U \setminus K)\right) \ .$$

If $f : Y \to \mathrm{pt}$ then we have that

$$f_!\mathcal{F} = \mathrm{colim}_{K \subseteq Y \text{ compact}} \mathrm{fib}\left(\mathcal{F}(Y) \to \mathcal{F}(Y \setminus K)\right)$$

is given by 'global sections with compact support' and written as $\Gamma_c(\mathcal{F})$. There is a natural map

$$f_!\mathcal{F} \to f_*\mathcal{F}$$

which is immediate from the definitions (as the map from the fiber to the first term) and which is an equivalence if f is proper.

Proposition 1.1.2. *If $i : U \to X$ is the inclusion of an open set, then the functor $i_!$ is given by 'extension by zero', that is $i_!\mathcal{F}$ is the sheafification of the presheaf*

$$V \mapsto \begin{cases} \mathcal{F}(V) & V \subseteq U \\ 0 & \textit{else} \end{cases}$$

Theorem 1.1.3 (Proper Base change)**.** *If we have a pullback diagram of locally compact Hausdorff spaces*

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

then for $\mathcal{F} \in \mathrm{Shv}(Y, \mathcal{D}\mathbb{Z})$ we have that

$$g^* f_!(\mathcal{F}) \simeq f'_! g'^*(\mathcal{F}).$$

In particular we have for $f : Y \to X$ and $x \in X$ that

$$(f_!\mathcal{F})_x = \Gamma_c(i^*\mathcal{F})$$

for $i : Y_x \to Y$ the inclusion of the fiber of the point. The functor $f_!$ is functor number 5 and it turns out that it has a mysterious right adjoint

$$f^! : \mathrm{Shv}(X; \mathcal{D}\mathbb{Z}) \to \mathrm{Shv}(Y; \mathcal{D}\mathbb{Z}) \ .$$

which is functor number 6 and called the exceptional inverse image functor. In general the functor $f^!$ is tricky to describe, but if $f : U \to X$ is the inclusion of an open subset then $f^! = f^*$ as one easily sees from Proposition 1.1.2 since extension by zero is more or less by definition left adjoint to f^*. We can summarize the situation by saying that we have for $f : Y \to X$ that

1. $f_! = f_*$ if f is proper
2. $f^! = f^*$ if f is an open immersion.

We claim that properties (1) and (2) already uniquely determine the adjunction $(f_!, f^!)$ for all maps f provided we also require functoriality, that is $(fg)_! = f_! g_!$. To see this we use the following assertion:

Lemma 1.1.4. *Every map $f : Y \to X$ of locally compact Hausdorff spaces can be factored as $Y \xrightarrow{i} \overline{Y} \xrightarrow{p} X$ where i is an open immersion and p is proper.*

Proof. We take the one point compactification Y' of Y. Then we consider the graph of f inside of $Y \times X$ and take its closure $\overline{Y}$ inside of $Y' \times X$. The projection $\overline{Y} \to X$ is then proper, and the inclusion $Y \to \overline{Y}$ open. $\square$

Now for a given factorization we have that

$$f_! = p_! i_! = p_* i_!$$

where $i_!$ is left adjoint to i^*. This uniquely determines the functor $f_!$.

Remark 1.1.5. One can wonder whether this is well-defined and how coherently this definition can be made. It is a remarkable observation by Gaitsgory-Rozenblyum and Liu-Zheng as well as Mann that one can in fact use this as a definition of $f_!$ and produce a highly coherent six functor formalism using that.

Remark 1.1.6. The ∞-category $\mathrm{Shv}(X; \mathcal{D}\mathbb{Z})$ for course makes sense for every topological space X, the conditions of being locally compact Hausdorff are not needed for that. The adjunction $f^* \dashv f_*$ also makes sense in this generality for each continuous map. However for the adjunction $f_! \dashv f^!$ to be defined and well behaved one then needs conditions on the map f that are automatically satisfies in the LCH case: it needs to be locally proper, see [SS14].

There is another way to recover the adjunction $f_! \dashv f^!$ from the adjunction $f^* \dashv f_*$ which is more categorical in nature than the geometric construction given above.

Let us describe the idea, which will be the central theme of this lecture course. For every presentable, stable ∞-category $\mathcal{C}$ there is a 'dual' category $\mathcal{C}^\vee$. One of the central themes of the first few weeks of the lecture will be to study this duality and particularly which categories are 'dualizable' (meaning that $\mathcal{C} \simeq (\mathcal{C}^\vee)^\vee$). The dual of the category of sheaves on a locally compact Hausdorff space is the ∞-category $\mathrm{coShv}(X; \mathcal{D}\mathbb{Z})$ of cosheaves with values in $\mathcal{D}\mathbb{Z}$, that is functors

$$\mathcal{F} : \mathrm{Open}(X) \to \mathcal{D}\mathbb{Z}$$

which satisfy 'codescent', that is $\mathcal{F}(\emptyset) = 0$, for two open sets $U, V \subseteq X$ we have that

$$\begin{tikzcd} \mathcal{F}(U \cap V) \arrow[r] \arrow[d] & \mathcal{F}(U) \arrow[d] \\ \mathcal{F}(V) \arrow[r] & \mathcal{F}(U \cup V) \end{tikzcd}$$

is a pushout and that for an increasing union of open subsets $\{U_i\}_{i\in I}$ indexed by a filtered poset I the map

$$\operatorname{colim}_I \mathcal{F}(U_i) \to \mathcal{F}\left(\bigcup_i U_i\right)$$

is an equivalence in $\mathcal{D}\mathbb{Z}$.

Theorem 1.1.7 (Lurie, Verdier duality)**.** *There is a canonical equivalence*

$$\mathbb{D} : \operatorname{Shv}(X, \mathcal{D}\mathbb{Z}) \simeq \operatorname{coShv}(X, \mathcal{D}\mathbb{Z})$$

sending $\mathcal{F} \in \operatorname{Shv}(X, \mathcal{D}\mathbb{Z})$ *to the cosheaf*

$$U \mapsto \Gamma_c(F|_U) \ .$$

Here we have used the functor $f_!$ implicitly in this equivalence, namely to define Γ_c. But we will see in the lecture course that this self-duality of $\operatorname{Shv}(X, \mathcal{D}\mathbb{Z})$ is a completely intrinsic property of the symmetric monoidal ∞-category $\operatorname{Shv}(X, \mathcal{D}\mathbb{Z})$: it is a locally rigid category. The rough idea is that for locally rigid categories $\mathcal{C}$ there is an equivalence to $\mathcal{C}^\vee$ informally induced by passing to internally dual objects. We will make this rigorous later in the course.

The point now is that for a continuous map $f : X \to Y$ the adjunction $f^* \dashv f_*$ dualizes to an adjunction on the categories of cosheaves:

$$f_+ : \operatorname{coShv}(X) \rightleftarrows \operatorname{coShv}(Y) : f^+ \ .$$

Concretely the functor f_+ is given by $f_+(\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$. Now using the duality of Theorem 1.1.7 for X and Y we get an induced adjunction between $\operatorname{Shv}(X)$ and $\operatorname{Shv}(Y)$. This is the adjunction $f_! \dashv f^!$. Said more abstractly: the adjunction $f_! \dashv f^!$ is the dual to $f^* \dashv f_*$ using the fact that categories of sheaves are canonically self-dual. The self-duality is induced by the tensor product.

Using the six functor formalism we we can define sheaf cohomology, compactly supported sheaf cohomology, sheaf homology and locally finite sheaf homology (aka Borel Moore homology) of a locally compact Hausdorff space X with coefficients in $\mathbb{Z}$ as

$$\begin{aligned} H^*(X,\mathbb{Z}) &\coloneqq p_* p^* \mathbb{Z} & H^*_{\mathrm{c}}(X,\mathbb{Z}) &\coloneqq p_! p^* \mathbb{Z} \\ H_*(X,\mathbb{Z}) &= p_! p^! \mathbb{Z} = p_+ p^+ \mathbb{Z} & H^{\mathrm{lf}}_*(X,\mathbb{Z}) &= p_* p^! \mathbb{Z} \end{aligned}$$

where $p : X \to \mathrm{pt}$ is the unique map to the point and $\mathbb{Z}$ denotes the constant sheaf/cosheaf with value $\mathbb{Z} \in \mathcal{D}\mathbb{Z}$ on the point. [1] For the definition of homology it is maybe useful to think in terms of cosheaves using the $(-)_+ \dashv (-)^+$ adjunction to be convinced that this is a reasonable definition of homology. We will see that homology and cohomology are dual to each other as a consequence of the general properties of the six functor formalism. More

[1]We are slightly conflating the object of $\mathcal{D}(\mathbb{Z})$ and its homology here for the purpose of exposition. We should really write the (co)chains instead of (co)homology.

precisely if X is locally nice (e.g. a CW complex) then cohomology is the dual of homology. In general locally finite homology is the dual of compactly supported cohomology, e.g. for the Cantor set where the first statement fails (see next Section).

This generalizes as follows: we define the *dualizing sheaf* of X as $\omega_X := p^!(\mathbb{Z})$. Then we can define a functor:

$$D : \mathrm{Shv}(X; \mathcal{D}\mathbb{Z}) \to \mathrm{Shv}(X; \mathcal{D}\mathbb{Z})^{\mathrm{op}} \qquad \mathcal{F} \mapsto \underline{\mathrm{Hom}}(\mathcal{F}, \omega_X)$$

and we refer to $D\mathcal{F}$ as the *Verdier dual of* $\mathcal{F}$. The functor D is left adjoint to D^{op}, that is

$$\mathrm{Map}_{\mathrm{Shv}(X;\mathcal{D}\mathbb{Z})}(\mathcal{F}, D\mathcal{G}) \simeq \mathrm{Map}_{\mathrm{Shv}(X;\mathcal{D}\mathbb{Z})}(\mathcal{G}, D\mathcal{F})$$

but generally the map $\mathcal{F} \to D^2\mathcal{F}$ is not an equivalence (it is for many sheaves though, such as $p^*\mathbb{Z}$ on nice spaces X). While mysterious on the side of sheaves, under the equivalence to cosheaves, $\underline{\mathrm{Hom}}(-, f^!(\mathbb{Z}))$ corresponds to $\underline{\mathrm{Hom}}(-, f^+(\mathbb{Z}))$, i.e. the dual of global sections. This means that under the equivalence $\mathrm{Shv}(X; \mathcal{D}\mathbb{Z}) \simeq \mathrm{coShv}(X; \mathcal{D}\mathbb{Z})$ the functor D sends a sheaf to the pointwise dual if the associated cosheaf. Using this we find that:

Proposition 1.1.8. *We have for* $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}(X; \mathcal{D}\mathbb{Z})$ *and* $C \in \mathcal{D}\mathbb{Z}$*:*

$$\begin{gathered} p_* D\mathcal{F} = D p_! \mathcal{F} \\ \underline{\mathrm{Hom}}(\mathcal{F}, D\mathcal{G}) = \underline{\mathrm{Hom}}(\mathcal{G}, D\mathcal{F}) \\ p^! DC = D p^* C \end{gathered}$$

Example 1.1.9. Combining the first and last assertion of Proposition 1.1.8 we obtain that

$$p_* p^! DC = D p_! p^* C$$

for any $C \in \mathcal{D}\mathbb{Z}$. Specifically for $C = \mathbb{Z}$ this shows the claim we already made above, namely that locally finite homology is always the dual of compactly supported cohomology.

Applying the first assertion to $\mathcal{F} = Dp^*C$ and also using the third we get

$$p_* D^2 p^* C = D p_! p^! DC \ .$$

If we assume that $p^*\mathbb{Z}$ agrees with its bidual (which is the case for sufficiently nice spaces such as CW complexes), then this yields the claim that the homology considered above is indeed a predual of cohomology.

Again we shall see that all these things make sense in an arbitrary locally rigid ∞-category. Verdier duality becomes particularly useful when we understand the dualizing complex:

Theorem 1.1.10 (Poincaré duality)**.** *Let* X *be a (homology) manifold of dimension* n*. Then* $p^!(-)$ *is equivalent to* $p^*(-) \otimes \omega_X$ *and* ω_X *is locally equivalent to* $\mathbb{Z}[n]$*.*[2]

[2]More precisely it is given by the n-fold shift of the orientation sheaf.

1.2 Efimov K-theory

For any small stable ∞-category $\mathcal{C}$ there is an algebraic K-theory spectrum[3]. We assume that the small stable ∞-categories are idempotent complete and denote the ∞-category of small, idempotent complete stable ∞-categories by $\mathrm{Cat}_\infty^{\mathrm{perf}}$. Then K-theory is a functor

$$\mathrm{K} : \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathrm{Sp}$$

where Sp is the ∞-category of spectra. We will review the definition of K-theory in the lecture. For a ring R we shall write $\mathrm{K}(R) := K(\mathcal{D}^{\mathrm{perf}}R)$.

A dualizable stable ∞-category on the other hand is a presentable stable ∞-category, i.e. a large category. One key fact is that there is a functor from small stable ∞-categories to dualizable ones, which sends $\mathcal{C}$ to its Ind-category $\mathrm{Ind}(\mathcal{C})$. Morally this freely adds filtered colimits (equivalently infinite sums) to $\mathcal{C}$. This defines a full faithful embedding

$$\mathrm{Ind} : \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathrm{Cat}_\infty^{\mathrm{dual}}$$

where the target is the category of dualizable, stable ∞-categories whose definition is the first major goal of the course. For example Ind takes the perfect derived ∞-category $\mathcal{D}^{\mathrm{perf}}R$ of any ring R (or more generally qcqs scheme) to the derived ∞-category $\mathcal{D}(R)$. As we have mentioned before, the object $\mathrm{Shv}(X, \mathcal{D}\mathbb{Z})$ for a locally compact Hausdorff space is an object in $\mathrm{Cat}_\infty^{\mathrm{dual}}$. It does not lie in the image of Ind as we will also see. Other examples of objects of interest in $\mathrm{Cat}_\infty^{\mathrm{dual}}$ are the categories of nuclear modules associated with analytic rings as defined by Clausen-Scholze.

Theorem 1.2.1 (Efimov)**.** *There is a functor*

$$\mathrm{K}^{\mathrm{cont}} : \mathrm{Cat}_\infty^{\mathrm{dual}} \to \mathrm{Sp}$$

that extends K-theory, i.e. such that $\mathrm{K}^{\mathrm{cont}} \circ \mathrm{Ind}$ *is equivalent to K-theory. This functor sends Verdier sequences to fiber sequences and is essentially uniquely determined by these properties.*

This result now allows us to take K-theory for the interesting categories such as sheaves or nuclear modules. It also shows that $\mathrm{K}^{\mathrm{cont}}(\mathcal{D}R) = \mathrm{K}(\mathcal{D}^{\mathrm{perf}}R) = \mathrm{K}(R)$. For the former the foundational result of Efimov is the following:

Theorem 1.2.2 (Efimov)**.** *For any locally compact Hausdorff space X there is an equivalence*

$$\mathrm{K}^{\mathrm{cont}}(\mathrm{Shv}(X, \mathcal{D}\mathbb{Z})) \simeq \Gamma_c\left(X, \underline{\mathrm{K}\mathbb{Z}}\right) \ .$$

Here $\underline{\mathrm{K}\mathbb{Z}}$ is the constant sheaf on the K-theory spectrum $\mathrm{K}\mathbb{Z}$ of the integers, considered as an object of

$$\mathrm{Shv}(X, \mathrm{Sp})$$

[3]For the experts: we always mean non-connective K-theory here

and we use that for sheaves of spectra there is an analogous six functor formalism as for sheaves with values in $\mathcal{D}\mathbb{Z}$. If we denote the map $X \to \mathrm{pt}$ by p we can also write $\Gamma_c(X, \underline{K\mathbb{Z}})$ as $p_! p^*(\mathrm{K}\mathbb{Z})$. We refer to this spectrum as the compactly supported $\mathrm{K}\mathbb{Z}$-cohomology of X. If X is compact then this is the equivalent to the cohomology

$$\Gamma(X, \underline{\mathrm{K}\mathbb{Z}}) = p_* p^* \mathrm{K}\mathbb{Z}$$

Here we have to be careful though, since this is sheaf cohomology. If we allow arbitrary compact Hausdorff spaces this might behave quite differently than the $\mathrm{K}\mathbb{Z}$ cohomology of the associated anima $\mathrm{Sing}(X) \in \mathrm{An}$. For example the Cantor set X has for any spectrum E that

$$\Gamma(X, \underline{E}) = \bigoplus_{\omega} E \qquad \text{and} \qquad E^{\mathrm{Sing}(X)} = \prod_{\omega} E\ .$$

However, if the space X is sufficiently nicely behaved, e.g. a CW-complex, then this distinction goes away. Slightly more generally we will see that for any topological space X that is locally of constant shape (we will explain what that means) there is an associated anima $\mathrm{Shape}(X) \in \mathrm{An}$ such that

$$\Gamma(X, \underline{E}) \simeq E^{\mathrm{Shape}(X)}$$

Being locally of constant shape for X is equivalent to the assertion that the functor

$$p^* : \mathrm{Shv}(\mathrm{pt}; \mathrm{An}) \to \mathrm{Shv}(X, \mathrm{An})$$

admits a left adjoint $p_\natural$ (recall that it always admits a right adjoint p_*). This condition and its analogue for sheaves of spectra will play a crucial role for us. We will see that in the stable case the left adjoint $p_\natural$ is automatically given by $p_!(- \otimes \omega_X)$.

Remark 1.2.3. If one is willing to work with pro-anima instead of anima then one can in fact define $\mathrm{Shape}(X)$ for any locally compact Hausdorff space X and get $\Gamma(X, \underline{E}) \simeq E^{\mathrm{Shape}(X)}$.

1.3 Completed Cosheaves

The ∞-category $\mathrm{Cat}_\infty^{\mathrm{dual}}$ of dualizable, stable ∞-categories has a lot of interesting structure which we will study in the lecture.

- $\mathrm{Cat}_\infty^{\mathrm{dual}}$ has all colimits and limits.
- There is the notion of Verdier sequences or short exact sequence that behaves a lot like short exact sequences of abelian groups.
- Every object $\mathcal{C} \in \mathrm{Cat}_\infty^{\mathrm{dual}}$ admits a 2-term 'resolution' by compactly generated stable ∞-categories, that is for fixed $\mathcal{C} \in \mathrm{Cat}_\infty^{\mathrm{dual}}$ there is a short exact sequence

$$0 \to \mathcal{C} \to \mathcal{D} \to \mathcal{E} \to 0$$

  with $\mathcal{D}$ and $\mathcal{E}$ compactly generated. Concretely we can choose $\mathcal{D}$ to be $\mathrm{Ind}(\mathcal{C}^{\omega_1})$ and $\mathcal{E}$ as $\mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}})$. We will explain what that means in the course.

In some sense we can think of compactly generated stable ∞-categories as 'injective objects' in $\mathrm{Cat}_\infty^{\mathrm{dual}}$.[4] In this sense we have an injective resolution and then for example the continuous K-theory functor $\mathrm{K}^{\mathrm{cont}} : \mathrm{Cat}_\infty^{\mathrm{dual}} \to \mathrm{Sp}$ of Efimov is defined using these resolutions, i.e. it is a sort of right derived functor of $\mathrm{K} : \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathrm{Sp}$.

- $\mathrm{Cat}_\infty^{\mathrm{dual}}$ has a tensor product $\otimes$ that make it symmetric monoidal. For example we have that $\mathrm{Shv}(X; \mathrm{Sp}) \otimes \mathrm{Shv}(Y; \mathrm{Sp}) \simeq \mathrm{Shv}(X \times Y; \mathrm{Sp})$. One can study dualizable objects within $\mathrm{Cat}_\infty^{\mathrm{dual}}$ and these turn out to be exactly the smooth and proper dualizable stable ∞-categories. The category $\mathrm{Shv}(X)$ is proper under very mild conditions on X but essentially never smooth.
- Since $\mathrm{Cat}_\infty^{\mathrm{dual}}$ has a tensor product we can speak about commutative algebra objects in $\mathrm{Cat}_\infty^{\mathrm{dual}}$. These are symmetric monoidal, presentable, stable ∞-categories with specific properties. Among those we will study a subclass called *(locally) rigid categories*. This notions extends the notion of rigidity for small symmetric monoidal categories. We will see that Verdier duality essentially is the statement that $\mathrm{Shv}(X; \mathrm{Sp})$ is locally rigid for any locally compact Hausdorff space X. It is rigid precisely if X is compact.
- The tensor product is closed, that is there is an inner hom $\underline{\mathrm{Hom}}^{\mathrm{dual}}(\mathcal{C}, \mathcal{D})$ for any pair of stable, dualizable ∞-categories. In general this inner hom is a bit hard to understand, but again one can use injective resolutions to get a handle on it. Specifically the ∞-categories of nuclear modules of Clausen-Scholze can be seen to be (a slight variant of) the inner hom in dualizable category between well-understood categories, e.g.

$$\widetilde{\mathrm{Nuc}}(\mathbb{Z}_p) \simeq \underline{\mathrm{Hom}}_{\mathcal{D}\mathbb{Z}}^{\mathrm{dual}}\left((\mathcal{D}\mathbb{Z})_p^\wedge, \mathcal{D}\mathbb{Z}\right) \simeq \lim_{n\to\infty}^{\mathrm{dual}} \mathcal{D}(\mathbb{Z}/p^n)$$

We will specifically study

$$\widehat{\mathrm{coShv}}(X; \mathcal{D}) \coloneqq \underline{\mathrm{Hom}}^{\mathrm{dual}}(\mathrm{Shv}(X; \mathrm{Sp}), \mathcal{D})$$

the 'dual' of the stable ∞-category of sheaves. We will give a concrete description of $\widehat{\mathrm{coShv}}(X; \mathcal{D})$ and relate it to Ind-sheaves (which are an ∞-categorical version of Kashiwara-Shapiras Ind sheaves). The main result about $\widehat{\mathrm{coShv}}(X; \mathcal{D})$ is the following:

Theorem 1.3.1 (Bartels–Efimov–Nikolaus)**.** *We have that* $\mathrm{K}^{\mathrm{cont}}(\widehat{\mathrm{coShv}}(X; \mathcal{D}\mathbb{Z}))$ *is the locally finite* $\mathrm{K}(\mathbb{Z})$*-homology of* X *(aka. Borel-Moore homology), that is:*

$$\mathrm{K}^{\mathrm{cont}}(\widehat{\mathrm{coShv}}(X; \mathcal{D}\mathbb{Z})) \simeq p_* p^! \mathrm{K}\mathbb{Z}$$

for $p : X \to \mathrm{pt}$.

[4]We will see that there is in fact a better notion of injective, but for the purpose of this introduction thinking if injective resolutions gives a good intuition.

Chapter 2

Categorical structures

2.1 Presentable ∞-categories

Presentable ∞-categories are big categories (*all* sets, *all* modules, etc.), that are in a sense still generated by small objects.

Definition 2.1.1. Let κ be a regular cardinal (e.g. $\kappa = \omega$, the first countable ordinal, or $\kappa = \omega_1$, the first uncountable ordinal).

1. An ∞-category I is *κ-filtered* if any map $K \to I$ from a κ-small simplicial set extends over the right cone $K * \Delta^0 \to I$.

2. For an ∞-category $\mathcal{C}$ with small colimits, an object $X \in \mathcal{C}$ is *κ-compact* if the canonical map
$$\operatorname{colim}_{i\in I} \operatorname{Map}_{\mathcal{C}}(X, Y_i) \to \operatorname{Map}_{\mathcal{C}}(X, \operatorname{colim}_{i\in I} Y_i)$$
is an equivalence for any κ-filtered small I and any functor $Y : I \to \mathcal{C}$. We write $\mathcal{C}^\kappa$ for the full subcategory on κ-compact objects.

We also call a functor $I \to \mathcal{C}$ with I κ-filtered a *κ-filtered diagram* in $\mathcal{C}$, speak of κ-filtered colimits, etc. If $\kappa = \omega$, we simply say filtered and compact.

Example 2.1.2.

1. The κ-compact objects in Set are precisely the κ-small sets, i.e. those with cardinality smaller than κ. The collection of κ-small subsets of a given set S forms a κ-filtered category, since the union of less than κ many κ-small subsets of S is still κ-small (this is where regularity of κ enters).

2. Similarly, κ-compact objects in $\operatorname{Mod}(R)$ are modules with a presentation with less than κ many generators and relations, and retracts of those.

3. κ-compact objects in $\mathcal{D}(R)$ are those which are equivalent to complexes of projectives with less than κ many generators in total.

4. κ-compact objects in An, the ∞-category of anima (homotopy types) are those anima which can be represented by simplicial sets with less than κ many nondegenerate simplices (or CW complexes with less than κ many cells), and retracts of those.

Lemma 2.1.3. *1. In* An*, κ-small limits commute with κ-filtered colimits.*

2. κ-small colimits of κ-compact objects are again κ-compact.

Proof. The first statement is [Lur17b, Proposition 5.3.3.3], and is a special property of An or more general ∞-topoi. As a quick reality check though, for a κ-small *product* we may check that

$$\mathrm{colim}_{i\in I}\prod_{j\in J} X_{ij} \to \prod_{j\in J}\mathrm{colim}_{i\in I} X_{ij}$$

is an equivalence: A point in the left term consists of a choice of i and for every j a point in X_{ij}. In the right term, we instead have for every j, a choice of point in $X_{i(j),j}$ for some $i(j)$ depending on j. The former is more restrictive, but if I is κ-filtered and J κ-small, then the $i(j)$ have a common upper bound in I, so any point in the target really comes from the source. (To turn this into a proof, we of course need to argue also about homotopies between points etc.)

The second follows from the first: Let $X : K \to \mathcal{C}$ be a κ-small diagram of κ-compact objects, and $Y : I \to \mathcal{C}$ a κ-filtered diagram. We may write

$$\begin{aligned} &\mathrm{Map}_{\mathcal{C}}(\mathrm{colim}_K X_k, \mathrm{colim}_I Y_i) \\ \simeq\, &\lim_K \mathrm{Map}_{\mathcal{C}}(X_k, \mathrm{colim}_I Y_i) \\ \simeq\, &\lim_K \mathrm{colim}_I \mathrm{Map}_{\mathcal{C}}(X_k, Y_i) \\ \simeq\, &\mathrm{colim}_I \lim_K \mathrm{Map}_{\mathcal{C}}(X_k, Y_i) \\ \simeq\, &\mathrm{colim}_I \mathrm{Map}_{\mathcal{C}}(\mathrm{colim}_K X_k, Y_i) \end{aligned}$$

□

We will now see a way to freely adjoin κ-filtered colimits to a given category. Recall first how to adjoin *all* small colimits:

Lemma 2.1.4. *Let $\mathcal{C}$ be a small ∞-category.*

1. *The Yoneda embedding $j : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$ is fully faithful.*

2. *For any $\mathcal{D}$ with all small colimits, restriction along j induces an equivalence between*

$$\mathrm{Fun}^{\mathrm{colim}}(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

where the left hand denotes small colimit-preserving functors. The inverse is given by left Kan extension along j.

Proof. [Lur17b, Proposition 5.1.3.1 and Theorem 5.1.5.6] □

This means that a colimit-preserving functor is determined by its restriction to the Yoneda image, and any functor on the Yoneda image may be extended to a colimit-preserving one by Kan extension. Note that $j : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$ does not preserve any nontrivial colimits, as colimits in the latter are formed pointwise.

Definition 2.1.5. For a small ∞-category $\mathcal{C}$, we define $\mathrm{Ind}_\kappa(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$ as the smallest full subcategory containing $j(\mathcal{C})$ and closed under κ-filtered colimits.

Remark 2.1.6. In $\mathrm{Ind}_\kappa(\mathcal{C})$, we have objects jX for every $X \in \mathcal{C}$, but also κ-filtered colimits, so we may form

$$\mathrm{colim}_{i \in I}\, jX_i$$

for a κ-filtered diagram $I \to \mathcal{C}$. Even if $\mathrm{colim}_{i\in I} X_i$ exists, this does not agree with $j\,\mathrm{colim}_{i\in I} X_i$, so we think of this as a new "formal" colimit we adjoin to $\mathrm{Ind}_\kappa(\mathcal{C})$. More specifically, these formal colimits are computed in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$, i.e. pointwise in An. In particular, the Yoneda Lemma implies that the jX are automatically κ-compact:

$$\begin{aligned} \mathrm{Map}_{\mathrm{Ind}_\kappa(\mathcal{C})}(jX, \mathrm{colim}_i Y_i) &\simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathrm{An})}(jX, \mathrm{colim}_i Y_i) \\ &\simeq \mathrm{colim}_i Y_i(X) \\ &\simeq \mathrm{colim}_i \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathrm{An})}(jX, Y_i) \\ &\simeq \mathrm{colim}_i \mathrm{Map}_{\mathrm{Ind}_\kappa(\mathcal{C})}(jX, Y_i). \end{aligned}$$

As a consequence, mapping spaces may be computed as

$$\begin{aligned} &\mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(\mathrm{colim}_{i\in I}\, jX_i, \mathrm{colim}_{i'\in I'}\, jY_{i'}) \\ =\, &\lim_{i\in I} \mathrm{colim}_{i'\in I'} \mathrm{Map}_{\mathcal{C}}(X_i, Y_{i'}) \\ =\, &\mathrm{colim}_{\ell\in\mathrm{Fun}(I,I')} \lim_{i\in I} \mathrm{Map}_{\mathcal{C}}(X_i, Y_{\ell(i)}), \end{aligned}$$

where the last step uses that filtered colimits distribute over limits in An, see e.g. [CH21, Corollary 7.17]. Moreover, it turns out that every object in $\mathrm{Ind}_\kappa(\mathcal{C})$ is in fact of the form $\mathrm{colim}_{i\in I}\, jX_i$ for some filtered diagram $I \to \mathcal{C}$.

Lemma 2.1.7. *For any $\mathcal{D}$ with κ-filtered colimits, restriction along j induces an equivalence*

$$\mathrm{Fun}^{\mathrm{colim}_{\kappa-\mathrm{filt}}}(\mathrm{Ind}_\kappa(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

Proof. [Lur17b, Proposition 5.3.5.10]. □

The inverse of the above equivalence associates to any functor $f : \mathcal{C} \to \mathcal{D}$ its Ind_κ-extension $F : \mathrm{Ind}_\kappa(\mathcal{C}) \to \mathcal{D}$. By the above Lemma, it is uniquely determined by the fact that F preserves κ-filtered colimits and $Fj \simeq f$.

Proposition 2.1.8. *Let $\mathcal{C}$ be a small ∞-category and $\mathcal{D}$ an ∞-category admitting κ-filtered colimits. Let $f : \mathcal{C} \to \mathcal{D}$ be any functor and $F : \mathrm{Ind}_\kappa(\mathcal{C}) \to \mathcal{D}$ its Ind_κ-extension. Then*

1. *If f is fully faithful and its image consists of κ-compact objects, then also F is fully faithful.*

2. *If additionally to (1), the image of f generates $\mathcal{D}$ under κ-filtered colimits, then F is an equivalence.*

Proof. [Lur17b, Proposition 5.3.5.11]. $\square$

If $\mathcal{C}$ already admits κ-small colimits, there is another more intrinsic description of $\mathrm{Ind}(\mathcal{C})$.

Lemma 2.1.9. *If $\mathcal{C}$ admits κ-small colimits, $\mathrm{Ind}_\kappa(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$ consists precisely of those functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{An}$ which preserve κ-small limits. In particular, $\mathrm{Ind}_\kappa(\mathcal{C})$ admits all small limits.*

Proof. See [Lur17b, Corollary 5.3.5.4] for the first claim. The addendum follows because the condition of preserving κ-small limits is clearly closed under taking small limits in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$ (which are computed pointwise). $\square$

Corollary 2.1.10. *If $\mathcal{C}$ admits κ-small colimits then $j : \mathcal{C} \to \mathrm{Ind}_\kappa(\mathcal{C})$ preserves them.*

Proof. If $X : K \to \mathcal{C}$ is a κ-small diagram,

$$\mathrm{Map}_{\mathrm{Ind}_\kappa(\mathcal{C})}(j \operatorname{colim}_{k\in K} X_k, F) = F(\operatorname{colim}_{k\in K} X_k),$$

and

$$\mathrm{Map}_{\mathrm{Ind}_\kappa(\mathcal{C})}(\operatorname{colim}_{k\in K} jX_k, F) = \lim_{k\in K} F(X_k).$$

Since $F \in \mathrm{Ind}_\kappa(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$ preserves κ-small limits, these are equivalent. $\square$

This also leads to another universal property of Ind_κ: Adjoining *all* colimits, relative to already having κ-small colimits. This is related to the fact that every colimit can canonically be written as κ-filtered colimit of κ-small colimits, hence the heuristic

$$\text{all colimits} \;=\; \kappa\text{-filtered colimits} \;+\; \kappa\text{-small colimits},$$

for example

$$\text{all colimits} \;=\; \text{filtered colimits} \;+\; \text{finite colimits},$$

Lemma 2.1.11. *If $\mathcal{C}$ admits κ-small colimits, $\mathrm{Ind}_\kappa(\mathcal{C})$ admits all small colimits, and for any $\mathcal{D}$ which admits small colimits, restriction along j gives an equivalence*

$$\mathrm{Fun}^{\mathrm{colim}}(\mathrm{Ind}_\kappa(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}^{\mathrm{colim}_{\kappa-\mathrm{sm}}}(\mathcal{C}, \mathcal{D}).$$

Proof. [Lur17b, Example 5.3.6.8]. $\square$

A lot of categories in daily life are Ind of something. For example, every set is a filtered colimit of finite sets, every group is a filtered colimit of finitely presented groups, every anima is a filtered colimit of (retracts of) anima represented by finite simplicial sets. In all those cases, we see that they are Ind of their compact objects.

Definition 2.1.12. Suppose that $\mathcal{C}$ admits small colimits and that $\mathcal{C}^\kappa$ is small. Consider the canonical colimit-preserving functor

$$k : \mathrm{Ind}_\kappa(\mathcal{C}^\kappa) \to \mathcal{C},$$

defined as the Ind_κ-extension of $\mathcal{C}^\kappa \subseteq \mathcal{C}$. Note that k is fully faithful by Proposition 2.1.8. We say that $\mathcal{C}$ is *κ-compactly generated* if k is an equivalence.

Definition 2.1.13. We call an ∞-category *presentable* if it admits all small colimits, and is κ-compactly generated for some κ.

Remark 2.1.14. It follows from Lemma 2.1.9 that presentable categories automatically admit all small limits.

Lemma 2.1.15. *Let $\mathcal{C}$ be an ∞-category which admits small colimits and where $\mathcal{C}^\kappa$ is small. The following are equivalent:*

1. *$\mathcal{C}$ is κ-compactly generated.*

2. *Every object of $\mathcal{C}$ can be written as small colimit of κ-compact objects.*

3. *If $X \to Y$ is a morphism in $\mathcal{C}$ such that*

$$\mathrm{Map}_{\mathcal{C}}(Z, X) \to \mathrm{Map}_{\mathcal{C}}(Z, Y)$$

is an equivalence for every κ-compact Z, then $X \to Y$ is an equivalence.

Proof. We first show $1 \Leftrightarrow 2$. Since k is fully faithful, k being an equivalence is equivalent to k being essentially surjective. If every object X of $\mathcal{C}$ can be written as colimit $\mathrm{colim}_K Z_k$ of κ-compact objects Z_k, we may assume it to be a κ-filtered colimit of κ-compact objects by rewriting it as $\mathrm{colim}_{K' \subseteq K} \mathrm{colim}_{K'} Z_k$, where K' ranges over the κ-filtered system of κ-small simplicial subsets of K. If $X = \mathrm{colim}_I Z_i$ is a κ-filtered colimit of κ-compact objects, k takes $\mathrm{colim}_{i \in I} jZ_i$ to X. Conversely, every object in $\mathrm{Ind}_\kappa(\mathcal{C})$ is of this form, and so if k is essentially surjective, every object in $\mathcal{C}$ is a κ-filtered colimit of κ-compact objects.

For $1 \Leftrightarrow 3$, we have the restricted Yoneda embedding $j' : \mathcal{C} \to \mathrm{Ind}_\kappa(\mathcal{C}^\kappa)$ taking $X \mapsto \mathrm{Map}_{\mathcal{C}}(-, X)$. (This is a κ-small limit preserving functor from $\mathcal{C}^{\kappa,\mathrm{op}} \to \mathrm{An}$, so lies in $\mathrm{Ind}_\kappa(\mathcal{C}^\kappa)$.) We claim that k is left adjoint to j'. Indeed, both $\mathrm{Map}_{\mathrm{Ind}_\kappa(\mathcal{C}^\kappa)}(F, j'X)$ and $\mathrm{Map}_{\mathcal{C}}(kF, X)$ are κ-filtered colimit preserving functors $\mathrm{Ind}_\kappa(\mathcal{C}^\kappa) \to \mathrm{An}^{\mathrm{op}}$ in F. So to produce an equivalence between them, it suffices to do so on the image of $j : \mathcal{C}^\kappa \to \mathrm{Ind}_\kappa(\mathcal{C}^\kappa)$, and we have $\mathrm{Map}_{\mathrm{Ind}_\kappa(\mathcal{C}^\kappa)}(jZ, j'X) \simeq \mathrm{Map}_{\mathcal{C}}(Z, X)$ by the Yoneda lemma.

Fully faithfullness of k gives that

$$\mathrm{Map}_{\mathrm{Ind}_\kappa(\mathcal{C}^\kappa)}(F, G) \to \mathrm{Map}_{\mathcal{C}}(kF, kG) \simeq \mathrm{Map}_{\mathrm{Ind}_\kappa(\mathcal{C})}(F, j'kG)$$

is an equivalence for any F, G. By Yoneda, this means that the unit $G \to j'kG$ is an equivalence. If 3 is satisfied, j' detects equivalences. Since the counit $kj'X \to X$ is taken by j' to the inverse equivalence to the unit $j'X \to j'kj'X$, this means that $kj'X \to X$ is an equivalence and so k is essentially surjective. Conversely, if k is an equivalence, of course its adjoint j' is too, and so in particular it detects equivalences. $\square$

Lemma 2.1.16. *Let $\mathcal{C}$ be presentable. Then every object is κ-compact for some κ, and each $\mathcal{C}^\kappa$ is small. In particular, every small set of objects of $\mathcal{C}$ lies in some $\mathcal{C}^\kappa$.*

Proof. If $\mathcal{C}$ is κ-compactly generated, we can write an object $c \in \mathcal{C}$ as a colimit of κ-compact objects. The colimit diagram has some size λ, and each κ-compact object is also $\max(\kappa, \lambda)$-compact, hence c is again $\max(\kappa, \lambda)$-compact. Moreover, a λ-compact object can be written as a λ-filtered colimit of λ-small colimits of objects in $\mathcal{C}^\kappa$, hence will lie in the closure of $\mathcal{C}^\kappa$ under λ-small colimits, which is again small. □

Corollary 2.1.17. *If $\mathcal{C}$ is κ-compactly generated for some κ, it is also λ-compactly generated for every $\lambda > \kappa$.*

Proof. Apply the above Lemma and the characterization (3) from Lemma 2.1.15. □

Warning 2.1.18. *The reader familiar with the theory of accessible categories and especially the exposition from [Lur17b, Chapter 5] might be surprised by the above statement; in the setting of accessible categories it is just plain wrong, and instead we only get that there* exists some *$\lambda > \kappa$ for which $\mathcal{C}$ is again λ-accessible, see [Lur17b, Proposition 5.4.2.9]. The reason for this is that if $\mathcal{C}$ is only κ-accessible, we don't have all colimits, and so in trying to do the same steps that lead to the proof of the above corollary, we would need to be able to write every κ-filtered colimit as a λ-filtered colimit of λ-small κ-filtered colimits (not just of arbitrary λ-small colimits!), which is just not possible in general.*

Proposition 2.1.19. *If $\mathcal{D}$ is presentable and $\mathcal{C}$ is small, then $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is presentable.*

Proof. The category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ still admits all colimits, computed pointwise. Moreover, we claim that $S = \{\mathrm{const}_d \mid d \in \mathcal{D}^\kappa\}$ is a jointly conservative set of κ-compact objects in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. The κ-compactness of const_d is clear from the adjunction $\mathrm{const} \dashv \lim_{\mathcal{C}}$ and the fact that κ-filtered colimits commute with κ-small limits in An.

Now suppose $\alpha : G \Rightarrow H$ is a natural transformation of functors $G, H : \mathcal{C} \to \mathcal{D}$ such that $\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, \alpha)$ is an equivalence for every $F : C \to \mathcal{D}^\kappa \subseteq \mathcal{D}$. Since $\mathcal{D}^\kappa$ is jointly conservative in $\mathcal{D}$, it suffices to check that $\mathrm{Map}_{\mathcal{D}}(d, \alpha_c)$ is an equivalence for all $d \in \mathcal{D}^\kappa$ and $c \in \mathcal{C}$. But this is the image of the equivalence $\mathrm{Map}_{\mathcal{D}}(\mathrm{const}_d, \alpha)$ under the functor $\mathrm{ev}_c : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathcal{C}$, hence an equivalence. □

One of the main reasons for the importance of presentable ∞-categories is the following:

Theorem 2.1.20 (Adjoint functor theorem)**.**

1. *If $\mathcal{C}$ is presentable and $\mathcal{C} \to \mathcal{D}$ preserves small colimits, then it admits a right adjoint.*

2. *A functor $\mathcal{D} \to \mathcal{C}$ between presentable ∞-categories admits a left adjoint if and only if it preserves limits and κ-filtered colimits for some κ. (The latter condition is also called* accessibility *of the functor.)*

Proof. [Lur17b, Corollary 5.5.2.9, 5.5.2.10]. □

For example, the adjoint functor theorem implies that the “diagonal” functor $\mathcal{D} \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ admits a right adjoint if $\mathcal{C}$ is small and $\mathcal{D}$ presentable, hence that presentable ∞-categories also admit small limits.

A good notion of morphisms between presentable ∞-categories is given by pairs of adjoints.

Definition 2.1.21. Pr^L denotes the (very big!) category whose objects are presentable ∞-categories, and whose morphisms are left adjoint (or colimit-preserving) functors.

Equivalently, one may define Pr^R, and passage to the right adjoint gives an equivalence $\mathrm{Pr}^{L,\mathrm{op}} \simeq \mathrm{Pr}^R$.

To get more examples for presentable ∞-categories, we consider the following notion:

Definition 2.1.22. A (left) *Bousfield localization* of an ∞-category $\mathcal{C}$ consists of a pair of adjoint functors

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, hook, shift left, "R"] \end{tikzcd}$$

where R is fully faithful.

Of course, this data is determined already by one of the two functors, by uniqueness of adjoints. It is therefore relatively easy to describe a Bousfield localization of $\mathcal{C}$, simply by giving the full subcategory $\mathcal{D}$.

$$\mathrm{Map}_{\mathcal{D}}(X, Y) \to \mathrm{Map}_{\mathcal{C}}(RX, RY) \simeq \mathrm{Map}_{\mathcal{D}}(LRX, Y)$$

is an equivalence, so $LRX \simeq X$. An object Y lies in the essential image of R if and only if $Y \to RLY$ is an equivalence. If W denotes the class of all morphisms in $\mathcal{C}$ which are sent to equivalences in $\mathcal{D}$ by L, then $\mathrm{Map}_{\mathcal{C}}(-, Y)$ takes W to equivalences if and only if Y is in the essential image of R: In one direction, this is just the equivalence, in the other, assume that $\mathrm{Map}_{\mathcal{C}}(-, Y)$ takes W to equivalences, then this applies in particular to $Y \to RLY$, so the identity on Y factors through $Y \to RLY$. But then $Y \to RLY \to Y$ and, using the adjunction, $RLY \to Y \to RLY$ are both the identity and $Y \simeq RLY$.

Lemma 2.1.23. *If*

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, hook, shift left, "R"] \end{tikzcd}$$

is a Bousfield localization, and W the class of morphisms in $\mathcal{C}$ which are sent to equivalences under L, then precomposition with L provides an equivalence

$$\mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}^{W-\mathrm{loc}}(\mathcal{C}, \mathcal{E})$$

where the right hand side denotes the full subcategory on functors taking W to equivalences. The equivalence also restricts to an equivalence

$$\mathrm{Fun}^{\mathrm{colim}}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}^{W-\mathrm{loc},\mathrm{colim}}(\mathcal{C}, \mathcal{E}).$$

Proof. [Lur17b, Proposition 5.2.7.12] □

This justifies the name localization. We may similarly specify a Bousfield localization by providing a collection of morphisms W in $\mathcal{C}$ and letting $\mathcal{D}$ be the full subcategory on all W-local objects, i.e. $Y \in \mathcal{C}$ where $\mathrm{Map}_{\mathcal{D}}(-, Y)$ takes W to equivalences.

Lemma 2.1.24. *If $\mathcal{C}$ is presentable, W is a (small!) set of morphisms in $\mathcal{C}$, and $\mathcal{D}$ the full subcategory of W-local objects, then $\mathcal{D}$ is also presentable and a Bousfield localization of $\mathcal{C}$. It is universally characterized by L inducing an equivalence*

$$\mathrm{Fun}^L(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}^{W-\mathrm{loc},L}(\mathcal{C}, \mathcal{E})$$

Proof. [Lur17b, Proposition 5.5.4.2(3) and Remark 5.5.1.6] □

Example 2.1.25. For a space X, $\mathrm{PShv}(X; \mathrm{An}) \supseteq \mathrm{Shv}(X; \mathrm{An})$ is a presentable Bousfield localization: Sheaves are exactly those presheaves which are local with respect to the morphisms

1. $\emptyset \to j(\emptyset)$
2. $j(U) \amalg_{j(U \cap V)} j(V) \to j(U \cup V)$
3. $\mathrm{colim}_{i \in I}\, j(U_i) \to j(\bigcup_{i \in I} U_i)$

which form a set.

Note that we see from this description also that colimit-preserving functors $\mathrm{Shv}(X; \mathrm{An}) \to \mathcal{E}$ are the same as W-local colimit-preserving functors $\mathrm{PShv}(X; \mathrm{An}) \to \mathcal{E}$, and hence the same as functors $\mathrm{Open}(X) \to \mathcal{E}$ with $F(\emptyset)$ initial, $F(U) \amalg_{F(U \cap V)} F(V) \simeq F(U \cup V)$ and $\mathrm{colim}_I\, F(U_i) \simeq F(\bigcup U_i)$, i.e. cosheaves with values in $\mathcal{E}$!

In fact, every presentable ∞-category is more or less of that form.

Proposition 2.1.26. *Let $\mathcal{C}_0$ be a small category admitting κ-small colimits. Then we have a Bousfield localization*

$$j_!^\kappa : \mathrm{Ind}(\mathcal{C}_0) \rightleftarrows \mathrm{Ind}_\kappa(\mathcal{C}_0) : j_!$$

where $j_!^\kappa$ is the Ind*-extension of $j^\kappa : \mathcal{C}_0 \subseteq \mathrm{Ind}_\kappa(\mathcal{C}_0)$, and $j_!$ is the* Ind_κ*-extension of $j : \mathcal{C}_0 \subseteq \mathrm{Ind}(\mathcal{C}_0)$.*

Proof. Since objects in the image of j are compact and hence κ-compact, it follows from Proposition 2.1.8 that $j_!$ is fully faithful. Note also that

$$j_!^\kappa j_! j^\kappa \simeq j_!^\kappa j \simeq j^\kappa$$

and hence $j_!^\kappa j_! \simeq \mathrm{id}$ by the universal property. By the local existence criterion for adjunctions, it suffices to see that the following composite is an equivalence

$$\mathrm{Map}_{\mathrm{Ind}(\mathcal{C}_0)}(X, j_! Y) \xrightarrow{j_!^\kappa} \mathrm{Map}_{\mathrm{Ind}_\kappa(\mathcal{C}_0)}(j_!^\kappa X, j_!^\kappa j_! Y) \simeq \mathrm{Map}_{\mathrm{Ind}_\kappa(\mathcal{C}_0)}(j_!^\kappa X, Y).$$

Assume first that $X = jX'$ and $Y = j^\kappa Y'$. Then the equivalence follows from the fact that j and j^κ are fully faithful, and 2-out-of-3 applied to $j^\kappa_! j \simeq j^\kappa$. Since jX is compact and $j_!$ preserves κ-filtered colimits, it follows that the equivalence holds for jX' and arbitrary Y. Since $j^\kappa_!$ preserves filtered colimits, it then follows that the equivalence also holds for arbitrary X. □

Corollary 2.1.27. *Let $\mathcal{C}$ be κ-compactly generated. Then we have a Bousfield localization*

$$L : \mathrm{Ind}(\mathcal{C}^\kappa) \rightleftarrows \mathcal{C} : R$$

where R is a restricted yoneda embedding sending c to $\mathcal{C}(-,c)|_{(\mathcal{C}^\kappa)^{\mathrm{op}}}$. Moreover, R preserves finite and κ-filtered colimits.

Proof. Use $\mathcal{C}_0 = \mathcal{C}^\kappa$ in the above Proposition, and note that under the identification $\mathcal{C} \simeq \mathrm{Ind}_\kappa(\mathcal{C}^\kappa)$, we have $\mathrm{colim}_i\, \mathcal{C}(-,c_i)|_{(\mathcal{C}^\kappa)^{\mathrm{op}}} = \mathcal{C}(-, \mathrm{colim}_i\, c_i)|_{(\mathcal{C}^\kappa)^{\mathrm{op}}}$ if the colimit is κ-filtered. □

Corollary 2.1.28. *An ∞-category $\mathcal{C}$ is presentable if and only if it arises as a left Bousfield localization of* $\mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathrm{An})$ *for some small ∞-category $\mathcal{C}_0$.*

Proof. If $\mathcal{C}$ is presentable, it is κ-compactly generated for some κ, and we have a left Bousfield localization $\mathrm{Ind}(\mathcal{C}^\kappa) \to \mathcal{C}$ by the above. But $\mathrm{Ind}(\mathcal{C}^\kappa)$ is by definition a left Bousfield localization of $\mathrm{Fun}((\mathcal{C}^\kappa)^{\mathrm{op}}, \mathrm{An})$, consisting of the objects local with respect to the maps

$$\mathrm{colim}_{i \in I}\, jX_i \to j(\mathrm{colim}_{i \in I}\, X_i)$$

for all finite diagrams I. These form a set. (Or more precisely, there is a set of representatives up to equivalence.) Since left Bousfield localizations compose, we are done. □

Remark 2.1.29. Exhibiting $\mathcal{C}$ as Bousfield localization of $\mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathrm{An})$ is a kind of generators-and-relations presentation of $\mathcal{C}$, since it says that colimit-preserving functors out of $\mathcal{C}$ are determined by an arbitrary functor out of the small ∞-category $\mathcal{C}_0$ (the generators), such that its colimit extension is taking the small class W (the relations) to equivalences.

Corollary 2.1.30. *If $\mathcal{C}$ and $\mathcal{D}$ are presentable, the category*

$$\mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$$

consisting of left adjoint (i.e. colimit-preserving) functors is itself presentable.

Proof. $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$ is clearly closed under small colimits in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ and therefore admits all small colimits. Writing $\mathcal{C}$ as presentable localization of $\mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathrm{An})$ at a set of morphisms W, we see that

$$\mathrm{Fun}^L(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}_0, \mathcal{D})$$

is the full subcategory on functors whose colimit extension $\mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathrm{An}) \to \mathcal{D}$ takes W to equivalences. If κ is bigger than the size of $\mathcal{C}_0$ and W, one sees that κ-filtered colimits and κ-compact objects in are formed pointwise here. So the κ-compact objects form a small category and $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$ is compactly generated. □

This means that $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$ provides an inner Hom to the category Pr^L. We will see later that there is a tensor product left adjoint to this Hom.

We close this discussion of presentability by an application of the adjoint functor theorem regarding generators of a category.

Lemma 2.1.31. *Let $\mathcal{C}$ be a presentable ∞-category and S a set of objects in $\mathcal{C}$. Then the following are equivalent:*

1. *The smallest full subcategory of $\mathcal{C}$ closed under small colimits and containing S is $\mathcal{C}$ itself.*

2. *If $X \to Y$ is a morphism such that $\mathrm{Map}_{\mathcal{C}}(Z, X) \to \mathrm{Map}_{\mathcal{C}}(Z, Y)$ is an equivalence for each $Z \in S$, then $X \to Y$ is an equivalence.*

Proof. Let $\mathcal{C}_0 \subseteq \mathcal{C}$ be the smallest full subcategory of $\mathcal{C}$ closed under colimits and containing S. Then $\mathcal{C}_0$ has arbitrary small colimits, and is κ-compactly generated where κ is such that all objects of S are κ-compact. So it is presentable, and $i : \mathcal{C}_0 \to \mathcal{C}$ has a right adjoint R. Since i is fully faithful, $X \to RiX$ is an equivalence for each $X \in \mathcal{C}_0$, and $Y \in \mathcal{C}$ lies in $\mathcal{C}_0$ if and only if $iRY \to Y$ is an equivalence. Now assume 2, this implies that R is conservative. But $R(iRY \to Y)$ is the inverse to the unit $RY \to RiRY$, so an equivalence, and so every Y is in the image and $\mathcal{C}_0 = \mathcal{C}$. Conversely, if 1 holds, $i : \mathcal{C}_0 \to \mathcal{C}$ is an equivalence, so R is. If $X \to Y$ induces an equivalence on $\mathrm{Map}_{\mathcal{C}}(Z, -)$ for all $Z \in S$, it does so for all $Z \in \mathcal{C}_0 = \mathcal{C}$, and Yoneda applies. $\square$

If the equivalent conditions of the Lemma hold, we say that S *generates* $\mathcal{C}$. Furthermore, if there exists some κ so that $S \subseteq \mathcal{C}^\kappa$, then $\mathcal{C}$ is κ-compactly generated.

Lemma 2.1.32. *If for any $Z \in S$, also $Z \otimes S^n = \mathrm{colim}_{S^n} Z \in S$ (for example if S is closed under finite colimits), then S generates $\mathcal{C}$ if and only if the following holds: For any morphism $X \to Y$ in $\mathcal{C}$ where in each diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

with $A, B \in S$, the dashed lift exists, $X \to Y$ is an equivalence.

Proof. If $\mathcal{C}$ is generated by S, maps out of S detect equivalences. Given $X \to Y$ with the lifting condition, it therefore suffices that $\mathrm{Map}_{\mathcal{C}}(A, X) \to \mathrm{Map}_{\mathcal{C}}(A, Y)$ is an equivalence for any $A \in S$. The lifting problem for the diagram

$$\begin{tikzcd} A \otimes S^{n-1} \arrow[r] \arrow[d] & X \arrow[d] \\ A \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

translates to a lifting problem for the diagram

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{C}}(A,X) \arrow[d] \\ \mathrm{pt} \arrow[r] \arrow[ur, dashed] & \mathrm{Map}_{\mathcal{C}}(A,Y). \end{tikzcd}$$

If such lifts exist always, this means that all relative homotopy groups of the pair $\mathrm{Map}_{\mathcal{C}}(A,X) \to \mathrm{Map}_{\mathcal{C}}(A,Y)$ are trivial, hence that this map is an equivalence.

Conversely, assume the lifting condition detects equivalences, and we need to prove that then $\mathcal{C}$ is generated by S. So we need to prove that the $\mathrm{Map}_{\mathcal{C}}(A,-)$ together detect equivalences. Let $X \to Y$ be a morphism inducing equivalences on all $\mathrm{Map}_{\mathcal{C}}(A,-)$. Since a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

is a point in the pullback $\mathrm{Map}(B,Y) \times_{\mathrm{Map}(A,Y)} \mathrm{Map}(A,X)$, but

$$\begin{tikzcd} \mathrm{Map}(B,X) \arrow[r] \arrow[d] & \mathrm{Map}(A,X) \arrow[d] \\ \mathrm{Map}(B,Y) \arrow[r] & \mathrm{Map}(A,Y) \end{tikzcd}$$

is a pullback diagram since the vertical maps are equivalences, any such square admits a lift $B \to X$. But this means that $X \to Y$ satisfies the lifting condition, and so $X \to Y$ is an equivalence. $\square$

We will also need to consider a version of Ind-completion for large categories.

Definition 2.1.33. For a large (but locally small) category $\mathcal{C}$, we define $\mathrm{Ind}(\mathcal{C})$ analogously to the small case as the full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$ generated by representables under *small* filtered colimits.

This $\mathrm{Ind}(\mathcal{C})$ will still be large but locally small, and will generally behave analogously to the small case, the exception being that it is generally not presentable since it has too many compact objects, namely all of $\mathcal{C}$ via the fully faithful Yoneda embedding:

$$j : \mathcal{C} \subseteq \mathrm{Ind}(\mathcal{C}).$$

Moreover, $\mathrm{Ind}(\mathcal{C})$ still satisfies the universal properties as in Lemma 2.1.7 and 2.1.11, compare [Lur18, 21.1.2.8]. In particular, Ind-extensions exist (uniquely) as usual.

Lemma 2.1.34. *Suppose that $\mathcal{C}$ admits filtered colimits.*

1. *The* Ind*-extension of the identity* $k : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ *is left adjoint to the Yoneda embedding:*

$$\begin{tikzcd} \mathrm{Ind}(\mathcal{C}) \arrow[r, shift left, "k"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, hook, shift left, "j"] \end{tikzcd}$$

2. *The* Ind*-extension of a functor* $F : \mathcal{D} \to \mathcal{C}$ *is given by* $\mathrm{Ind}(\mathcal{D}) \xrightarrow{\mathrm{Ind}(F)} \mathrm{Ind}(\mathcal{C}) \xrightarrow{k} \mathcal{C}$.

Proof. By definition of the Ind-extension, we have a natural equivalence $kj \simeq \mathrm{id}$. Now by the local existence criterion it suffices to check that

$$\mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(X, jY) \xrightarrow{k} \mathrm{Map}_{\mathcal{C}}(kX, kjY) \simeq \mathrm{Map}_{\mathcal{C}}(kX, Y)$$

is an equivalence. Since k preserves filtered colimits, we can pull them out in the X-variable on both sides. Hence by definition of $\mathrm{Ind}(\mathcal{C})$ it suffices to check the equivalence on representables jX_0, which follows from the equivalence $kj \simeq \mathrm{id}$.

For the second point, denote by $F_! : \mathrm{Ind}(\mathcal{D}) \to \mathcal{C}$ the Ind-extension of F. Since both $F_!$ and $k \circ \mathrm{Ind}(F)$ preserve filtered colimits, we can check their equivalence after restricting along $j : \mathcal{D} \subseteq \mathrm{Ind}(\mathcal{D})$. But then $F_! j \simeq F \simeq kjF \simeq k\,\mathrm{Ind}(F)j$. □

We will often refer to the functor $k : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ as the "colimit functor", as it sends the formal filtered colimits $\mathrm{colim}_i\, jX_i$ in $\mathrm{Ind}(\mathcal{C})$ to actual filtered colimits $\mathrm{colim}_i\, X_i$ in $\mathcal{C}$. The following Lemma shows in particular that for κ-compactly generated categories, this k factors through a similar colimit functor $\mathrm{Ind}(\mathcal{C}^\kappa) \to \mathcal{C}$.

Lemma 2.1.35. *Suppose* $\mathcal{C}$ *is* κ*-compactly generated, and denote by* $i : \mathcal{C}^\kappa \subseteq \mathcal{C}$ *the inclusion. Recall the adjunction* $L \dashv R$ *from Corollary 2.1.27, and denote by* $\mathrm{Ind}(i) : \mathrm{Ind}(\mathcal{C}^\kappa) \rightleftarrows \mathrm{Ind}(\mathcal{C}) : i^*$ *the usual adjunction on* Ind*-categories induced by* i*. Then we have a commutative diagram*

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook, "j"] \arrow[dr, hook, "R"'] & \mathrm{Ind}(\mathcal{C}) \arrow[d, "i^*"'] \arrow[dr, "k"] & \mathrm{Ind}(\mathcal{C}^\kappa) \arrow[l, hook', "\mathrm{Ind}(i)"'] \arrow[d, "L"] \\ & \mathrm{Ind}(\mathcal{C}^\kappa) \arrow[r, "L"'] & \mathcal{C} \end{tikzcd}$$

Proof. Note first that by definition the adjunction $\mathrm{Ind}(i) \dashv i^*$ is restricted from the usual left-Kan-extension/restriction adjunction $\mathrm{Lan}_{i^{\mathrm{op}}} : \mathrm{Fun}((\mathcal{C}^\kappa)^{\mathrm{op}}, \mathrm{An}) \hookrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$ to the respective Ind-categories. Indeed, since both left Kan extension and restriction preserve colimits, it suffices to see that they send representables into the Ind-categories. Left Kan extension even preserves representable functors. For the restriction, note that $i : \mathcal{C}^\kappa \to \mathcal{C}$ preserves finite colimits, hence $\mathcal{C}(i(-), c) : (\mathcal{C}^\kappa)^{\mathrm{op}} \to \mathrm{An}$ preserves finite limits and is therefore contained in $\mathrm{Ind}(\mathcal{C}^\kappa)$ by Lemma 2.1.9.

Now L is by definition the Ind-extension of i, hence the previous Lemma gives $L \simeq k\,\mathrm{Ind}(i)$. Passing to right adjoints, we see that also the leftmost triangle commutes. Finally, to see that $Li^* \simeq k$, note first that both sides preserve filtered colimits, so it suffices to check this after restricting along j, where it becomes $Li^*j \simeq LR \simeq \mathrm{id} \simeq kj$. □

Lemma 2.1.36. *Let $\mathcal{C}, \mathcal{D}$ admit filtered colimits. If $F : \mathcal{C} \to \mathcal{D}$ preserves filtered colimits, then the Beck-Chevalley transformation $k \operatorname{Ind}(F) \Rightarrow Fk$ is an equivalence, making the following diagram commute:*

$$\begin{tikzcd} \operatorname{Ind}(\mathcal{C}) \arrow[r, "\operatorname{Ind}(F)"] \arrow[d, "k"'] & \operatorname{Ind}(\mathcal{D}) \arrow[d, "k"] \\ \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd}$$

Proof. The Beck-Chevalley transformation is defined as the composite

$$k \operatorname{Ind}(F) \xRightarrow{k \operatorname{Ind}(F)\eta} k \operatorname{Ind}(F) jk \simeq kjFk \underset{\simeq}{\xRightarrow{\varepsilon Fk}} Fk$$

where the middle equivalence comes from the naturality of j. It thus suffices to see that $k \operatorname{Ind}(F)\eta$ is an equivalence. Note that since $k \operatorname{Ind}(F)jk \simeq Fk$ preserves filtered colimits, it suffices to check the equivalence after restricting along j by Lemma the universal property of the Ind-completion. However, ηj is an equivalence by the triangle identities for the adjunction $k \dashv j$, hence we are done. □

2.2 Compactly assembled ∞-categories

Recall that an object X in an ∞-category $\mathcal{C}$ is called compact, if the functor

$$\operatorname{Map}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathrm{An}$$

commutes with filtered colimits. Here the convention is that if we drop the cardinal κ then it is always implicitly assumed to be ω.

Let us instead call an object *weakly compact* if every map $X \to \operatorname{colim}_{i \in I} Z_i$ factors through finite stage Z_i, or equivalently:

Definition 2.2.1. $X \in \mathcal{C}$ is called *weakly compact* if

$$\pi_0 \operatorname{colim}_{i \in I} \operatorname{Map}_{\mathcal{C}}(X, Z_i) \to \pi_0 \operatorname{Map}_{\mathcal{C}}(X, \operatorname{colim}_{i \in I} Z_i)$$

is surjective for any filtered diagram $Z : I \to \mathcal{C}$.

Lemma 2.2.2. *If filtered colimits in $\mathcal{C}$ commute with finite limits, weakly compact objects are compact.*

Proof. Write $Z = \operatorname{colim}_{i \in I} Z_i$. If $X \to Z_i$ and $X \to Z_j$ are two maps lifting the same $X \to Z$, they provide a map $X \to Z_i \times_Z Z_j$. Writing this as a filtered colimit of $Z_i \times_{Z_k} Z_j$ (over the $k \in I$ with $i, j \to k$, we see that both maps become homotopic in some Z_k. So the map is actually bijective on π_0. Now if we inductively know that

$$\operatorname{colim}_{i \in I} \operatorname{Map}_{\mathcal{C}}(X, Z_i) \to \operatorname{Map}_{\mathcal{C}}(X, \operatorname{colim}_{i \in I} Z_i)$$

is an equivalence on π_k, for $k \leq n$ and *any* Z_i, then for any $f : X \to Z_i$, we may form $Z'_j = \mathrm{eq}(\ X \rightrightarrows Z_j\ )$ (indexed over $I_{i/}$), and since

$$\mathrm{Map}_{\mathcal{C}}(X, Z'_j) \simeq \mathrm{Map}_{\mathcal{C}}(X, X) \times \Omega_f \mathrm{Map}_{\mathcal{C}}(X, Z_j),$$

one deduces that the map for Z is even an equivalence on π_k for $k \leq n+1$. $\square$

We now would like to formulate a corresponding notion for morphisms.

Definition 2.2.3. A morphism $f : X \to Y$ in an ∞-category is called *weakly compact*, if for every morphism $Y \to Z = \mathrm{colim}_{i \in I} Z_i$ where I is filtered the composite $X \to Y \to \mathrm{colim}_{i\in I} Z_i$ factors over a finite stage $Z_{i_0} \to Z$.

One could ask for a more structured version of this akin to the definition of compact objects, i.e. that such a factorisation exists in families of maps in some sense.

Definition 2.2.4. A morphism $f : X \to Y$ is *strongly compact* if for every filtered colimit $Z = \mathrm{colim}_{i\in I} Z_i$ there exists a lift as indicated:

$$\begin{tikzcd} \mathrm{colim}_i \mathrm{Map}_{\mathcal{C}}(Y, Z_i) \arrow[r, "f^*"] \arrow[d] & \mathrm{colim}_i \mathrm{Map}_{\mathcal{C}}(X, Z_i) \arrow[d] \\ \mathrm{Map}_{\mathcal{C}}(Y, Z) \arrow[r, "f^*"] \arrow[ur, dashed] & \mathrm{Map}_{\mathcal{C}}(X, Z) \end{tikzcd}$$

Note that this lift here is of course up to homotopy, so in the ∞-category of anima.

Clearly strongly compact implies weakly compact: In the definition of compact we just ask for such a lift on a single point of $\mathrm{Map}_{\mathcal{C}}(Y, Z)$ and ignore the upper triangle. The converse is (probably) not true in general, even if filtered colimits in $\mathcal{C}$ commute with finite limits. We will however develop below the notion of *compactly assembled* categories, and somewhat surprisingly will see that in those categories the two notions coincide.

Remark 2.2.5. If $\mathcal{C}$ is a poset, then weakly compact objects / morphisms coincide with their strong counterparts, and classically one says if $x < y$ is compact then x is way below y. Indeed, all mapping spaces are (-1)-truncated, i.e. either empty or the point, so taking π_0 doesn't change them and a surjection is automatically an equivalence, showing that weakly compact objects are compact. Note also that any diagram of (-1)-truncated spaces commutes, and so the only situation in which we could not find a lift for the above diagram is when $\mathrm{Map}_{\mathcal{C}}(Y, Z) = *$ and $\mathrm{colim}_i \mathrm{Map}_{\mathcal{C}}(X, Z_i) = \emptyset$. But then $\mathrm{Map}_{\mathcal{C}}(X, Z_i) = \emptyset$ for each i, which shows that $X \to Y$ cannot be weakly compact. Thus weakly compact morphisms in posets are automatically strongly compact.

Example 2.2.6.

1. Assume that a morphism $X \to Y$ factors over a weakly compact object $K \in \mathcal{C}$, i.e. is of the form $X \to K \to Y$. Then it is weakly compact. To see this we simply observe that for a given morphisms $Y \to Z = \mathrm{colim}_{i\in I} Z_i$ the composition $K \to Y \to Z$ already has to factor over a finite stage by weak compactness of K. In

fact, if K is compact, $X \to Y$ is even strongly compact, since we can get a lift in the diagram by considering

$$\begin{tikzcd}
\operatorname{colim}_i \operatorname{Map}_{\mathcal{C}}(Y, Z_i) \arrow[r] \arrow[d] & \operatorname{colim}_i \operatorname{Map}_{\mathcal{C}}(K, Z_i) \arrow[r] \arrow[d, "\simeq"] & \operatorname{colim}_i \operatorname{Map}_{\mathcal{C}}(X, Z_i) \arrow[d] \\
\operatorname{Map}_{\mathcal{C}}(Y, Z) \arrow[r] & \operatorname{Map}_{\mathcal{C}}(K, Z) \arrow[r] & \operatorname{Map}_{\mathcal{C}}(X, Z)
\end{tikzcd}$$

and noting that the morphism in the middle is an equivalence, so we get a lift by following the inverse of this morphism.

2. If $\mathcal{C}$ is compactly generated, then the converse is also true, namely that the weakly compact morphisms agree with strongly compact morphisms and are precisely those which factor over a compact object. To see this let $f : X \to Y$ be weakly compact. We write $Y = \operatorname{colim}_{i \in I} Y_i$ as a filtered colimit of compact objects. Then the map $f : X \to Y$ factors by definition of compactness as $X \to Y_i \to Y$ and this gives the desired factorization.

The whole idea of compact morphisms is to generalize the previous example to the non compactly generated case. We will see that there are many ∞-categories which don't have many compact objects, but a lot of compact morphisms.

Example 2.2.7. Consider the poset $\operatorname{Open}(X)$ of open subsets of a topological space X. In view of Remark 2.2.5 we will just speak of compact objects and compact maps in $\operatorname{Open}(X)$. It is easy to see that an open $U \subseteq X$ is compact as a topological space if and only if it is compact as an object in $\operatorname{Open}(X)$.

Now consider an inclusion of open subsets $U \subseteq V$, and suppose that we can find a compact space K with $U \subseteq K \subseteq V$. Then $U \subseteq V$ is a compact morphism in $\operatorname{Open}(X)$. Indeed, for every morphism $V \subseteq \bigcup W_i$ with W_i a filtered system of opens in X we find an i_0 such that K and thus also U is already contained in W_{i_0}. Under some assumptions on X the converse is also true, as the following lemma shows (and Remark 2.2.9 shows that some hypotheses are necessary).

Lemma 2.2.8. *Let X be a topological space which is either*

1. *locally compact (every point admits a neighborhood basis of compact sets), or*
2. *a T_3-space (we can separate points and closed sets by disjoint open neighborhoods).*

Then an inclusion of open sets $U \subseteq V$ in X is a compact morphism in $\operatorname{Open}(X)$ *if and only if there exists a compact set K with $U \subseteq K \subseteq V$.*

Proof. We have already seen the "if" direction above.

Now if X is locally compact, we pick for each $x \in V$ a compact neighborhood $x \in K_x \subseteq V$. The interiors of the K_x form an open covering of V, so by compactness of $U \subseteq V$, we see that U is covered by a finite union of the K_x, which is still compact.

If X is a T_3-space, recall first that this is equivalent to every point admitting a neighborhood basis of closed sets. We pick for each $x \in V$ a neighborhood W_x of x with $\overline{W_x} \subseteq V$. Using that $U \subseteq V$ is a compact morphism, we see that U is covered by finitely W_x and hence $\overline{U} \subseteq V$. We now show that $\overline{U}$ is compact. Given any open cover $\{U_i\}$ of $\overline{U}$, we can repeat the above trick to write $U_i = \bigcup_{x \in U_i} W_x^i$ for W_x^i an open neighborhood of x with $\overline{W_x^i} \subseteq U_i$. Then $V \subseteq X = (X \setminus \overline{U}) \cup \bigcup_i \bigcup_{x \in U_i} W_x^i$, and hence $U \subseteq \bigcup_{\ell=1}^n W_{x_\ell}^{i_\ell}$. Thus $\overline{U} \subseteq \bigcup_{\ell=1}^n U_{i_\ell}$, showing compactness of $\overline{U}$. $\square$

This example suggests that a compact morphism should factor over a compact object in some larger category, an intuition which will later be made precise.

Remark 2.2.9. There exists a non-compact Hausdorff space X and an open dense subset $U \subseteq X$ such that $U \subseteq X$ is a compact morphism in $\mathrm{Open}(X)$, see [Joh82, p.309]. In other words, the assumptions in the above Lemma really are necessary, as the consequence can fail even in Hausdorff spaces.

We have the following easy assertions:

Lemma 2.2.10. *1. An object $X \in \mathcal{C}$ is weakly/strongly compact iff the identity $X \to X$ is weakly/strongly compact.*

2. If $f : X \to Y$ is weakly/strongly compact then for arbitrary morphisms $W \to X$ and $Y \to Z$ the composition $W \to X \to Y \to Z$ is also weakly/strongly compact.

3. If $F : \mathcal{C} \hookrightarrow \mathcal{D}$ is a fully faithful functors that preserves filtered colimits, then it reflects weakly/strongly compact morphisms.

Proof. 1. If X is weakly/strongly compact, $X \to X$ is weakly/strongly compact. It remains to show that if $X \to X$ is weakly/strongly compact, X is weakly/strongly compact. For the weak statement, observe that we directly see that any $X \to \operatorname{colim} Z_i$ factors through a finite stage, and for the strong statement, consider the diagram

$$\begin{tikzcd}
\operatorname{colim} \mathrm{Map}_{\mathcal{C}}(X, Z_i) \arrow[r] \arrow[d] & \operatorname{colim} \mathrm{Map}_{\mathcal{C}}(X, Z_i) \arrow[d] \\
\mathrm{Map}_{\mathcal{C}}(X, \operatorname{colim} Z_i) \arrow[r] \arrow[ur, dashed] & \mathrm{Map}_{\mathcal{C}}(X, \operatorname{colim} Z_i)
\end{tikzcd}$$

which encodes directly that the dashed map is a homotopy inverse to the vertical map.

2. Given $Z \to \operatorname{colim}_I U_i$, by weak compactness of $X \to Y$ we find a lift $X \to U_i$ of the map $X \to Y \to Z \to \operatorname{colim}_I U_i$. Precomposing with $W \to X$, we get the desired lift of $W \to Z \to \operatorname{colim}_I U_i$. The strong statement is similarly obtained by composing diagrams.

3. We give the argument for weakly compact morphisms and leave the one for strongly compact ones as an exercise. Suppose $f : X \to Y$ is a morphism in $\mathcal{C}$ such that

Ff is weakly compact, and that we have a map $Y \to \operatorname{colim}_i Z_i$. Then there exists a factorization

$$\begin{tikzcd} FX \arrow[r, dashed] \arrow[d, "Ff"'] & FZ_i \arrow[d] \\ FY \arrow[r] & \operatorname{colim}_i FZ_i \end{tikzcd}$$

and since F is fully faithful we also see that $X \to Y \to \operatorname{colim}_i Z_i$ factors through $Z_i \to \operatorname{colim}_i Z_i$.

□

Definition 2.2.11. We say that an object X of an ∞-category $\mathcal{C}$ is called weakly/strongly *compactly exhaustible* if it can be written as a sequential colimit

$$X = \operatorname{colim}(X_0 \to X_1 \to X_2 \to ...)$$

where all the transition maps $X_i \to X_{i+1}$ are weakly/strongly compact.

Remark 2.2.12. In view of Remark 2.2.5 the notion of weakly and strongly compactly exhaustible object agrees in any poset.

Remark 2.2.13. Classically one says that U is a compactly exhausted topological space if there exists a sequence of compact topological subspaces $K_0 \subseteq K_1 \subseteq ... \subseteq U$ with union U and such that each K_i is contained in the interior of K_{i+1}. As we saw in Remark 2.2.9, in general this is **not** the same as being compactly exhaustible in $\mathrm{Open}(X)$. However, Lemma 2.2.8 shows that if we assume X to be a locally compact or T_3, then these notions do agree.

Example 2.2.14 (Almost mathematics)**.** Assume A is a local ring with maximal ideal $\mathfrak{m} \subseteq A$ with $\mathfrak{m} \otimes_A^L \mathfrak{m} = \mathfrak{m}$ (for example $\mathfrak{m}^2 = \mathfrak{m}$ and $\mathfrak{m}$ flat). The kernel of

$$\mathrm{Mod}(A) \to \mathrm{Mod}(A/\mathfrak{m})$$

forms a full subcategory $\mathrm{aMod}_{\mathfrak{m}}(A)$ closed under colimits, generated by $\mathfrak{m}$. Compact objects in $\mathrm{aMod}_{\mathfrak{m}}(A)$ are exactly the ones which are finitely presented as modules. By Nakayama, these are all trivial, so there are no nonzero compact objects. However, in the example $A = \mathbb{Z}_p[p^{1/p^\infty}]$ any of the inclusions $p^v\mathfrak{m} \to \mathfrak{m}$ with $v > 0$ factors through a finitely generated free module p^vA, and so is a (strongly) compact morphism. In particular, we may write $\mathfrak{m}$ as colimit of

$$p\mathfrak{m} \subseteq p^{1/p}\mathfrak{m} \subseteq p^{1/p^2}\mathfrak{m} \subseteq \dots,$$

so $\mathrm{aMod}_{\mathfrak{m}}(A)$ is generated by (strongly) compactly exhaustibles.

Clearly every compact object is (strongly) compactly exhaustible, but the converse does not hold. We now can state the main result about compactly assembled ∞-categories:

Theorem 2.2.15 (Clausen, Lurie)**.** *For a presentable ∞-category $\mathcal{C}$ the following are equivalent:*

1. *$\mathcal{C}$ is generated under colimits by strongly compactly exhaustible objects.*

2. *Filtered colimits in $\mathcal{C}$ are exact and $\mathcal{C}$ is generated under colimits by weakly compactly exhaustible objects.*

3. *The colimit functor $k : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint.*

4. *$\mathcal{C}$ is ω_1-compactly generated and the colimit functor $\mathrm{Ind}(\mathcal{C}^{\omega_1}) \to \mathcal{C}$ admits a left adjoint.*

5. *$\mathcal{C}$ is a retract in Pr^{L} of a compactly generated ∞-category.*

6. *Filtered colimits in $\mathcal{C}$* distribute *over small limits, i.e. we have*

$$\lim_K \operatorname{colim}_I F \simeq \operatorname{colim}_{I^K} \lim_K F$$

for K arbitrary and I filtered. [1]

Here $\mathrm{Ind}(\mathcal{C})$ of the (locally small but) large category $\mathcal{C}$ was introduced and discussed in Definition 2.1.33 and the following lemmas. In short, as in the small case, it is defined as the full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$ generated by representables under small filtered colimits, and satisfies the same universal property.

We will prove this result in the next section. But for the moment let us draw some corollaries and give some examples.

Definition 2.2.16 (Lurie, Clausen)**.** An ∞-category is called *compactly assembled* if it is presentable and satisfies the equivalent conditions of Theorem 2.2.15.

Note that by Theorem 2.2.15 every compactly assembled ∞-category is ω_1-compactly generated. The converse is not true, but we have:

Example 2.2.17. Every compactly generated ∞-category is compactly assembled. This follows by Theorem 2.2.15(5).

Example 2.2.18. The partially ordered set $[0,1]$ has all suprema and is therefore presentable. Its only compact object is 0, but it is compactly assembled: Every "positive length" morphism is compact. Observe that Theorem 2.2.15 says that $[0,1]$ must be a retract of a compactly generated category. Indeed, if C is the Cantor set, there is a surjective continuous increasing map $f : C \to [0,1]$, and an increasing map $g : [0,1] \to C$ with $g(x) = \inf f^{-1}(x)$. Both preserve suprema and are therefore morphisms in Pr^L, and $f \circ g = \mathrm{id}$. Finally, if we think of the Cantor set as decimal numbers in base 3 all of whose digits are 0 or 2, compact objects are exactly the ones that end in infinitely many 0's, and these are dense, so C is compactly generated.

[1] Equivalently, it suffices to ask that filtered colimits *commute* with finite limits and distribute over small products. The former is a version of Grothendieck's AB5 axiom and the latter is a version of Grothendieck's AB6 axiom.

Remark 2.2.19. A poset P that is compactly assembled as a category is classically called a continuous poset, see [GHK$^+$03] or [Joh82, Chapter VII]. In this case one says if $x < y$ is compact that x is way below y, denoted $x \ll y$. This inspired Joyal and Johnstone's [JJ82] 1-categorical treatment of compactly assembled ordinary categories, which they call continuous categories (and drop the presentability condition).

Proposition 2.2.20. *Let X be a Hausdorff space. Then the following are equivalent:*

1. $\mathrm{Shv}(X) = \mathrm{Shv}(X; \mathrm{An})$ *is compactly assembled.*

2. $\mathrm{Open}(X)$ *is compactly assembled.*

3. X *is locally compact.*

Proof. For (1) $\Longrightarrow$ (2), we will use that $j : \mathrm{Open}(X) \to \mathrm{Shv}(X)$ preserves limits and filtered colimits, so it admits a left adjoint $L : \mathrm{Shv}(X) \to \mathrm{Open}(X)$, which takes a sheaf $\mathcal{F}$ to the union of all opens U where $\mathcal{F}(U)$ is nonempty. Since j preserves filtered colimits, one readily checks that L preserves strongly compact morphisms and hence strongly compactly exhaustible objects. Moreover, j is fully faithful so L is essentially surjective, which then shows that also $\mathrm{Open}(X)$ must be generated under colimits by strongly compactly exhaustible objects, i.e. (2) holds.

For the implication (2) $\Longrightarrow$ (3), recall (see e.g. [Joh82, Chapter II] or [MLM92, Chapter IX]) that the sobrification reflection $\mathrm{sob} : \mathrm{Top} \to \mathrm{Top}$ arises from the idempotent adjunction between spaces and locales, given by $\mathrm{Open} : \mathrm{Top} \to \mathrm{Loc} : \mathrm{pt}$. Moreover, Hausdorff spaces are automatically sober, and thus we can recover $X \cong \mathrm{pt}(\mathrm{Open}(X))$. It was shown in [Joh82, Proposition VII.4.5] that if $\mathrm{Open}(X)$ is compactly assembled (in his terminology: a continuous lattice, or equivalently (by definition) a locally compact locale) then $\mathrm{pt}(\mathrm{Open}(X)) = X$ is locally compact.

Finally, we show (3) $\Longrightarrow$ (1). If $U \subseteq K \subseteq V \subseteq X$ where U, V are open and K is compact, then we have for a filtered colimit $\mathcal{F} = \mathrm{colim}_{i \in I} \mathcal{F}_i$ in $\mathrm{Shv}(X)$:

$$\begin{tikzcd}
\mathrm{colim}_i \mathrm{Map}(\underline{V}, \mathcal{F}_i) \arrow[r] \arrow[d] & \mathrm{colim}_i \Gamma(K; \mathcal{F}_i|_K) \arrow[r] \arrow[d, "\simeq"] & \mathrm{colim}_i \mathrm{Map}(\underline{U}, \mathcal{F}_i) \arrow[d] \\
\mathrm{Map}(\underline{V}, \mathcal{F}) \arrow[r] & \Gamma(K; \mathcal{F}|_K) \arrow[r] & \mathrm{Map}(\underline{U}, \mathcal{F})
\end{tikzcd}$$

This immediately shows that $\underline{U} \to \underline{V}$ is strongly compact. By Remark 2.2.13, this shows that $\underline{U}$ for any compactly exhaustible open is strongly compactly exhaustible. Now we claim that in any locally compact space, every open set is the union of open compactly exhaustible subsets. Given this, cover an open U by such compactly exhaustible U_i. Since finite unions of compacts are compact, also compactly exhaustible open sets are closed under finite unions. Hence we can assume without loss of generality that the collection $\{U_i\}$ is closed under finite unions. This lets us write U as filtered colimit of compactly exhaustible opens in $\mathrm{Open}(X)$. As $j : \mathrm{Open}(X) \to \mathrm{Shv}(X)$ preserves filtered colimits, we can in turn write $\underline{U}$ as filtered colimit of the strongly compactly exhaustible $\underline{U_i}$. Since $\mathrm{Shv}(X)$ is generated by the $\underline{U}$, this yields that it is compactly assembled.

Regarding the claim, note that it suffices to show that every point of U admits a compactly exhaustible neighborhood in U. So let $x \in U$. Since X is locally compact, we find a compact neighborhood $x \in K_1 \subseteq U$. Now generally, by local compactness of X, whenever L is a compact set sitting in an open set V, we can find for each $\ell \in L$ a compact neighborhood $\ell \in L_\ell \subseteq V$. By compactness, L is covered by finitely many of the interiors of the L_ℓ, and thus we find $L \subseteq V' \subseteq L' \subseteq U$ for an open V' and compact L'. Applying this inductively starting with $K_1 \subseteq U$ we obtain $K_n \subseteq U_n \subseteq K_{n+1} \subseteq U$ with each K_n compact and each U_n open. Then $U_\omega := \bigcup_{n\geq 1} U_n \subseteq U$ is a compactly exhaustible neighborhood of x. □

Remark 2.2.21. It is not hard to show without any assumptions on X that $\mathrm{Open}(X)$ is compactly assembled if and only if the sobrification $\mathrm{sob}(X)$ of X is locally compact. However, the Hausdorffness is necessary for the implication (3) $\Longrightarrow$ (1), where it is used to guarantee that sections $\Gamma(K, -) : \mathrm{Shv}(X) \to \mathrm{An}$ on a compact subspace preserve filtered colimits. Since $\Gamma(K; -) : \mathrm{Shv}(X) \to \mathrm{An}$ is corepresented by $\underline{K}$, one sees easily that K being compact is necessary. The Hausdorffness assumption is more subtle though. On the one hand, global sections on the Sierpinski space preserve filtered colimits even though it is only T_0, and on the other one can show that the interval with two endpoints $X = [0,1] \amalg_{[0,1)} [0,1]$ is compact and sober, but $\Gamma(X, -)$ does not preserve filtered colimits.

Nevertheless, in the 1-categorical setting, Joyal and Johnstone have shown a stronger statement via a more elaborate proof strategy. For a locale X, they prove that $\mathrm{Shv}(X; \mathrm{Set})$ is compactly assembled if and only if X is "metastably locally compact", see [JJ82, Theorem 5.9].

Proposition 2.2.22. *Let X be a quasi-separated topological space, i.e. where the collection of compact open sets is stable under finite intersections. Then the following are equivalent:*

1. $\mathrm{Shv}(X)$ *is compactly generated.*
2. $\mathrm{Open}(X)$ *is compactly generated.*
3. *X is coherent in the sense of [Lur17b], i.e. compact open sets are stable under finite intersections and form a basis for X.*
4. *The sobrification* $\mathrm{sob}(X)$ *of X is the underlying space of a scheme.*

Proof. The implication (1) $\Longrightarrow$ (2) is shown in the same way as the implication (1) $\Longrightarrow$ (2) in Proposition 2.2.20; we have a compact-object preserving left adjoint $L : \mathrm{Shv}(X) \to \mathrm{Open}(X)$ of j.

For the implication (2) $\Longrightarrow$ (3), note first that an open subset $U \subseteq X$ is compact if and only if it is a compact object in $\mathrm{Open}(X)$. Now let $U \subseteq X$ be open. In $\mathrm{Open}(X)$, we can write U as a (filtered) colimit of compacts $U = \mathrm{colim}_i U_i$, which means that $U = \bigcup_i U_i$. This proves that the compact open sets form a basis of X.

The implication (3) $\Longrightarrow$ (1) is [Lur17b, Proposition 6.5.4.4]. Moreover, since $\mathrm{Spec}\, R$ is coherent and $\mathrm{Shv}(X) \simeq \mathrm{Shv}(\mathrm{sob}(X))$, we also get (4) $\Longrightarrow$ (1) and that (1) implies that $\mathrm{sob}(X)$ is coherent. Thus $\mathrm{sob}(X)$ is locally spectral in the sense of [Hoc69], and Theorem 9 of op. cit. yields (4). □

Example 2.2.23. Since a scheme X is automatically sober, we see that the category $\mathrm{Shv}(|X|)$ of sheaves on the underlying space X is compactly generated. Tensoring with the compactly generated category of abelian groups and taking modules[2] we also get that the category of $\mathcal{O}_X$-module sheaves on $|X|$ is compactly generated.

A variant of the previous example is this: let k be an algebraically closed field and $V \subseteq k^n$ be an affine variety, i.e. the set of solutions of polynomial equations equipped with the classical Zariski topology (we don't require it to be irreducible here). Then the category of sheaves on the underlying topological space is compactly generated, since the sobrification of V is the scheme $\mathrm{Spec}(\mathcal{O}_V)$. More generally if R is a Jacobson ring, then the maximal spectrum $\mathrm{mSpec}(R)$ has the sobrification $\mathrm{Spec}(R)$, thus the categories of sheaves are compactly generated.

Example 2.2.24. Note that a Hausdorff space X is automatically quasi-separated, and hence $\mathrm{Shv}(X)$ is compactly generated if and only if X is totally disconnected.

Recently Harr [Har23a] has studied the above question with coefficients in an arbitrary compactly generated stable category $\mathcal{C}$. For example, under a hypercompleteness assumption on a locally compact Hausdorff space X, he proves in [Har23a, Proposition 3.4] that $\mathrm{Shv}(X;\mathcal{C})$ is compactly generated if and only if X is totally disconnected. He does this by first showing another interesting result which completely characterizes the compact objects in $\mathrm{Shv}(X;\mathcal{C})$ as those sheaves which are locally constant with compact stalks and compact support, c.f. [Har23a, Theorem 2.3].

2.3 Proof of Theorem 2.2.15

In this section, we prove Theorem 2.2.15. We will roughly follow the strategy in the following graph:

$$\begin{tikzcd}
(1) \arrow[rr, Rightarrow] & & (2) \arrow[dl, Rightarrow] & \\
& (3) \arrow[ul, Rightarrow] \arrow[rr, Leftrightarrow] \arrow[dr, Rightarrow, "\text{using (2)}"] & & (6) \\
(5) \arrow[ur, Rightarrow] & & (4) \arrow[ll, Rightarrow] &
\end{tikzcd}$$

2.3.1 $1 \Rightarrow 2$

Lemma 2.3.1. *Assume that $X \in \mathcal{C}$ is strongly compactly exhaustible as witnessed by a sequential colimit $X = \mathrm{colim}_{\mathbb{N}} X_n$. Let Y be the colimit of an arbitrary filtered diagram $Y = \mathrm{colim}_I Y_i$. Then the functor $k : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ induces an equivalence*

$$\mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(\mathrm{colim}_n\, jX_n, \mathrm{colim}_i\, jY_i) \xrightarrow{\simeq} \mathrm{Map}_{\mathcal{C}}(\mathrm{colim}_n X_n, \mathrm{colim}_i Y_i).$$

[2]More generally, if $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L)$ and $A \in \mathrm{CAlg}(\mathcal{C})$, then the forgetful functor $\mathrm{Mod}_A(\mathcal{C}) \to \mathcal{C}$ is conservative and preserves limits and filtered colimits, hence its left-adjoint $\mathcal{C} \to \mathrm{Mod}_A(\mathcal{C})$ preserves compact generators.

Note that this yields an equivalence $\mathrm{Map}_{\mathcal{C}}(\mathrm{colim}_n X_n, \mathrm{colim}_i Y_i) \simeq \lim_n \mathrm{colim}_i \mathrm{Map}_{\mathcal{C}}(X_n, Y_i)$.

Proof. We have a canonical commutative diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(\mathrm{colim}_n jX_n, \mathrm{colim}_i jY_i) \arrow[r, "k"] \arrow[d, "\simeq"] & \mathrm{Map}_{\mathcal{C}}(\mathrm{colim}_n X_n, \mathrm{colim}_i Y_i) \arrow[d, "\simeq"'] \\
\lim_n \mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(jX_n, \mathrm{colim}_i jY_i) \arrow[r, "\lim_n k"] & \lim_n \mathrm{Map}_{\mathcal{C}}(X_n, \mathrm{colim}_i Y_i) \\
\lim_n \mathrm{colim}_i \mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(jX_n, jY_i) \arrow[r, "\lim_n \mathrm{colim}_i k", "\simeq"'] \arrow[u, "\simeq"] & \lim_n \mathrm{colim}_i \mathrm{Map}_{\mathcal{C}}(X_n, Y_i) \arrow[u]
\end{tikzcd}$$

By 2-out-of-3 it remains to see that the bottom right vertical map is an equivalence. We can use strong compactness of the morphisms to factor each $\mathrm{Map}_{\mathcal{C}}(X_{n+1}, \mathrm{colim}_i Y_i) \to \mathrm{Map}_{\mathcal{C}}(X_n, \mathrm{colim}_i Y_i)$ over $\mathrm{colim}_i \mathrm{Map}_{\mathcal{C}}(X_n, Y_i)$. Thus the limit in the codomain agrees with the limit over

$$\cdots \to \mathrm{Map}(X_2, Y) \to \mathrm{colim}_i \mathrm{Map}_{\mathcal{C}}(X_1, Y_i) \to \mathrm{Map}(X_1, Y) \to \mathrm{colim}_i \mathrm{Map}_{\mathcal{C}}(X_0, Y_i) \to \mathrm{Map}(X_0, Y)$$

which in turn agrees with the limit over

$$\cdots \to \mathrm{colim}_i \mathrm{Map}_{\mathcal{C}}(X_1, Y_i) \to \mathrm{colim}_i \mathrm{Map}_{\mathcal{C}}(X_0, Y_i).$$

□

Corollary 2.3.2. *For a strongly compactly exhaustible object X the presentation as a filtered colimit $X = \mathrm{colim}_i X_i$ is unique as an Ind-object.*

Proof. Let $X = \mathrm{colim}_n X'_n$ witness that X is strongly compactly exhaustible, and suppose we can also write X as a filtered colimit $X = \mathrm{colim}_i X_i$. The previous lemma gives an equivalence

$$\mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(\mathrm{colim}_n jX'_n, \mathrm{colim}_i jX_i) \xrightarrow[\simeq]{k} \mathrm{Map}_{\mathcal{C}}(X, X)$$

which allows us to lift the identity on X to an equivalence of the Ind-objects. □

Recall that the reason why in a compactly generated category filtered colimits commute with finite limits is that we can check equivalences by mapping out of compact objects, which allows us to reduce to the case of Anima. The right vertical composite equivalence in the above lemma tells us that in our current setting, where we can only check equivalences by mapping out of strongly compactly exhaustible objects, we can still reduce to Anima.

Lemma 2.3.3 (1 ⇒ 2 in Theorem 2.2.15)**.** *If $\mathcal{C}$ is generated by strongly compactly exhaustible objects, it is generated by weakly compactly exhaustible objects and filtered colimits in $\mathcal{C}$ are exact.*

Proof. Since strongly compactly exhaustible objects are in particular weakly compactly exhaustible, the only nontrivial implication to show is that filtered colimits in $\mathcal{C}$ are exact. If K is finite and I is filtered, and $F : K \times I \to \mathcal{C}$ some functor, we need to show that

$$\operatorname{colim}_I \lim_K F \to \lim_K \operatorname{colim}_I F$$

is an equivalence. Since strongly compactly exhaustible objects generate $\mathcal{C}$ by assumption, it suffices to show that the above map induces an equivalence on $\operatorname{Map}_{\mathcal{C}}(X, -)$ for $X = \operatorname{colim}_{\mathbb{N}} X_n$ strongly compactly exhausted. Then

$$\begin{aligned} \operatorname{Map}_{\mathcal{C}}(X, \operatorname{colim}_I \lim_K F) &\simeq \lim_n \operatorname{colim}_I \lim_K \operatorname{Map}_{\mathcal{C}}(X_n, F) \\ &\simeq \lim_K \lim_n \operatorname{colim}_I \operatorname{Map}_{\mathcal{C}}(X_n, F) \\ &\simeq \operatorname{Map}_{\mathcal{C}}(X, \lim_K \operatorname{colim}_I F), \end{aligned}$$

where in the first and last step we used the right vertical equivalence in Lemma 2.3.1 and that Hom preserves limits in the second variable, and in the second step we used that $\lim_K$ commutes with colim_I and $\lim_n$. □

2.3.2 $2 \Rightarrow 3$

In order to prove the existence of a left adjoint of the colimit functor $k : \operatorname{Ind}(\mathcal{C}) \to \mathcal{C}$, we will see that it suffices to establish the analogue of Lemma 2.3.1 for objects which are weakly compactly exhausted, under the additional assumption that in $\mathcal{C}$ filtered colimits are exact.

For that, we will first recast the definition of weakly compact morphisms in terms of Ind, and introduce a variant.

Lemma 2.3.4. *A morphism $X \to Y$ in $\mathcal{C}$ is weakly compact if and only if for each $Z \in \operatorname{Ind}(\mathcal{C})$ and any map $Y \to kZ$, we have a lift in the following diagram.*

$$\begin{tikzcd} jX \arrow[r, dashed] \arrow[d] & Z \arrow[d] \\ jY \arrow[r] & jkZ. \end{tikzcd}$$

Proof. We can write $Z = \operatorname{colim}_I jZ_i$ and hence $kZ = \operatorname{colim}_I Z_i$ for I filtered. Since jX is compact in $\operatorname{Ind}(\mathcal{C})$, the wanted factorisation is precisely the same as a finite stage $i \in I$ and a factorisation

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[d] & Z_i \arrow[d] \\ Y \arrow[r] & \operatorname{colim}_I Z_i. \end{tikzcd}$$

in $\mathcal{C}$. □

Definition 2.3.5. Let K be some simplicial set, and $X, Y \in \mathrm{Fun}(K, \mathcal{C})$. We call $X \to Y$ *locally weakly compact* if for any $Z \in \mathrm{Fun}(K, \mathrm{Ind}(\mathcal{C}))$ we have a factorisation

$$\begin{tikzcd} j_*X \arrow[r, dashed] \arrow[d] & Z \arrow[d] \\ j_*Y \arrow[r] & j_*k_*Z \end{tikzcd}$$

in $\mathrm{Fun}(K, \mathrm{Ind}(\mathcal{C}))$.

Remark 2.3.6. If Z is of the form $\mathrm{colim}_I\, jZ_i$, i.e. represented by a diagram $\mathrm{Fun}(K \times I, \mathcal{C})$ with I filtered, then

$$\mathrm{Map}_{\mathrm{Fun}(K,\mathrm{Ind}(\mathcal{C}))}(j_*X, Z) = \mathrm{colim}_{\ell \in \mathrm{Fun}(K,I)} \mathrm{Map}_{\mathrm{Fun}(K,\mathcal{C})}(X, Z_{\ell(-)})$$

where $Z_{\ell(-)}$ denotes the composite $K \xrightarrow{\ell, Z} I \times \mathrm{Fun}(I, \mathcal{C}) \to \mathcal{C}$, and we use Remark 2.1.6. So local compactness here means that $X \to Y \to \mathrm{colim}_I\, Z_i$ factors "locally" through a finite stage, in that there is some $\ell : K \to I$ so that each $X_k \to Y_k \to \mathrm{colim}_i\, Z_{i,k}$ factors through a finite stage $Z_{\ell(k),k}$ depending on k.

Lemma 2.3.7. *Suppose that K is a finite simplicial set and $X \to Y$ in $\mathrm{Fun}(K, \mathcal{C})$ is locally weakly compact. Then $\mathrm{colim}_K X \to \mathrm{colim}_K Y$ is weakly compact.*

Proof. Let $\mathrm{colim}_i\, Z_i$ be a filtered colimit in $\mathcal{C}$. We want to factor the composite

$$\begin{tikzcd} \mathrm{colim}_K X \arrow[d] & \\ \mathrm{colim}_K Y \arrow[r] & \mathrm{colim}_i\, Z_i \end{tikzcd}$$

through a finite stage Z_i. Adjoining over the colim_K and rewriting $\mathrm{const}_K \mathrm{colim}_i\, Z_i \simeq k_* \mathrm{colim}_i\, \mathrm{const}_K\, jZ_i$, we can apply $j_* : \mathrm{Fun}(K, \mathcal{C}) \to \mathrm{Fun}(K, \mathrm{Ind}(\mathcal{C}))$, to get the diagram

$$\begin{tikzcd} j_*X \arrow[r, dashed] \arrow[d] & \mathrm{colim}_i\, \mathrm{const}_K\, jZ_i \arrow[d] \\ j_*Y \arrow[r] & j_*k_*\, \mathrm{colim}_i\, \mathrm{const}_K\, jZ_i \end{tikzcd}$$

where now a lift exists by local weak compactness of $X \to Y$. Moreover, since K is finite, j_*X is a compact object in $\mathrm{Fun}(K, \mathrm{Ind}(\mathcal{C}))$[3], hence we obtain a further factorization of the lift through a finite stage $j_*X \to \mathrm{const}_K\, jZ_i$. Applying k_* and adjoining back the const_K, we get the desired lift of the original map through Z_i. □

[3]More generally, if $F : K \to \mathcal{D}^\omega$ is a functor with K a finite simplicial set, then $F \in \mathrm{Fun}(K, \mathcal{D})^\omega$. Indeed, in this case $\mathrm{Tw}(K)$ is still finite, and hence $\mathrm{Nat}(F, \mathrm{colim}_i\, G_i) \simeq \lim_{(x \to y) \in \mathrm{Tw}(K)} \mathrm{Map}_{\mathcal{D}}(F(x), \mathrm{colim}_i\, G_i(y)) \simeq \mathrm{colim}_i \lim_{(x \to y) \in \mathrm{Tw}(K)} \mathrm{Map}_{\mathcal{D}}(F(x), G_i(y)) \simeq \mathrm{colim}_i\, \mathrm{Nat}(F, G_i)$, using that filtered colimits commute with finite limits in An and the formula for mapping spaces in a functor category, c.f. [GHN15, Proposition 5.1].

Lemma 2.3.8. *Assume filtered colimits are exact in $\mathcal{C}$. If K is n-dimensional and $X \to Y$ in $\mathrm{Fun}(K, \mathcal{C})$ is the composite of $n+1$ pointwise weakly compact maps, then $X \to Y$ is locally weakly compact.*

Proof. We prove the following inductive version: Assume we have a map $X \to Y$ in $\mathrm{Fun}(K, \mathcal{C})$ and already a lift

$$\begin{tikzcd} jX \arrow[r, dashed] \arrow[d] & Z \arrow[d] \\ jY \arrow[r] & jkZ \end{tikzcd}$$

in $\mathrm{Fun}(K^{(n-1)}, \mathrm{Ind}(\mathcal{C}))$, i.e. on the $(n-1)$-skeleton. Then we claim that after precomposing with a pointwise weakly compact map $X' \to X$, we obtain a lift on all of K. Write Z as a filtered colimit $Z = \mathrm{colim}_i\, jZ_i$. The obstruction to extending the above lift over K, i.e. over the remaining n-simplices, is as follows. For an n-simplex $k_0 \to \ldots \to k_n$ in K, we have an S^{n-2} worth of maps $jX_{k_0} \to \mathrm{colim}_i\, jZ_{i,k_n}$, with a provided homotopy identifying them on the colimit, i.e. a map

$$jX_{k_0} \to (\mathrm{colim}_i\, jZ_{i,k_n})^{S^{n-2}} \times_{j(\mathrm{colim}_i\, Z_{i,k_n})^{S^{n-2}}} j(\mathrm{colim}_i\, Z_{i,k_n}),$$

which we need factor through $\mathrm{colim}\, jZ_{k_n}$. Since filtered colimits commute with finite limits in $\mathcal{C}$ and $\mathrm{Ind}(\mathcal{C})$ and j preserves limits (and $(-)^{S^{n-2}}$ is a finite limit) we may identify this with

$$jX_{k_0} \to \mathrm{colim}_i \left(jZ_{i,k_n}^{S^{n-2}} \times_{j(\mathrm{colim}_i\, Z_{i,k_n}^{S^{n-2}})} j(\mathrm{colim}_i\, Z_{i,k_n}) \right).$$

By compactness of jX_{k_0} we can factor through a finite stage

$$jX_{k_0} \to jZ_{i,k_n}^{S^{n-2}} \times_{j(\mathrm{colim}_i\, Z_{i,k_n}^{S^{n-2}})} j(\mathrm{colim}_i\, Z_{i,k_n}).$$

Applying k and using that filtered colimits commute with finite limits in $\mathcal{C}$ lets us rewrite this as a map

$$X_{k_0} \to Z_{i,k_n}^{S^{n-2}} \times_{\mathrm{colim}_i\, Z_{i,k_n}^{S^{n-2}}} \mathrm{colim}_i\, Z_{i,k_n} = \mathrm{colim}_{i \to \ell}\, Z_{i,k_n}^{S^{n-2}} \times_{Z_{\ell,k_n}^{S^{n-2}}} Z_{\ell,k_n}.$$

If we precompose with a levelwise compact map $X' \to X$, we obtain a factorisation

$$X'_{k_0} \to Z_{i,k_n}^{S^{n-2}} \times_{Z_{\ell,k_n}^{S^{n-2}}} Z_{\ell,k_n}$$

which yields the commutative diagram

$$\begin{tikzcd} jX'_{k_0} \arrow[rrr] \arrow[d, dashed] & & & jX_{k_0} \arrow[d] \\ jZ_{i,k_n}^{S^{n-2}} \times_{jZ_{\ell,k_n}^{S^{n-2}}} jZ_{\ell,k_n} \arrow[rrr] \arrow[d] & & & jZ_{i,k_n}^{S^{n-2}} \times_{j(\mathrm{colim}_i\, Z_{i,k_n}^{S^{n-2}})} j(\mathrm{colim}_i\, Z_{i,k_n}) \arrow[d] \\ jZ_{\ell,k_n} \arrow[r] & \mathrm{colim}_i\, jZ_{i,k_n} \arrow[rr] & & \mathrm{colim}_i\, jZ_{i,k_n}^{S^{n-2}} \times_{j(\mathrm{colim}_i\, Z_{i,k_n}^{S^{n-2}})} j(\mathrm{colim}_i\, Z_{i,k_n}) \end{tikzcd}$$

(commutativity of the bottom rectangle can be readily checked by projecting to the factors of the pullback in the codomain). Thus we obtain the desired lift, and by induction on the skeleta of K, this proves the full statement. $\square$

Corollary 2.3.9. *Assume filtered colimits are exact in $\mathcal{C}$, and $X_\bullet \in \mathrm{Fun}(\mathbb{N}, \mathcal{C})$ is a sequential diagram of weakly compact morphisms. Then*

$$\pi_0 \operatorname{Map}_{\operatorname{Ind}(\mathcal{C})}(\operatorname{colim}_{\mathbb{N}} jX_n, Y) \to \pi_0 \operatorname{Map}_{\mathcal{C}}(\operatorname{colim}_{\mathbb{N}} X_n, kY)$$

is an equivalence for any Ind-object Y.

Proof. Write $Y = \operatorname{colim}_I jY_i \in \operatorname{Ind}(\mathcal{C})$, and let $\operatorname{const} Y \in \operatorname{Fun}(\mathbb{N}, \operatorname{Ind}(\mathcal{C}))$ be the constant diagram. We also have $jX_\bullet \in \operatorname{Fun}(\mathbb{N}, \operatorname{Ind}(\mathcal{C}))$. The map $jX_{\bullet-1} \to jX_\bullet$ is pointwise compact, and since $\mathbb{N}$ is equivalent to a 1-dimensional diagram, $jX_{\bullet-2} \to jX_\bullet$ is locally compact in $\operatorname{Fun}(\mathbb{N}, \operatorname{Ind}(\mathcal{C}))$. So we have a lift in the diagram

$$\begin{tikzcd} jX_{\bullet-2} \arrow[r, dashed] \arrow[d] & \operatorname{const} Y \arrow[d] \\ jX_\bullet \arrow[r] & \operatorname{const} jkY, \end{tikzcd}$$

where the bottom map comes from the map $X_\bullet \to kY = \operatorname{colim}_I Y_i$. The top map corresponds to a map $\operatorname{colim}_{\mathbb{N}} jX_{\bullet-2} \to Y$ in $\operatorname{Ind}(\mathcal{C})$. Since $\operatorname{colim}_{\mathbb{N}} jX_{\bullet-2} \to \operatorname{colim}_{\mathbb{N}} jX_\bullet$ is an equivalence, this proves surjectivity.

For injectivity, we work with $\operatorname{Fun}(\mathbb{N} \times \Delta^1, \operatorname{Ind}(\mathcal{C}))$. Giving two maps $\operatorname{colim}_{\mathbb{N}} jX_\bullet \to Y$ in $\operatorname{Ind}(\mathcal{C})$ lifting $\operatorname{colim}_{\mathbb{N}} X_\bullet \to kY = \operatorname{colim}_I Y_i$ then corresponds to a dashed lift in

$$\begin{tikzcd} jX_\bullet \arrow[r, dashed] \arrow[d] & \operatorname{const} Y \arrow[d] \\ jX_\bullet \arrow[r] & \operatorname{const} jkY, \end{tikzcd}$$

on restrictions to $\mathbb{N} \times \partial\Delta^1$. As in the proof of Lemma 2.3.8, this lift extends to all of $\mathbb{N} \times \Delta^1$ after precomposing with $jX_{\bullet-2} \to jX_\bullet$. Under the adjunction between $\operatorname{colim}_{\mathbb{N}}$ and const, this yields that any pair of maps $\operatorname{colim}_{\mathbb{N}} jX_\bullet \to Y$ lifting the given $\operatorname{colim}_{\mathbb{N}} X_\bullet \to kY$ is homotopic (since $jX_{\bullet-2} \to jX_\bullet$ is an equivalence under $\operatorname{colim}_{\mathbb{N}}$). $\square$

With this we can show that we can replace the strongly compactly exhaustible X in Lemma 2.3.1 by a weakly compactly exhaustible one, as long as filtered colimits are exact in $\mathcal{C}$.

Lemma 2.3.10. *Assume filtered colimits are exact in $\mathcal{C}$ and $X = \operatorname{colim} X_n$ is weakly compactly exhausted. For $Y \in \operatorname{Ind}(\mathcal{C})$, we obtain an equivalence*

$$k : \operatorname{Map}_{\operatorname{Ind}(\mathcal{C})}(\operatorname{colim}_{\mathbb{N}} jX_n, Y) \xrightarrow{\simeq} \operatorname{Map}_{\mathcal{C}}(\operatorname{colim}_{\mathbb{N}} X_n, kY).$$

Proof. We claim the following general fact: Let $F : \mathcal{C} \to \mathcal{D}$ be a functor preserving finite limits, and let $X \in \mathcal{C}$ be an object such that

$$F : \pi_0 \operatorname{Map}_{\mathcal{C}}(X, Y) \to \pi_0 \operatorname{Map}_{\mathcal{D}}(F(X), F(Y))$$

is an isomorphism for all $Y \in \mathcal{C}$. Then

$$F : \operatorname{Map}_{\mathcal{C}}(X, Y) \to \operatorname{Map}_{\mathcal{D}}(F(X), F(Y))$$

is an equivalence for all $Y \in \mathcal{C}$. To see this, we prove inductively that

$$F : \pi_i(\operatorname{Map}_{\mathcal{C}}(X, Y); f) \to \pi_i(\operatorname{Map}_{\mathcal{D}}(F(X), F(Y)); f)$$

is an isomorphism for all Y, f and i. Assume this is known for all Y, f and $i \leq n$. Then for some Y and f consider $Y' = \operatorname{eq}(f, f : X \to Y)$. Since

$$\Omega_f \operatorname{Map}_{\mathcal{C}}(X, Y) \to \operatorname{Map}_{\mathcal{C}}(X, Y') \to \operatorname{Map}_{\mathcal{C}}(X, X)$$

is a split fiber sequence (and same for the corresponding sequence in $\mathcal{D}$), we have that F gives a diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \pi_{n+1}(\operatorname{Map}_{\mathcal{C}}(X, Y); f) \arrow[r] \arrow[d] & \pi_n(\operatorname{Map}_{\mathcal{C}}(X, Y')) \arrow[r] \arrow[d] & \pi_n(\operatorname{Map}_{\mathcal{C}}(X, X)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi_{n+1}(\operatorname{Map}_{\mathcal{D}}(FX, FY); f) \arrow[r] & \pi_n(\operatorname{Map}_{\mathcal{D}}(FX, FY')) \arrow[r] & \pi_n(\operatorname{Map}_{\mathcal{D}}(FX, FX)) \arrow[r] & 0.
\end{tikzcd}$$

By assumption, the right vertical maps are isomorphisms, so also the left vertical map.

Since by Corollary 2.3.9, the π_0 condition is satisfied in the present situation, the theorem follows. $\square$

Corollary 2.3.11. *If filtered colimits are exact in $\mathcal{C}$, a morphism $X \to Y$ which factors as*

$$X = X_0 \to X_1 \to \ldots \to Y$$

with all $X_n \to X_{n+1}$ weakly compact (i.e. into "infinitely many weakly compact morphisms"), is strongly compact.

Proof. Let $Z = \operatorname{colim}_{i \in I} jZ_i$ be some Ind-object. For some $Y \to kZ$, we have

$$\begin{tikzcd}
\operatorname{Map}_{\operatorname{Ind}(\mathcal{C})}(jY, Z) \arrow[r] \arrow[d] & \operatorname{Map}_{\operatorname{Ind}(\mathcal{C})}(\operatorname{colim} jX_n, Z) \arrow[r] \arrow[d, "\simeq"] & \operatorname{Map}_{\operatorname{Ind}(\mathcal{C})}(jX, Z) \arrow[d] \\
\operatorname{Map}_{\mathcal{C}}(Y, kZ) \arrow[r] & \operatorname{Map}_{\mathcal{C}}(\operatorname{colim} X_n, Z) \arrow[r] & \operatorname{Map}_{\mathcal{C}}(X, kZ)
\end{tikzcd}$$

$\square$

Corollary 2.3.12. *If filtered colimits are exact in $\mathcal{C}$, then weakly compactly exhaustible objects are ω_1-compact.*

Proof. If $X = \operatorname{colim} X_n$ is weakly compactly exhausted, and $\operatorname{colim}_I Y_i$ is an ω_1-filtered colimit, we have by Lemma 2.3.10

$$\begin{aligned}
&\operatorname{Map}_{\mathcal{C}}(X, \operatorname{colim}_I Y_i)\\
=&\operatorname{Map}_{\operatorname{Ind}(\mathcal{C})}(\operatorname{colim}_{\mathbb{N}} jX_n, \operatorname{colim}_I jY_i)\\
=&\operatorname{colim}_I \operatorname{Map}_{\operatorname{Ind}(\mathcal{C})}(\operatorname{colim}_{\mathbb{N}} jX_n, jY_i)\\
=&\operatorname{colim}_I \operatorname{Map}_{\mathcal{C}}(X, Y_i)
\end{aligned}$$

since countable colimits of compact objects are ω_1-compact. □

Lemma 2.3.13 (2 ⇒ 3 in Theorem 2.2.15)**.** *If $\mathcal{C}$ is generated by weakly compactly exhaustible objects and filtered colimits in $\mathcal{C}$ are exact, then $k : \operatorname{Ind}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint.*

Proof. By the pointwise criterion for existence of adjoints, it suffices to show that for each $X \in \mathcal{C}$, $\operatorname{Map}_{\mathcal{C}}(X, k(-))$ is a representable functor $\operatorname{Ind}(\mathcal{C}) \to \mathrm{An}$, i.e. there exists $X' \in \operatorname{Ind}(\mathcal{C})$ with an equivalence

$$\operatorname{Map}_{\operatorname{Ind}(\mathcal{C})}(X', Y) \cong \operatorname{Map}_{\mathcal{C}}(X, kY)$$

natural in Y. The collection of X for which such X' exists is closed under colimits, since limits of representable functors are representable. It also contains compactly exhaustible objects by Lemma 2.3.10. So it contains every $X \in \mathcal{C}$ and the claim follows. □

We also use the notion of locally weakly compact morphisms in $\operatorname{Fun}(K, \mathcal{C})$ to show the following:

Lemma 2.3.14. *If filtered colimits are exact in $\mathcal{C}$, weakly compactly exhaustible objects are closed under countable colimits.*

Proof. Every countable colimit can be written as a sequential colimit of finite colimits: Enumerating all simplices of a countable simplicial set K, the simplicial subset $K_n \subseteq K$ spanned by the first n simplices is finite, and so

$$\operatorname{colim}_K F = \operatorname{colim}_{\mathbb{N}} \operatorname{colim}_{K_n} F|_{K_n}.$$

To prove closure under finite colimits it suffices to show closure under pushouts, since the initial object is clearly compact. Given a diagram $B \leftarrow A \to C$ of weakly compactly exhaustible objects, we may write $A = \operatorname{colim} A_n$, $B = \operatorname{colim} B_n$, $C = \operatorname{colim} C_n$, and then use Lemma 2.3.10 and Remark 2.1.6 to lift $A \to B$ to a natural transformation $A_n \to B_{i(n)}$. Here we may assume $i : \mathbb{N} \to \mathbb{N}$ to be cofinal, and hence reindex B_n to have an actual natural transformation $A_n \to B_n$, same for $A_n \to C_n$.

We thus have a diagram $B_\bullet \leftarrow A_\bullet \to C_\bullet$, i.e. a sequential diagram $\operatorname{Fun}(K, \mathcal{C})$ where $K = \bullet \leftarrow \bullet \to \bullet$, consisting of pointwise weakly compact maps. Since K is 1-dimensional, the composite of any two successive maps is locally weakly compact in $\operatorname{Fun}(K, \mathcal{C})$, and so by Lemma 2.3.7 the composite of any two successive maps in $B_\bullet \amalg_{A_\bullet} C_\bullet$ is weakly compact. So $B \amalg_A C$ is weakly compactly exhausted.

For sequential colimits we proceed analogously, applying Lemma 2.3.10 inductively to write a sequential diagram $A_0 \to A_1 \to \dots$ of compactly exhaustible objects as a sequential colimit of sequential diagrams $A_i = \operatorname{colim}_n A_{i,n}$ where the maps $A_{i,n} \to A_{i,n+1}$ are compact. This has the diagonal entries as a cofinal subdiagram, and the maps between them are compact since compact morphisms form a 2-sided ideal. □

2.3.3 $3 \Rightarrow 1$ and $3 \Leftrightarrow 6$

We now prove that if $k : \operatorname{Ind}(\mathcal{C}) \to \mathcal{C}$ has a left adjoint $\widehat{\jmath}$, $\mathcal{C}$ is generated by strongly compactly exhaustible objects. To do so, we first derive basic properties of such a left adjoint, and then characterize compact morphisms in terms of it.

Lemma 2.3.15. *If $k : \operatorname{Ind}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint $\widehat{\jmath}$, it is fully faithful, and* $\mathrm{id} \to k \circ \widehat{\jmath}$ *is an equivalence.*

Proof. It is easy to verify that generally a left (right) adjoint is fully faithful if and only if the associated unit (counit) is an equivalence. Moreover, it is a general fact that in an adjoint triple $\widehat{\jmath} \dashv k \dashv j$ the $\widehat{\jmath}$ is fully faithful if and only if j is. One way to see this is to check that for all objects X, Y we obtain a commutative diagram

$$\begin{tikzcd}
\operatorname{Map}(k\widehat{\jmath}X, Y) \arrow[r, equal] \arrow[d, "(\eta^{\widehat{\jmath}}_X)^*"'] & \operatorname{Map}(\widehat{\jmath}X, jY) \arrow[d, equal] \\
\operatorname{Map}(X, Y) & \operatorname{Map}(X, kjY) \arrow[l, "(\varepsilon^j_Y)_*"]
\end{tikzcd}$$

where $\eta^{\widehat{\jmath}} : \mathrm{id} \Rightarrow k\widehat{\jmath}$ is the unit of $\widehat{\jmath} \dashv k$ and $\varepsilon^j : kj \Rightarrow \mathrm{id}$ is the counit of $k \dashv j$, and the unlabeled equivalences come from the adjunctions. Then use 2-out-of-3 and the Yoneda lemma. □

Remark 2.3.16. In what follows, we will sometimes refer to "the canonical transformation $\widehat{\jmath} \to j$". More precisely, one can check that the following solid square commutes and hence define said transformation as the diagonal composite

$$\begin{tikzcd}
\hat{\jmath}kj \arrow[r, "\simeq", "\hat{\jmath}\varepsilon^j"'] \arrow[d, "\varepsilon^{\widehat{\jmath}} j"'] & \hat{\jmath} \arrow[d, "\eta^j \hat{\jmath}"] \arrow[dl, dashed] \\
j \arrow[r, "j\eta^{\widehat{\jmath}}", "\simeq"'] & jk\hat{\jmath}
\end{tikzcd}$$

Lemma 2.3.17. *If $k : \operatorname{Ind}(\mathcal{C}) \to \mathcal{C}$ has a left adjoint $\widehat{\jmath}$, and $X \to Y$ is a morphism in $\mathcal{C}$, the following are equivalent:*

1. *$X \to Y$ is strongly compact.*

2. *$X \to Y$ is weakly compact.*

3. $jX \to jY$ *factors through* $\widehat{\jmath}Y$ *in* $\mathrm{Ind}(\mathcal{C})$.

4. $\widehat{\jmath}X \to \widehat{\jmath}Y$ *factors through* jX *in* $\mathrm{Ind}(\mathcal{C})$.

Proof. If $X \to Y$ is strongly compact, it is also weakly compact. Recall that a morphism $X \to Y$ is weakly compact if and only if for any $Z \in \mathrm{Ind}(\mathcal{C})$, we have a dashed lift in

$$\begin{tikzcd} jX \arrow[r, dashed] \arrow[d] & Z \arrow[d] \\ jY \arrow[r] & jkZ. \end{tikzcd}$$

In particular, we may apply this to $Z = \widehat{\jmath}Y$ to obtain a factorisation $jX \to \widehat{\jmath}Y \to jY$. Given such a factorization, we see that $X \to Y$ is strongly compact since for any $Z \in \mathrm{Ind}(\mathcal{C})$ we get a commutative diagram as follows and hence a diagonal lift for the outer rectangle:

$$\begin{tikzcd} \mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(jY, Z) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(\widehat{\jmath}Y, Z) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(jX, Z) \arrow[d] \\ \mathrm{Map}_{\mathcal{C}}(Y, kZ) \arrow[r] & \mathrm{Map}_{\mathcal{C}}(Y, kZ) \arrow[r] & \mathrm{Map}_{\mathcal{C}}(X, kZ) \end{tikzcd}$$

Finally we show (3) $\implies$ (4); the converse is dual. The diagram on the left below corresponds under the adjunction $\widehat{\jmath} \dashv k$ to the diagram on the right below:

$$\begin{tikzcd} \widehat{\jmath}X \arrow[r] \arrow[d] & \widehat{\jmath}Y \arrow[d] \\ jX \arrow[r] \arrow[ur, dashed] & jY \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d, "\simeq"'] & k\widehat{\jmath}Y \arrow[d, "\simeq"] \\ kjX \arrow[r] \arrow[ur, dashed] & kjY \end{tikzcd}$$

Now in (3) we are given a lift making and a homotopy making the bottom triangle commute. Together with the homotopy making the outer square commute, it is clear in the right picture that this determines a homotopy making top triangle commute, giving (4). $\square$

Lemma 2.3.18 (3 $\Leftrightarrow$ 6 of Theorem 2.2.15)**.** *For presentable* $\mathcal{C}$*, the following are equivalent:*

1. $k : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ *admits a left adjoint.*

2. *In* $\mathcal{C}$*, filtered colimits distribute over small limits, i.e.*

$$\mathrm{colim}_{I^K} \lim_K F \simeq \lim_K \mathrm{colim}_I F.$$

3. *In* $\mathcal{C}$*, filtered colimits are exact and distribute over small products, i.e.*

$$\mathrm{colim}_{\prod_J I} \prod_J F \simeq \prod_J F\, \mathrm{colim}_I F.$$

Proof. $1 \Rightarrow 2$: If $k : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint, it preserves limits. Let $F : K \times I \to \mathcal{C}$ be a diagram with I filtered. Then since limits and filtered colimits in $\mathrm{Ind}(\mathcal{C})$ are formed pointwise, the Ind object $\lim_K \mathrm{colim}_I\, jF(k,i)$ agrees with $\mathrm{colim}_{I^K}\, j \lim_K F(k,i)$, using that filtered colimits distribute over small limits in An, see e.g. [CH21, Corollary 7.17]. Applying k and using the assumption that it commutes with limits, we learn that

$$\lim_K \mathrm{colim}_I\, F(k,i) \simeq \mathrm{colim}_{I^K} \lim_K F(k,i)$$

as desired.

$2 \Rightarrow 3$: If K is finite, the diagonal map $I \to I^K$ is cofinal (this is essentially the definition of filtered). So filtered colimits distributing over finite limits is equivalent to filtered colimits commuting with finite limits. Distributing over products is of course also a special case.

$3 \Rightarrow 1$: Similar to $1 \Rightarrow 2$, the condition 3 implies that $k : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ commutes with finite limits and products, so general limits. Say $\mathcal{C}$ is κ-compactly generated, so that by Lemma 2.1.35 the functor k factors as the right Bousfield localization $i^* : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C}^\kappa)$ followed by the left Bousfield localization $L : \mathrm{Ind}(\mathcal{C}^\kappa) \to \mathcal{C}$. We claim that L also commutes with limits, hence admits a further left adjoint, so that also the composite $k = Li^*$ does.

For any diagram $F : I \to \mathrm{Ind}(\mathcal{C}^\kappa)$ there is a comparison map $\phi : L(\lim F) \to \lim LF$. Since i^* is a right Bousfield localization, so is the postcomposition $(i^*)_* : \mathrm{Fun}(I, \mathrm{Ind}(\mathcal{C})) \to \mathrm{Fun}(I, \mathrm{Ind}(\mathcal{C}^\kappa))$, and in particular it is essentially surjective, so that $F = i^*F'$ for some $F' : I \to \mathrm{Ind}(\mathcal{C})$. Then the composite $k(\lim F') = Li^*(\lim F') = L(\lim F) \xrightarrow{\phi} \lim LF = \lim kF'$ is an equivalence since k preserves limits, hence by 2-out-of-3 also ϕ is an equivalence. $\square$

Lemma 2.3.19 ($3 \Rightarrow 1$ of Theorem 2.2.15)**.** *If $k : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint $\widehat{\jmath}$, $\mathcal{C}$ is generated by strongly compactly exhaustible objects.*

Proof. Note that Lemma 2.3.18 above implies that filtered colimits in $\mathcal{C}$ are exact. So weakly compactly exhaustible objects in $\mathcal{C}$ are closed under finite colimits by Lemma 2.3.14, and agree with strongly compactly exhaustible objects by Lemma 2.3.17. So by Lemma 2.1.32, it suffices to prove that if $X \to Y$ is a morphism in $\mathcal{C}$ such that

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

admits a lift whenever A, B are strongly compactly exhaustible, then $X \to Y$ is an equivalence. Since $\widehat{\jmath}$ is fully faithful, it suffices to check that $\widehat{\jmath}X \to \widehat{\jmath}Y$ is an equivalence in $\mathrm{Ind}(\mathcal{C})$. Since $\mathrm{Ind}(\mathcal{C})$ is generated by the Yoneda image which is closed under finite colimits, by another use of Lemma 2.1.32 we need to prove that any diagram

$$\begin{tikzcd} jA \arrow[r] \arrow[d] & \widehat{\jmath}X \arrow[d] \\ jB \arrow[r] \arrow[ur, dashed] & \widehat{\jmath}Y \end{tikzcd}$$

admits a dashed lift, for arbitrary $A, B \in \mathcal{C}$.

Now observe that $\mathrm{Ind}(\mathcal{C})^{\Delta^1}$ is compactly generated (since objects of the form $\emptyset \to jB$ and $jA \to jA$ are generators by adjunctions, and are compact; also note that the compact objects are exactly those of the form $jA \to jB$). So any object $F \to G$ of $\mathrm{Ind}(\mathcal{C})^{\Delta^1}$ is a filtered colimit of arrows $jA \to jB$ from $\mathcal{C}^{\Delta^1}$. Applying this to $\widehat{\jmath}X \to \widehat{\jmath}Y$, we may write this as

$$(\widehat{\jmath}X \to \widehat{\jmath}Y) = \mathrm{colim}_I(jA_i \to jB_i).$$

Next, observe that applying k yields

$$(X \to Y) = \mathrm{colim}_I(A_i \to B_i),$$

and applying (colimit-preserving) $\widehat{\jmath}$ again yields

$$(\widehat{\jmath}X \to \widehat{\jmath}Y) = \mathrm{colim}_I(\widehat{\jmath}A_i \to \widehat{\jmath}B_i).$$

So the canonical maps

$$\mathrm{colim}_I(\widehat{\jmath}A_i \to \widehat{\jmath}B_i) \to \mathrm{colim}_I(jA_i \to jB_i) \to (\widehat{\jmath}X \to \widehat{\jmath}Y)$$

are equivalences. Since $jA \to jB$ is compact in $\mathrm{Ind}(\mathcal{C})^{\Delta^1}$, the original map to $(\widehat{\jmath}X \to \widehat{\jmath}Y)$ now factors as

$$\begin{tikzcd} jA \arrow[r] \arrow[d] & \widehat{\jmath}A_i \arrow[r] \arrow[d] & jA_i \arrow[r] \arrow[d] & \widehat{\jmath}X \arrow[d] \\ jB \arrow[r] & \widehat{\jmath}B_i \arrow[r] & jB_i \arrow[r] & \widehat{\jmath}Y \end{tikzcd}$$

for some $i \in I$. But we can apply the same argument to the rightmost square again, since $(jA_i \to jB_i)$ is compact. Inductively we thus factor the original diagram as

$$\begin{tikzcd} jA = jA_0 \arrow[r] \arrow[d] & \widehat{\jmath}A_1 \arrow[r] \arrow[d] & jA_1 \arrow[r] \arrow[d] & \dots & \widehat{\jmath}X \arrow[d] \\ jB = jB_0 \arrow[r] & \widehat{\jmath}B_1 \arrow[r] & jB_1 \arrow[r] & \dots & \widehat{\jmath}Y. \end{tikzcd}$$

By fully faithfullness of j and Lemma 2.3.17 the $jA_n \to jA_{n+1}$ are induced by strongly compact morphisms $A_n \to A_{n+1}$ in $\mathcal{C}$, and likewise we get $B_n \to B_{n+1}$. The horizontal colimits here agree with $\mathrm{colim}_n \widehat{\jmath}A_n = \widehat{\jmath}\,\mathrm{colim}\, A_n$ and $\widehat{\jmath}\,\mathrm{colim}\, B_n$. Since $\widehat{\jmath}$ is fully faithful and we may lift against strongly compactly exhaustible objects by assumption, we therefore find a compatible lift $\mathrm{colim}\,\widehat{\jmath}B_n \to \widehat{\jmath}X$. Precomposing, we find a lift out of the original $jB_0 = jB$. $\square$

2.3.4 $3 \Leftrightarrow 4 \Leftrightarrow 5$

Lemma 2.3.20 (3 $\Rightarrow$ 4 of Theorem 2.2.15)**.** *If $k : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint, then $\mathcal{C}$ is ω_1-compactly generated and $\mathrm{Ind}(\mathcal{C}^{\omega_1}) \to \mathcal{C}$ admits a left adjoint.*

Proof. By Lemma 2.3.19 and 2.3.3, $\mathcal{C}$ is generated by weakly compactly exhaustible objects and filtered colimits are exact. By Corollary 2.3.12, this implies that $\mathcal{C}$ is generated by ω_1-compact objects. As in the proof of Lemma 2.3.18, this means that the colimit functor $\mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ factors through $\mathrm{Ind}(\mathcal{C}^{\omega_1})$, and its left adjoint factors through $\mathrm{Ind}(\mathcal{C}^{\omega_1})$ as well, giving the desired adjoint. $\square$

Lemma 2.3.21 (4 $\Rightarrow$ 5 of Theorem 2.2.15)**.** *If $\mathcal{C}$ is ω_1-compactly generated and $\mathrm{Ind}(\mathcal{C}^{\omega_1}) \to \mathcal{C}$ admits a left adjoint, $\mathcal{C}$ is a retract in Pr^L of a compactly generated category.*

Proof. The right adjoint of $k : \mathrm{Ind}(\mathcal{C}^{\omega_1}) \to \mathcal{C}$ is given by the restricted Yoneda embedding j', which is fully faithful since $\mathcal{C}$ is ω_1-compactly generated. As in the proof of Lemma 2.3.15, this implies that the left adjoint $\widehat{\jmath}$ is fully faithful, and hence $k \circ \widehat{\jmath} \simeq \mathrm{id}_{\mathcal{C}}$. This is the desired retraction. $\square$

Lemma 2.3.22 (5 $\Rightarrow$ 3 of Theorem 2.2.15)**.** *If $\mathcal{C}$ is a retract in Pr^L of a compactly generated category, then $k : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ has a left adjoint.*

Proof. If $\mathcal{C}$ is compactly generated, it is of the form $\mathrm{Ind}(\mathcal{C}_0)$. In that case, a left adjoint can be described as $\mathrm{Ind}(j)$, the Ind-extension of the functor $\mathcal{C}_0 \to \mathrm{Ind}(\mathrm{Ind}(\mathcal{C}_0))$. Indeed, both

$$\mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(\mathrm{Ind}(j)(-), Y), \quad \mathrm{Map}_{\mathcal{C}}(-, kY)$$

are functors $\mathcal{C} \to \mathrm{An}^{\mathrm{op}}$ which preserve filtered colimits, and they agree on the objects coming from $\mathcal{C}_0$, so we have the desired adjunction.

If more generally $\mathcal{C}$ is a retract of a compactly generated $\mathcal{C}'$, with both functors $\mathcal{C} \xrightarrow{i} \mathcal{C}' \xrightarrow{r} \mathcal{C}$ colimit-preserving, the functor $k : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ is a retract:

$$\begin{tikzcd}
\mathrm{Ind}(\mathcal{C}) \arrow[r, "\mathrm{Ind}(i)"] \arrow[d, "k"] & \mathrm{Ind}(\mathcal{C}') \arrow[r, "\mathrm{Ind}(r)"] \arrow[d, shift left, "k"] & \mathrm{Ind}(\mathcal{C}) \arrow[d, "k"] \\
\mathcal{C} \arrow[r, "i"] & \mathcal{C}' \arrow[r, "r"] \arrow[u, shift left, "\widehat{\jmath}"] & \mathcal{C}
\end{tikzcd}$$

Using the middle adjunction, we have the following commutative diagram for $X \in \mathcal{C}$ and $Y \in \mathrm{Ind}(\mathcal{C})$:

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(X, kY) \arrow[r, "i"] \arrow[d, "\mathrm{id}"'] & \mathrm{Map}_{\mathcal{C}'}(iX, ikY) \arrow[r, "\simeq"] & \mathrm{Map}_{\mathcal{C}'}(iX, k\,\mathrm{Ind}(i)Y) \arrow[d, "\simeq"] \\
\mathrm{Map}_{\mathcal{C}}(X, kY) & \mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(\mathrm{Ind}(r)\widehat{\jmath}iX, Y) \arrow[l, "k"] & \mathrm{Map}_{\mathrm{Ind}(\mathcal{C}')}(\widehat{\jmath}iX, \mathrm{Ind}(i)Y) \arrow[l, "\mathrm{Ind}(r)"]
\end{tikzcd}$$

using that $k \circ \widehat{\jmath} \simeq \mathrm{id}$. If all morphisms here were invertible, we would have exhibited $\mathrm{Ind}(r) \circ \widehat{\jmath} \circ i$ as left adjoint of k. We don't have that, but we have exhibited $\mathrm{Map}_{\mathcal{C}}(X, k(-))$ as retract of $\mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(\mathrm{Ind}(r)\widehat{\jmath}iX, -)$. By Yoneda, this shows that $\mathrm{Map}_{\mathcal{C}}(X, k(-))$ is itself corepresentable, by a retract of $\mathrm{Ind}(r)\widehat{\jmath}iX$, and so we have that k admits a left adjoint. $\square$

2.4 Properties of compactly assembled ∞-categories

Having proved the equivalence of all characterisations of compactly assembled ∞-categories, we record a number of observations:

Proposition 2.4.1. *In a compactly assembled ∞-category $\mathcal{C}$, the following are equivalent for a morphism $X \to Y$:*

1. *$X \to Y$ is weakly compact.*
2. *$X \to Y$ is strongly compact.*
3. *$jX \to jY$ factors over $\widehat{\jmath}Y$.*
4. *$\widehat{\jmath}X \to \widehat{\jmath}Y$ factors over jX.*
5. *$\widehat{\jmath}X \to \widehat{\jmath}Y$ is compact in $\mathrm{Ind}(\mathcal{C}^{\omega_1})$.*

Proof. The equivalence of the first four points was shown in Lemma 2.3.17. For the last point, observe that $\mathrm{Ind}(\mathcal{C}^{\omega_1})$ is compactly generated, so $\widehat{\jmath}X \to \widehat{\jmath}Y$ is compact if and only it factors over some jZ. Since the map $\widehat{\jmath}X \to jZ$ canonically factors through jX, this is equivalent to (4). $\square$

Remark 2.4.2. Thus, in a compactly assembled category, we drop the adjective weakly or strongly and just consider compact maps and compactly exhaustible objects.

However, as the above Proposition shows, being compact is something which is witnessed by the *datum* of a lift of $jX \to \widehat{\jmath}Y$ and hence it makes sense to remember this datum when talking about compact maps.

Definition 2.4.3. Let $\mathcal{C}$ be compactly assembled. A (compactly) assembled map $X \to Y$ in $\mathcal{C}$ is given by a compact map *together* with a lift $jX \to \widehat{\jmath}Y$. We write

$$\mathrm{Map}^{\mathrm{ca}}_{\mathcal{C}}(X, Y) \coloneqq \mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(jX, \widehat{\jmath}Y)$$

for the *space of compactly assembled maps.*

Note that this is *not* a full subspace of $\mathrm{Map}_{\mathcal{C}}(X, Y)$, so for example a homotopy between compact maps is more data than just a homotopy between the underlying maps.

Proposition 2.4.4. *In a compactly assembled category $\mathcal{C}$, every compact morphism $X \to Y$ may be factored as a composite of two compact morphisms $X \to K \to Y$ where K is ω_1-compact. This extends to a $\mathbb{Q} \cap [0, 1]$-indexed diagram X_α (with $X_0 = X$ and $X_1 = Y$) such that each "positive-length" morphism is compact, and each X_α for $0 < \alpha < 1$ is ω_1-compact.*

Proof. Suppose we already know that we can factor $X \to Y$ into two compact maps $X \to Z \to Y$. We can repeat this to obtain a factorization $X \to X_0 \to Y_0 \to Y$ with all maps compact. Now since $\mathcal{C}$ is ω_1-compactly generated, we can express Y_0 as filtered colimit of ω_1-compact objects. As $X_0 \to Y_0$ is compact, it factors through a finite stage $X_0 \to K \to Y_0$ where K is ω_1-compact. Since compact maps form a 2-sided ideal, we now see that $X \to K \to Y$ is the desired factorization of $X \to Y$ into two compact maps through an ω_1-compact object.

For the second statement, observe that $\mathbb{Q} \cap [0,1]$ can be realized as ascending union of discrete subposets, where in each step we just add finitely many points between two points. Beginning with $\{0,1\}$ and $X \to Y$, we may inductively extend over each of these discrete subposets, and then obtain a diagram indexed over their colimit.

Finally, we prove the first claim, that a compact map may be factored into two compact maps. By assumption we have a lift $jX \to \widehat{\jmath}Y$, and we may write $\widehat{\jmath}Y = \operatorname{colim}_{i \in I} jY_i$ as a filtered colimit of representables. Applying k, we find that $\operatorname{colim}_{i \in I} Y_i \simeq Y$, and applying $\widehat{\jmath}$, we find that $\operatorname{colim}_{i \in I} \widehat{\jmath}Y_i \simeq \widehat{\jmath}Y$, so

$$\operatorname{colim}_{i \in I} \widehat{\jmath}Y_i \to \operatorname{colim}_{i \in I} jY_i \to \widehat{\jmath}Y$$

are equivalences. Now the map $jX \to \widehat{\jmath}Y$ factors as

$$jX \to \widehat{\jmath}Y_i \to jY_i \to \widehat{\jmath}Y,$$

which witnesses compactness of both $X \to Y_i$ and $Y_i \to Y$ by Proposition 2.4.1. □

Example 2.4.5. In a general ∞-category, compact morphisms do not necessarily factor through multiple compact morphisms. For example, consider the poset

$$\mathbb{N} + \mathbb{N} \times \mathbb{N} + \{\infty\},$$

where $\mathbb{N} \times \mathbb{N}$ carries the canonical partial ordering (componentwise instead of lexicographical). Then every subset has a supremum, so this is a presentable category. The morphism

$$(0,0) \to \infty$$

is compact, since every family x_i of elements with supremum ∞ contains elements from $\mathbb{N} \times \mathbb{N} + \{\infty\}$, which receive a morphism from $(0,0)$. Since both $(0,0)$ and ∞ can be expressed as suprema of elements strictly smaller than themselves, they are not compact. Any factorisation of $(0,0) \to \infty$ into two compact morphisms must therefore take the form

$$(0,0) \to (a,b) \to \infty$$

with $a > 0$ or $b > 0$. But the morphisms $(a,b) \to \infty$ with $a > 0$ or $b > 0$ are never compact. For example, if $a > 0$, we have $\infty = \sup(0,n)$ but $(a,b) \not\leq (0,n)$ for any n. So $(0,0) \to \infty$ is an example of a compact morphism that cannot be factored into compact morphisms.

Proposition 2.4.6. *In a compactly assembled ∞-category, the following are equivalent for an object X:*

1. X is weakly compactly exhaustible.

2. X is strongly compactly exhaustible.

3. X is ω_1-compact.

4. X may be written as colimit of a $\mathbb{Q}_{\geq 0}$-indexed diagram of compact maps.

5. X may be written as colimit of a $\mathbb{Q}$-indexed diagram of compact maps.

Furthermore, in any of the $\mathbb{N}$, $\mathbb{Q}_{\geq 0}$ or $\mathbb{Q}$-indexed diagrams above, we may choose all objects to be ω_1-compact.

Proof. Since strongly and weakly compact morphisms agree, (1) and (2) are equivalent. We have also seen in Corollary 2.3.12 that compactly exhaustible objects are always ω_1-compact. Conversely, let X be ω_1-compact. Write $X = \operatorname{colim}_{i\in I} X_i$ where X_i are weakly compactly exhaustible. Since the compactly exhaustible objects are closed under countable (i.e. ω_1-small) colimits by Lemma 2.3.14, we may assume I to be ω_1-filtered here. But then the identity on X factors through one of the X_i. So X is a retract of a compactly exhaustible object, but retracts can be written as countable colimits as well.

If we have $X = \operatorname{colim}(X_0 \to X_1 \to \dots)$ compactly exhausted, we may first factor each morphism into compact morphisms through ω_1-compact objects by Proposition 2.4.4, and hence assume by cofinality that the X_i are ω_1-compact. Factoring, we may extend each $X_i \to X_{i+1}$ to a $\mathbb{Q}\cap[i, i+1]$-indexed diagram of ω_1-compact objects and compact morphisms, and therefore have extended the diagram to a $\mathbb{Q}_{\geq 0}$ diagram with the same colimit. Finally, $\mathbb{Q}_{\geq 0}$ and $\mathbb{Q}$ contain each other as cofinal subsets ($\mathbb{Q}_{>0} \subseteq \mathbb{Q}_{\geq 0}$ is isomorphic to $\mathbb{Q}$). □

So we may view compactly assembled ∞-categories as a special kind of ω_1-compactly generated ones: The ones where in addition, every ω_1-compactly generated object can be compactly exhausted.

Next, we investigate what we can say about Ind-extensions of functors out of compactly assembled categories. To this end, let $\mathcal{C}$ be compactly assembled and $\mathcal{D}$ a category admitting filtered colimits. Given any functor $F : \mathcal{C} \to \mathcal{D}$, note that $k\operatorname{Ind}(F)$ is the Ind-extension of F. Indeed, clearly $k\operatorname{Ind}(F)$ preserves filtered colimits, and furthermore $k\operatorname{Ind}(F)j \simeq kjF \simeq F$. We use the superscript filt to denote full subcategories $\operatorname{Fun}^{\text{filt}} \subseteq \operatorname{Fun}$ on functors preserving filtered colimits, so that Ind-extension is the inverse equivalence to

$$j^* : \operatorname{Fun}^{\text{filt}}(\operatorname{Ind}(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \operatorname{Fun}(\mathcal{C}, \mathcal{D}). \tag{2.1}$$

Lemma 2.4.7. *Let $\mathcal{C}$ be compactly assembled and $\mathcal{D}$ a category admitting filtered colimits. A functor $F : \mathcal{C} \to \mathcal{D}$ commutes with filtered colimits if and only if its* Ind*-extension*

$$\operatorname{Ind}(\mathcal{C}) \to \mathcal{D}$$

is local with respect to $\hat{\jmath}X \to jX$, i.e. takes those morphisms to equivalences. In other words, the equivalence of (2.1) *restricts to an equivalence*

$$j^* \simeq \hat{\jmath}^* : \operatorname{Fun}^{\text{filt}}_{\hat{\jmath}\to j}(\operatorname{Ind}(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \operatorname{Fun}^{\text{filt}}(\mathcal{C}, \mathcal{D}).$$

Proof. If the Ind-extension $F' : \mathrm{Ind}(\mathcal{C}) \to \mathcal{D}$ of $F : \mathcal{C} \to \mathcal{D}$ is local, we have $F \simeq F' \circ j \simeq F' \circ \widehat{\jmath}$, but the latter is a composition of two filtered-colimit preserving functors.

Conversely, assume F preserves filtered colimits. Writing $\widehat{\jmath} X \simeq \mathrm{colim}_I\, jX_i$, we learn that $X \simeq \mathrm{colim}_I\, X_i$ by applying k, and

$$F'(\widehat{\jmath} X) \simeq \mathrm{colim}_I\, F(X_i) \simeq F(X) \simeq F'(jX)$$

as F preserves filtered colimits, so F' inverts the morphisms $\widehat{\jmath} X \to jX$. □

Proposition 2.4.8. *Let $\mathcal{C}$ be compactly assembled and $\mathcal{D}$ a category admitting filtered colimits. Then $\mathrm{Fun}^{\mathrm{filt}}(\mathcal{C}, \mathcal{D})$ is a Bousfield colocalization of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ with right adjoint $\mathrm{asm}_{\mathrm{filt}} : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}^{\mathrm{filt}}(\mathcal{C}, \mathcal{D})$ sending F to $k \circ \mathrm{Ind}(F) \circ \widehat{\jmath}$. In particular, given a functor $F : \mathcal{C} \to \mathcal{D}$, the counit*

$$k \circ \mathrm{Ind}(F) \circ \widehat{\jmath} \to F$$

is the filtered colimit assembly map*, i.e. a terminal object in $\mathrm{Fun}^{\mathrm{filt}}(\mathcal{C}, \mathcal{D})_{/F}$.*

Proof. It is a general fact that the adjoint triple $\widehat{\jmath} \dashv k \dashv j$ yields an adjunction $\widehat{\jmath} k \dashv jk$, and hence $(jk)^* \dashv (\widehat{\jmath} k)^*$ which in turn restricts to an adjunction

$$(jk)^* : \mathrm{Fun}^{\mathrm{filt}}_{\widehat{\jmath} \to j}(\mathrm{Ind}(\mathcal{C}), \mathcal{D}) \rightleftarrows \mathrm{Fun}^{\mathrm{filt}}(\mathrm{Ind}(\mathcal{C}), \mathcal{D}) : (\widehat{\jmath} k)^*. \tag{$*$}$$

Indeed, if $F : \mathrm{Ind}(\mathcal{C}) \to \mathcal{D}$ preserves filtered colimits, then so does $F\widehat{\jmath} k$, and furthermore one checks that whiskering with k from the left inverts the canonical transformation $\widehat{\jmath} \to j$ (this follows from the triangle identities, c.f. Remark 2.3.16). If F furthermore inverts $\widehat{\jmath} \to j$, then $Fjk \simeq F\widehat{\jmath} k$ still preserves filtered colimits. Then $F\eta : F \to Fjk$ is a morphism in $\mathrm{Fun}^{\mathrm{filt}}(\mathrm{Ind}(\mathcal{C}), \mathcal{D})$. The equivalence j^* from (2.1) sends this to $F\eta j$, which is an equivalence by the triangle identities. Thus $(\eta^*)_F = F\eta$ is an equivalence. This proves that the adjunction $(*)$ exists and we can identify the left adjoint with the inclusion $\mathrm{Fun}^{\mathrm{filt}}_{\widehat{\jmath} \to j} \subseteq \mathrm{Fun}^{\mathrm{filt}}$. Finally, we have the commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{filt}}(\mathrm{Ind}(\mathcal{C}), \mathcal{D}) \arrow[r, "j^*", "\simeq"'] & \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \\
\mathrm{Fun}^{\mathrm{filt}}_{\widehat{\jmath} \to j}(\mathrm{Ind}(\mathcal{C}), \mathcal{D}) \arrow[u, hook] \arrow[r, "\simeq", "j^*"'] & \mathrm{Fun}^{\mathrm{filt}}(\mathcal{C}, \mathcal{D}) \arrow[u, hook]
\end{tikzcd}$$

The top horizontal equivalence has inverse given by Ind-extension sending $F \mapsto k\,\mathrm{Ind}(F)$, and the bottom horizontal equivalence is clearly naturally equivalent to $\widehat{\jmath}^*$. The left inclusion admits a right adjoint given by $(\widehat{\jmath} k)^*$, and thus the right inclusion admits a composite right adjoint $\mathrm{asm}_{\mathrm{filt}}$ sending F to $k\,\mathrm{Ind}(F)\widehat{\jmath} k \widehat{\jmath} \simeq k\,\mathrm{Ind}(F)\widehat{\jmath}$, as desired. □

This way of passing from a functor to a colimit-preserving one in a universal way is generally known as "assembly" of the functor, because the new functor is typically "assembled" from the restriction of the old functor to some class of objects. For example, if $\mathcal{C}$ is even compactly generated, the universal filtered-colimit preserving functor over F is just the Ind-extension of $F|_{\mathcal{C}^\omega}$. The above formula describes the assembly of F as the unique filtered-colimit preserving functor which on compactly exhausted objects is described by

$$\mathrm{colim}(X_0 \to X_1 \to \dots) \mapsto \mathrm{colim}(FX_0 \to FX_1 \to \dots)$$

Corollary 2.4.9. *For a compactly assembled category $\mathcal{C}$,* $\mathrm{Map}_{\mathcal{C}}^{\mathrm{ca}}(X,Y)$ *coincides with the filtered colimit assembly of* $\mathrm{Map}_{\mathcal{C}}(X,-)$*.*

Proof. The assembly of $\mathrm{Map}_{\mathcal{C}}(X,-)$ takes Y to

$$(k \circ \mathrm{Ind}(\mathrm{Map}_{\mathcal{C}}(X,-)))(\hat{\jmath}Y).$$

The functor $k \circ \mathrm{Ind}(\mathrm{Map}_{\mathcal{C}}(X,-)) : \mathcal{C} \to \mathrm{An}$ is the Ind-extension of the functor $\mathrm{Map}_{\mathcal{C}}(X,-)$. This coincides with $\mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(jX,-)$ since it has the correct value on representables and preserves filtered colimits in $\mathrm{Ind}(\mathcal{C})$. So the assembly takes Y to

$$\mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(jX, \hat{\jmath}Y) = \mathrm{Map}_{\mathcal{C}}^{\mathrm{ca}}(X,Y)$$

as claimed. □

2.5 The category of presentable ∞-categories

Lemma 2.5.1. Pr^L *has small limits, and they are formed "underlying" (i.e. the forgetful functor* $\mathrm{Pr}^L \to \mathrm{Cat}_\infty$ *preserves limits). Analogously,* Pr^R *has small limits and they are formed underlying.*

Proof. [Lur17b, Theorems 5.5.3.13 and 5.5.3.18]. □

Corollary 2.5.2. *Colimits in* Pr^L *are not formed underlying. Instead, they are formed by passing to the opposite diagram of right adjoints (along the contravariant equivalence* $\mathrm{Pr}^L \simeq (\mathrm{Pr}^R)^{\mathrm{op}}$*) and passing to the limit instead.*

Example 2.5.3. Even though the coproduct $\mathcal{C} \amalg \mathcal{D}$ formed in Cat_∞ is just the disjoint union, $\mathcal{C} \amalg \mathcal{D}$ formed in Pr^L agrees with the product $\mathcal{C} \times \mathcal{D}$, and more generally $\coprod_I \mathcal{C}_i \simeq \prod_I \mathcal{C}_i$ for *any* set I. An explanation for the different behaviour is that a presentable ∞-category needs to have small colimits, and in the disjoint union of $\mathcal{C}$ and $\mathcal{D}$, we for example don't have a coproduct of objects coming from different components.

Definition 2.5.4. We write Pr^L_κ for the (non-full) subcategory of Pr^L consisting of all κ-compactly generated categories with morphisms given by left adjoint functors $F : \mathcal{C} \to \mathcal{D}$ which take $\mathcal{C}^\kappa$ into $\mathcal{D}^\kappa$.

Lemma 2.5.5. Pr^L_κ *is equivalent to the full subcategory of the ∞-category* $\mathrm{Cat}_\infty^{\mathrm{rex}(\kappa)}$ *of small categories with κ-small colimits, and κ-small colimit preserving functors spanned by the idempotent complete ∞-categories.*[4]

Proof. The inverse equivalences are given by Ind_κ and $(-)^\kappa$. [Lur17a, Lemma 5.3.2.9]. □

Lemma 2.5.6. *For a pair of adjoint functors $L : \mathcal{C} \to \mathcal{D} : R$, we have:*

[4]Note that for $\kappa > \omega$ idempotent completeness is automatic since splitting idempotents can be achieved by a sequential colimits, so in this case $\mathrm{Pr}^L_\kappa \simeq \mathrm{Cat}_\infty^{\mathrm{rex}(\kappa)}$. But for $\kappa = \omega$ this makes a difference.

1. *If the right adjoint R preserves κ-filtered colimits, L preserves κ-compact objects.*

2. *If $\mathcal{C}$ is κ-compactly generated and L preserves κ-compact objects, R preserves κ-filtered colimits.*

Proof. If X is κ-compact and R preserves κ-filtered colimits, then $\mathrm{Map}_{\mathcal{D}}(LX, -) \simeq \mathrm{Map}_{\mathcal{C}}(X, R(-))$ commutes with κ-filtered colimits, so LX is κ-compact. For the other statement, let $\mathcal{C}$ be κ-compactly generated and Y_i a κ-filtered diagram. To check that $\operatorname{colim} RY_i \simeq R(\operatorname{colim} Y_i)$ it suffices to apply $\mathrm{Map}_{\mathcal{C}}(X, -)$ for κ-compact X, which leads to $\operatorname{colim} \mathrm{Map}_{\mathcal{D}}(LX, Y_i)$ on both sides since LX is κ-compact. □

Lemma 2.5.7. *The forgetful functor $\mathrm{Pr}^L_\kappa \to \mathrm{Pr}^L$ (and hence also the functor $\mathrm{Ind}_\kappa : \mathrm{Cat}_\infty^{\mathrm{rex}(\kappa)} \to \mathrm{Pr}^L$) preserves colimits.*

Proof. Let $\mathcal{C}_i \to \mathcal{C}$ be a colimit cone in Pr^L over a diagram in Pr^L_κ. We need to prove that it is a colimit cone in Pr^L_κ. Passing to right adjoints, it suffices to check that the limit of κ-compactly generated categories along κ-filtered colimit preserving right adjoint functors is itself κ-compactly generated, and universal among κ-filtered colimit preserving left adjoint functors into the diagram. The first statement follows since the right adjoint functors out of the limit are jointly conservative, and so their left adjoints (which preserve κ-compact objects) take generators to generators collectively. The other statement follows since κ-filtered colimits and limits in the limit of categories are formed pointwise. □

Example 2.5.8. For a ring R, we have the category of perfect complexes $\mathcal{D}(R)^\omega$. We have a functor $BG \to \mathrm{Cat}_\infty^{\mathrm{rex}}$ encoding the trivial G-action on $\mathcal{D}(R)^\omega$. By the above, its colimit is given by the compact objects in the colimit of the trivial G-action on its Ind-category $\mathcal{D}(R)$, viewed as diagram $BG \to \mathrm{Pr}^L$. This colimit may be computed as the limit of the right adjoint diagram, which is the functor category $\mathrm{Fun}(BG, \mathcal{D}(R)) \simeq \mathcal{D}(R[G])$[5]. So the colimit of the original diagram $BG \to \mathrm{Cat}_\infty^{\mathrm{rex}}$ is $\mathcal{D}(R[G])^\omega$.

Example 2.5.9. The functor $\mathrm{Pr}^L_\kappa \to \mathrm{Pr}^L$ does *not* preserve limits As an example, let $\kappa = \omega$ [6] and recall the almost mathematics situation from Example 2.2.14, where we have a pullback diagram in Pr^L

$$\begin{tikzcd}
\mathrm{aMod}_{\mathfrak{m}}(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Mod}(A) \arrow[d] \\
0 \arrow[r] & \mathrm{Mod}(A/\mathfrak{m})
\end{tikzcd}$$

This is not a pullback diagram in Pr^L_ω, as the top left corner is not even compactly generated. (In fact, the pullback in Pr^L_ω is 0, as any compact-object preserving functor into the kernel of $\mathrm{Mod}(A) \to \mathrm{Mod}(A/\mathfrak{m})$ is zero.) We will however see later that this is a limit in the category of compactly assembled ∞-categories, but $\mathrm{Pr}^L_{\mathrm{ca}} \to \mathrm{Pr}^L$ does also not generally preserve limits.

[5] One way to see this is to use the Schwede-Shipley Theorem, see e.g. [Lur17a, Theorem 7.1.2.1].

[6] Lucas Mann has shown that for *uncountable* regular κ, the inclusion $\mathrm{Pr}^L_\kappa \subset \mathrm{Pr}^L$ preserves κ-small limits, see [Luc22, Corollary A.2.7].

To close this subsection, we provide a useful description of colimit inclusions in Pr^L, and use it to prove some closure properties of fully faithful functors,

Lemma 2.5.10. *Consider a sifted diagram $\mathcal{C}_\bullet : I \to \mathrm{Pr}^L$ with left adjoints $f_{i,j} : \mathcal{C}_i \to \mathcal{C}_j$. If each right adjoint $f_{i,j}^R$ preserves I-indexed colimits (e.g. I is filtered and $\mathcal{C}_\bullet$ lands in $\mathrm{Pr}^L_{\mathrm{ca}}$), then the colimit inclusions $\lambda_i : \mathcal{C}_i \to \mathcal{C}$ of $\mathcal{C} = \mathrm{colim}_i^{\mathrm{Pr}^L} \mathcal{C}_i = \lim_i^{\mathrm{Cat}_\infty} \mathcal{C}_i$ can be concretely described by the formula*

$$\mathrm{pr}_j \lambda_i = \mathrm{colim}_{\ell \in I_{\{i,j\}/}} f_{j,\ell}^R f_{i,\ell}$$

where $I_{\{i,j\}/}$ denotes the category of objects in I equipped with morphisms from i and j. Here the colimit is taken along maps induced by the units $\mathrm{id} \Rightarrow f_{\ell,m}^R f_{\ell,m}$ for $\ell \to m$, and the map to $\mathrm{pr}_j \lambda_i$ is induced by the units $\mathrm{id} \Rightarrow \mathrm{pr}_\ell \lambda_\ell$.

Proof. By siftedness of I we note that $I_{\{i,j\}/} \to I$ is always cofinal, hence the right adjoints also preserve these colimits. Given this, the proof for the general case is a straightforward adaption of the case $I = \mathbb{N}$, hence we will prove only the latter for notational clarity.

So consider a sequential diagram $\mathcal{C}_1 \xrightarrow{f_{1,2}} \mathcal{C}_2 \xrightarrow{f_{2,3}} \mathcal{C}_3 \to \cdots$ and write the composite functors as $f_{k,n}$ for $k \leq n$. Then $\mathcal{C} = \mathrm{colim}_n^{\mathrm{Pr}^L} = \lim_n^{\mathrm{Cat}_\infty} \mathcal{C}_n$ has projections $\mathrm{pr}_n : \mathcal{C} \to \mathcal{C}_n$ which admit the left adjoint colimit inclusions $\lambda_n : \mathcal{C}_n \to \mathcal{C}$. Fix n and consider $F_k \coloneqq \mathrm{colim}_{\ell \geq k,n} f_{k,\ell}^R f_{n,\ell} : \mathcal{C}_n \to \mathcal{C}_k$. Since $f_{k-1,k}^R$ preserves filtered colimits, we have canonical equivalences $f_{k,k+1}^R F_{k+1} \simeq F_k$, so that the F_k assemble into $\Lambda_n : \mathcal{C}_n \to \mathcal{C}$ with $\mathrm{pr}_k \Lambda_n = F_k$. To show that $\Lambda_n = \lambda_n$, we prove that Λ_n is left adjoint to pr_n. This follows from the following equivalences, which are natural in $x \in \mathcal{C}_n$ and $y = (y_n)_n \in \mathcal{C}$

$$\begin{aligned}
\mathcal{C}(\Lambda_n x, y) &= \lim_{k \geq n} \mathcal{C}_k(\mathrm{colim}_{\ell \geq k} f_{k,\ell}^R f_{n,\ell} x, y_k) \\
&= \lim_{k \geq n} \lim_{\ell \geq k} \mathcal{C}_k(f_{k,\ell}^R f_{n,\ell} x, y_k) \\
&= \lim_{k \geq n} \mathcal{C}_k(f_{n,k} x, y_k) \\
&= \lim_{k \geq n} \mathcal{C}_n(x, f_{n,k}^R y_k) \\
&= \mathcal{C}_n(x, y_n).
\end{aligned}$$

□

Remark 2.5.11. If we assume that each $f_{i,j} : \mathcal{C}_i \to \mathcal{C}_j$ is fully faithful, then we don't need any assumptions on the right adjoints for the above proof to work, and this shows that the colimit inclusions λ_i of a sifted colimit of fully faithful functors in Pr^L are themselves all fully faithful. However, given furthermore a cone $\mathcal{C}_\bullet \Rightarrow \mathrm{const}\,\mathcal{D}$ where each $\mathcal{C}_i \hookrightarrow \mathcal{D}$ is fully faithful, the induced map $\mathrm{colim}_i \mathcal{C}_i \to \mathcal{D}$ will generally *not* be fully faithful, unless we again assume that the right adjoints preserve I-indexed colimits, as we show below. For an example showing that one really needs this assumption on the right adjoints for the induced map $\mathrm{colim}_i \mathcal{C}_i \to \mathcal{D}$ to also be fully faithful, see [Efi24, Remark 1.71].

Corollary 2.5.12. *Let I be sifted and denote by $\mathrm{Pr}^L_I \subset \mathrm{Pr}^L$ the wide subcategory on functors whose right adjoints preserve I-indexed colimits. Note that this is closed under colimits in Pr^L. If we let $\mathcal{F} \subseteq \mathrm{Ar}(\mathrm{Pr}^L_I)$ denote the full subcategory on fully faithful functors, then $\mathcal{F}$ is closed under I-indexed colimits. In particular, fully faithful functors are closed under filtered colimits in $\mathrm{Pr}^L_{\mathrm{ca}}$.*

Proof. For ease of notation, let us again consider a pointwise fully faithful transformation $\alpha : \mathcal{C}_\bullet \Rightarrow \mathcal{D}_\bullet$ of diagrams $\mathcal{C}_\bullet, \mathcal{D}_\bullet : \mathbb{N} \to \mathrm{Pr}^L$, where the right adjoints of all involved functors preserve sequential colimits. Write $f_{k,n} : \mathcal{C}_k \to \mathcal{C}_n$ respectively $g_{k,n} : \mathcal{D}_k \to \mathcal{D}_n$ for the transition maps of $\mathcal{C}_\bullet$ respectively $\mathcal{D}_\bullet$ and $\lambda_n^{\mathcal{C}}, \mathrm{pr}_n^{\mathcal{C}}$ respectively $\lambda_n^{\mathcal{D}}, \mathrm{pr}_n^{\mathcal{D}}$ for the colimit inclusions and their right adjoints of $\mathcal{C}_\bullet$ respectively $\mathcal{D}_\bullet$. Moreover, let $\alpha_\infty : \mathcal{C}_\infty \to \mathcal{D}_\infty$ denote the induced functor on the colimit. We want to check that $\eta^{\alpha_\infty} : \mathrm{id}_{\mathcal{C}_\infty} \Rightarrow \alpha_\infty^R \alpha_\infty$ is an equivalence. Since α^R and each $\mathrm{pr}_n^{\mathcal{C}}$ preserve sequential colimits and $\mathcal{C}_\infty$ is generated under sequential colimits by the images of the $\lambda_n^{\mathcal{C}}$, it suffices to check that $\mathrm{pr}_k^{\mathcal{C}} \eta^{\alpha_\infty} \lambda_n^{\mathcal{C}} : \mathrm{pr}_k^{\mathcal{C}} \lambda_n^{\mathcal{C}} \Rightarrow \mathrm{pr}_k^{\mathcal{C}} \alpha_\infty^R \alpha_\infty \lambda_n^{\mathcal{C}}$ is an equivalence for all k, n. Note that for each $\ell \geq k, n$ we have a commutative diagram of functors as on the left, and taking colimits then yields the right square by Lemma 2.5.10:

$$\begin{tikzcd}
\mathrm{pr}_k^{\mathcal{C}} \lambda_n^{\mathcal{C}} \arrow[r, "\eta^{\alpha_\infty}"] \arrow[d, equal] & \mathrm{pr}_k^{\mathcal{C}} \alpha_\infty^R \alpha_\infty \lambda_n^{\mathcal{C}} \arrow[r, equal] & \alpha_k^R \mathrm{pr}_k^{\mathcal{D}} \lambda_\ell^{\mathcal{D}} \alpha_n \arrow[d, equal] \\
f_{k,\ell}^R \mathrm{pr}_\ell^{\mathcal{C}} \lambda_\ell^{\mathcal{C}} f_{n,\ell} & & \alpha_k^R g_{k,\ell}^R \mathrm{pr}_\ell^{\mathcal{D}} \lambda_\ell^{\mathcal{D}} g_{n,\ell} \alpha_n \\
f_{k,\ell}^R f_{n,\ell} \arrow[u, "\eta^{\mathrm{pr}_\ell^{\mathcal{C}}}"] \arrow[r, "\simeq", "\eta^{\alpha_\ell}"'] & f_{k,\ell}^R \alpha_\ell^R \alpha_\ell f_{n,\ell} \arrow[r, equal] & \alpha_k^R g_{k,\ell}^R g_{n,\ell} \alpha_n \arrow[u, "\eta^{\mathrm{pr}_\ell^{\mathcal{D}}}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{pr}_k^{\mathcal{C}} \lambda_n^{\mathcal{C}} \arrow[r, "\eta^{\alpha_\infty}"] & \mathrm{pr}_k^{\mathcal{C}} \alpha_\infty^R \alpha_\infty \lambda_n^{\mathcal{C}} \\
\mathrm{colim}_{\ell \geq n,k} f_{k,\ell}^R f_{n,\ell} \arrow[u, "\simeq"] \arrow[r, "\simeq", "\mathrm{colim}_\ell \eta^{\alpha_\ell}"'] & \mathrm{colim}_{\ell \geq n,k} f_{k,\ell}^R \alpha_\ell^R \alpha_\ell f_{n,\ell} \arrow[u, "\simeq"']
\end{tikzcd}$$

$\square$

2.6 The category of compactly assembled ∞-categories

Proposition 2.6.1. *1. A filtered-colimit preserving functor $F : \mathcal{C} \to \mathcal{D}$ between compactly assembled ∞-categories preserves compact morphisms if and only if it commutes with $\widehat{\jmath}$, more precisely that the natural transformation $\widehat{\jmath} \circ F \to \mathrm{Ind}(F) \circ \widehat{\jmath}$ makes the diagram*

$$\begin{tikzcd}
\mathrm{Ind}(\mathcal{C}) \arrow[r, "\mathrm{Ind}(F)"] & \mathrm{Ind}(\mathcal{D}) \\
\mathcal{C} \arrow[u, "\widehat{\jmath}"] \arrow[r, "F"] & \mathcal{D} \arrow[u, "\widehat{\jmath}"']
\end{tikzcd}$$

commute.

2. A colimit-preserving functor $F : \mathcal{C} \to \mathcal{D}$ between presentable ∞-categories, where $\mathcal{C}$ is compactly assembled, preserves strongly compact morphisms if and only if its right adjoint R commutes with filtered colimits.

Proof. For the first statement, first assume that F preserves compact morphisms. If $X = \mathrm{colim}_{\mathbb{N}} X_n$ is compactly exhausted, $FX = \mathrm{colim}_{\mathbb{N}} FX_n$ is, too, and we have $\widehat{\jmath}FX \simeq \mathrm{colim}\, jFX_n \simeq \mathrm{Ind}(F)\,\mathrm{colim}\, jX_n \simeq \widehat{\jmath}X$. So the canonical transformation

$$\widehat{\jmath} \circ F \to \mathrm{Ind}(F) \circ \widehat{\jmath}$$

is an equivalence on compactly exhaustible objects, and since both functors commute with filtered colimits, also general objects. Conversely, if F commutes with $\widehat{\jmath}$, a witness $jX \to \widehat{\jmath}Y$

of compactness of $X \to Y$ is taken by $\operatorname{Ind}(F)$ to a morphism $jFX \to \hat{\jmath}FY$ witnessing compactness of $FX \to FY$.

For the second statement, first assume that the right adjoint R preserves filtered colimits. One may directly check from the definition of strongly compact morphisms that F preserves strongly compact morphisms. Conversely, assume that F preserves strongly compact morphisms. To check that

$$\operatorname{colim}_{i\in I} RZ_i \to R(\operatorname{colim}_{i\in I} Z_i)$$

is an equivalence for any filtered diagram, it suffices to check this after $\operatorname{Map}_{\mathcal{C}}(X,-)$ for a strongly compactly exhausted $X = \operatorname{colim}_{n\in\mathbb{N}} X_n$. But we have

$$\begin{aligned}
\operatorname{Map}_{\mathcal{C}}(X, \operatorname{colim}_{i\in I} RZ_i) &= \operatorname{Map}_{\operatorname{Ind}(\mathcal{C})}(\operatorname{colim}_{\mathbb{N}} jX_n, \operatorname{colim}_I jRZ_i)\\
&= \operatorname{Map}_{\operatorname{Ind}(\mathcal{C})}(\operatorname{colim}_{\mathbb{N}} jFX_n, \operatorname{colim}_I jZ_i)\\
&= \operatorname{Map}_{\mathcal{C}}(FX, \operatorname{colim} Z_i) = \operatorname{Map}_{\mathcal{C}}(X, R\operatorname{colim} Z_i).
\end{aligned}$$

□

Definition 2.6.2. A left adjoint functor between compactly assembled categories that satisfies the equivalent conditions of Proposition 2.6.1 is called compactly assembled. We denote by $\mathrm{Pr}^L_{\mathrm{ca}}$ the non-full subcategory of Pr^L spanned by the compactly assembled categories and compactly assembled functors.

We note that since each compactly assembled category is ω_1-compactly generated, and compactly assembled functors preserve ω_1-compact objects, we find that $\mathrm{Pr}^L_{\mathrm{ca}}$ is actually a non-full subcategory of $\mathrm{Pr}^L_{\omega_1}$. We have an equivalence

$$\mathrm{Pr}^L_{\omega_1} \simeq \mathrm{Cat}_\infty^{\mathrm{rex}(\omega_1)}$$

where the latter is the ∞-category of small ∞-categories that admit ω_1-small (i.e countable) colimits and functors that preserve them (see [Lur17b, Proposition 5.5.7.8]). The equivalence is implemented by taking ω_1-compact objects and vice versa by taking $\operatorname{Ind}_{\omega_1}$. It follows that we can think of $\mathrm{Pr}^L_{\mathrm{ca}}$ also equivalently as some category of small categories. We would like to make this perspective explicit now.

Definition 2.6.3. A small ∞-category is called compactly assembled in the small sense if it admits countable colimits and every object is a sequential colimit along compact morphisms[7]. A functor between such is called compactly assembled in the small sense if it preserves ω_1-small colimits and compact morphisms. We denote the ∞-category of small compactly assembled ∞-categories by $\mathrm{Cat}_\infty^{\mathrm{ca}}$.

Proposition 2.6.4. *We have an equivalence* $\mathrm{Pr}^L_{\mathrm{ca}} \simeq \mathrm{Cat}_\infty^{\mathrm{ca}}$.

Proof. If $\mathcal{C}$ is compactly assembled, then $\mathcal{C}^{\omega_1}$ admits countable colimits (this doesn't use anything), and all objects in $\mathcal{C}^{\omega_1}$ can be written as sequential colimit of ω_1-compact objects

[7]Compact here is meant to be checked against all countable filtered colimits

along compact morphisms by Proposition 2.4.6. So $\mathcal{C}^{\omega_1}$ is compactly assembled in the small sense.

Conversely, assume $\mathcal{C}$ is ω_1-compactly generated and $\mathcal{C}^{\omega_1}$ is compactly assembled in the small sense. If $X = \operatorname{colim} X_n$ is compactly exhausted in $\mathcal{C}^{\omega_1}$, then

$$\operatorname{Map}_{\operatorname{Ind}(\mathcal{C}^{\omega_1})}(\operatorname{colim}_n jX_n, Y) \simeq \operatorname{Map}_{\mathcal{C}}(\operatorname{colim}_n X_n, kY).$$

Indeed, every Y in $\operatorname{Ind}(\mathcal{C}^{\omega_1})$ can be written as ω_1-filtered colimit of ω_1-compact Y, and both sides commute with ω_1-filtered colimits in Y. So it suffices to check this for Y an ω_1-compact object in $\operatorname{Ind}(\mathcal{C}^{\omega_1})$. These can always be represented as countable filtered diagrams, and the same argument as used to prove Lemma 2.3.10 proves the above statement in that case. We thus get a well-defined functor $\widehat{\jmath} : \mathcal{C}^{\omega_1} \to \operatorname{Ind}(\mathcal{C}^{\omega_1})$ taking a compactly exhausted $\operatorname{colim} X_n$ to $\operatorname{colim} jX_n$, and its $\operatorname{Ind}_{\omega_1}$-extension provides a functor $\mathcal{C} \to \operatorname{Ind}(\mathcal{C}^{\omega_1})$ left adjoint to the colimit functor. □

We will not really use this perspective here, since we believe that in most examples of compactly assembled categories, such as $\operatorname{Shv}(X)$, the presentable ∞-category is the more natural object to define than its ω_1-compact objects. Note that we have a fully faithful inclusion

$$i : \operatorname{Pr}^L_\omega \subseteq \operatorname{Pr}^L_{\mathrm{ca}}$$

since every compactly generated ∞-category is compactly assembled and the morphisms are the same by Proposition 2.6.1.

Theorem 2.6.5. *The category $\operatorname{Pr}^L_{\mathrm{ca}}$ admits all colimits and the inclusion functor $\operatorname{Pr}^L_{\mathrm{ca}} \to \operatorname{Pr}^L$ creates colimits.*

Proof. Consider a diagram

$$I \to \operatorname{Pr}^L_{\mathrm{ca}} \qquad i \mapsto \mathcal{C}_i$$

and take the colimit of the composition $I \to \operatorname{Pr}^L_{\mathrm{ca}} \to \operatorname{Pr}^L$. We denote this colimit by $\mathcal{C}$. Equivalently, $\mathcal{C}$ is the limit of the right adjoint diagram in Pr^R. To argue that the original diagram is a colimit in $\operatorname{Pr}^L_{\mathrm{ca}}$ (i.e. that $\mathcal{C}$ is compactly assembled, it is a diagram of compactly assembled functors, and is an initial such cone), we may equivalently check that the right adjoint diagram is a limit diagram in the category whose objects are compactly assembled ∞-categories, and whose morphisms are filtered-colimit preserving right adjoint functors.

Since limits and filtered colimits in a limit of categories along such functors are formed levelwise, the characterisation of compactly assembled categories from Theorem 2.2.15(6) is obviously stable under limits along such functors. □

Proposition 2.6.6. *For every regular cardinal κ the functor $(-)^\kappa : \operatorname{Pr}^L_{\mathrm{ca}} \to \operatorname{Cat}_\infty^{\mathrm{rex,idem}}$ preserves κ-filtered colimits.*

Proof. For uncountable κ, this functor factors as

$$\operatorname{Pr}^L_{\mathrm{ca}} \subset \operatorname{Pr}^L_\kappa \xrightarrow[\simeq]{(-)^\kappa} \operatorname{Cat}_\infty^{\mathrm{rex}(\kappa)} \subset \operatorname{Cat}_\infty,$$

where the middle equivalence is Lemma 2.5.5. The first functor preserves all colimits, since in both cases they are computed in Pr^L, and that the last functor preserves κ-filtered colimits is shown in [Lur17b, Proposition 5.5.7.11].

The case $\kappa = \omega$ requires a lot more work.[8] For ease of notation, we will consider sequential colimits; the case for general filtered colimits is entirely analogous. So consider a sequential diagram $\mathcal{C}_1 \xrightarrow{f_{1,2}} \mathcal{C}_2 \xrightarrow{f_{2,3}} \mathcal{C}_3 \to \cdots$ in $\mathrm{Pr}^L_{\mathrm{ca}}$. For $k \leq n$, we denote by $f_{k,n}$ the composite $\mathcal{C}_k \to \mathcal{C}_n$ in the given diagram. Its colimit is computed in Pr^L, i.e. $\mathcal{C} = \lim(\cdots \to \mathcal{C}_3 \xrightarrow{f^R_{2,3}} \mathcal{C}_2 \xrightarrow{f^R_{1,2}} \mathcal{C}_1)$ computed in Cat_∞. The colimit inclusions $\lambda_n : \mathcal{C}_n \to \mathcal{C}$ are by definition left adjoint to the projections $\mathrm{pr}_n : \mathcal{C} \to \mathcal{C}_n$.

To see that the canonical comparison functor $\Phi : \mathrm{colim}_n\, \mathcal{C}_n^\omega \to \mathcal{C}^\omega$ is fully faithful, note that we have a commutative triangle

$$\begin{tikzcd} & \mathrm{colim}_n\, \mathcal{C}_n^\omega \arrow[dl, "\Phi"'] \arrow[d, hook] \\ (\mathrm{colim}_n\, \mathcal{C}_n)^\omega \arrow[r, hook] & \mathrm{colim}_n\, \mathcal{C}_n^{\omega_1} \end{tikzcd}$$

Here the vertical functor is fully faithful functors are closed under filtered colimits in Cat, and the horizontal one is similarly obtained by applying $(-)^\omega$ to the functor $\mathrm{colim}_n\, \mathcal{C}_n \to \mathrm{colim}_n\, \mathrm{Ind}(\mathcal{C}_n^{\omega_1}) \simeq \mathrm{Ind}(\mathrm{colim}_n\, \mathcal{C}_n^{\omega_1})$, which is fully faithful by Corollary 2.5.12. Thus Φ is fully faithful by 2-out-of-3.

To see that Φ is essentially surjective, consider the diagram where the horizontal compositions are all identities and the vertical lines are colimit diagrams:

$$\begin{tikzcd} \mathcal{C}_1 \arrow[r, hook, "\hat{\jmath}"] \arrow[d, "f_1"'] & \mathrm{Ind}(\mathcal{C}_1^{\omega_1}) \arrow[r, "k"] \arrow[d, "F_1"] & \mathcal{C}_1 \arrow[d, "f_1"] \\ \mathcal{C}_2 \arrow[r, hook, "\hat{\jmath}"] \arrow[d] & \mathrm{Ind}(\mathcal{C}_2^{\omega_1}) \arrow[r, "k"] \arrow[d] & \mathcal{C}_2 \arrow[d] \\ \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\ \mathcal{C} \arrow[r, "J"] & \mathrm{Ind}(\mathrm{colim}_n\, \mathcal{C}_n^{\omega_1}) \arrow[r, "K"] & \mathcal{C} \end{tikzcd}$$

where $F_n = \mathrm{Ind}(f_n^{\omega-1})$. Let $\Lambda_n : \mathrm{Ind}(\mathcal{C}_n^{\omega_1}) \to \mathrm{Ind}(\mathrm{colim}_n\, \mathcal{C}_n^{\omega_1})$ denote the colimit inclusions. Now suppose that $x \in \mathcal{C}^\omega$. We need to show that $x = \lambda_n y$ for some $y \in \mathcal{C}_n^\omega$. Since $J : \mathcal{C} \to \mathrm{Ind}(\mathrm{colim}_n\, \mathcal{C}_n^{\omega_1})$ preserves compacts, we see that Jx lies in $\mathrm{colim}_n\, \mathcal{C}_n^{\omega_1}$, hence $Jx = \Lambda_n j x_n$ for some $x_n \in \mathcal{C}_n^{\omega_1}$. By commutativity we see that also

$$x = KJx = K\Lambda_n j x_n = \lambda_n k j x_n = \lambda_n x_n \qquad \text{and} \qquad Jx = J\lambda_n x_n = \Lambda_n \hat{\jmath} x_n.$$

[8] In the stable setting, there is actually a fairly short proof, using the canonical sequence $\mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C}^{\omega_1}) \to \mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}}(\mathcal{C}))$ to reduce to the compactly generated case, see [Efi24, Proposition 1.72]. However, this crucially uses that in the stable case the above cofiber sequence is also a fiber sequence, which fails in the unstable case, see Example 3.3.10.

So we are in the following situation

$$
\begin{tikzcd}
\mathcal{C}_n \arrow[r, "\widehat{\jmath}"] \arrow[d, "\lambda_n"'] & \operatorname{Ind}(\mathcal{C}_n^{\omega_1}) \arrow[r, "k"] \arrow[d, "\Lambda_n"'] & \mathcal{C}_n \arrow[d, "\lambda_n"] \\
\mathcal{C} \arrow[r, "J"'] & \operatorname{Ind}(\operatorname{colim}_n \mathcal{C}_n^{\omega_1}) \arrow[r, "K"'] & \mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
x_n \arrow[r, mapsto] \arrow[d, mapsto] & \widehat{\jmath}x_n \arrow[r] & jx_n \arrow[r, mapsto] \arrow[d, mapsto] & x_n \arrow[d, mapsto] \\
x \arrow[rr, mapsto] & & Jx \arrow[r, mapsto] & x
\end{tikzcd}
$$

Ideally we now want to show that the ω_1-compact x_n is actually already compact, because then it is exactly the preimage of x we were looking for. However, this will generally only be true after moving further along the sequential colimit, i.e. after increasing n.

Namely, as indicated in the diagram above, the canonical map $\widehat{\jmath}x_n \to jx_n$ in $\mathcal{C}_n^{\omega_1}$ is sent by $\Lambda_n : \operatorname{Ind}(\mathcal{C}_n^{\omega_1}) \to \operatorname{Ind}(\operatorname{colim}_n \mathcal{C}_n^{\omega_1})$ to the identity on Jx. Recall from Proposition 2.4.6 that we can write $\widehat{\jmath}x_n = \operatorname{colim}_k jx_n^k$ in $\operatorname{Ind}(\mathcal{C}_n^{\omega_1})$ where each $x_n^k \to x_n^{k+1}$ is a compact map in $\mathcal{C}_n^{\omega_1}$. In particular, since $Jx = \Lambda_n jx_n \simeq \Lambda_n \widehat{\jmath}x_n = \operatorname{colim}_k \Lambda_n jx_n^k$ is compact, we see that the identity on $\Lambda_n jx_n$ factors through some $\Lambda_n jx_n^k \to \Lambda_n jx_n^{k+1}$. Since this happens inside $\operatorname{colim}_n \mathcal{C}_n^{\omega_1}$ and is witnessed by a finite amount of data, it already occurs at some finite stage. So after possibly increasing n, we can assume that the identity on jx_n factors through $jx_n^k \to jx_n^{k+1}$. By fully faithfulness of j this comes from a factorization of the identity on x_n through the compact map $x_n^k \to x_n^{k+1}$, so that x_n is compact with $\lambda_n x_n = x$, as desired. $\square$

Example 2.6.7. The functor $(-)^{\omega_1} : \mathrm{Pr}^L_{\mathrm{ca}} \to \mathrm{Cat}_\infty$ does not commute with sequential colimits. For example, let $\mathcal{C}_n = \prod_{i=1}^n \mathrm{An}$. Then $\operatorname{colim}_n^{\mathrm{ca}} \mathcal{C}_n = \prod_{\mathbb{N}} \mathrm{An}$ has ω_1-compact objects given by $\prod_{\mathbb{N}} \mathrm{An}^{\omega_1}$, which disagrees with $\operatorname{colim}_n \prod_{i=1}^n \mathrm{An}^{\omega_1}$.

Finally, let us mention the following important structural result, first proved by Maxime Ramzi:

Theorem 2.6.8 ([Ram, Theorem A])**.** *The ∞-category $\mathrm{Pr}^L_{\mathrm{ca}}$ is ω_1-compactly generated.*

2.7 Limits of compactly assembled categories

As a result of the presentability of $\mathrm{Pr}^L_{\mathrm{ca}}$ we can deduce that it also admits all small limits. But limits are in fact hard to understand (presentability only gives an abstract existence proof). The problem is that the functor $\mathrm{Pr}^L_{\mathrm{ca}} \to \mathrm{Pr}^L$ does not preserve limits. Given a diagram $\mathcal{C}_\bullet : I \to \mathrm{Pr}^L_{\mathrm{ca}}$, we at least obtain a canonical comparison functor $\Phi : \lim_I^{\mathrm{ca}} \mathcal{C}_i \to \lim_I \mathcal{C}_i$ (living in Pr^L) from the limit in $\mathrm{Pr}^L_{\mathrm{ca}}$ to the limit in Pr^L. Now there are 3 increasingly rarer situations where this comparison functor can fail to be an equivalence:

1. $\lim_I \mathcal{C}_i$ is not even compactly assembled.

2. $\lim_I \mathcal{C}_i$ is compactly assembled, but Φ is not strongly continuous.

3. Φ lives in $\mathrm{Pr}^L_{\mathrm{ca}}$, but is not an equivalence.

In general, all three situations can and do occur. We will see in Proposition 2.7.12 that the last situation can only happen for infinite limits. However, for even the simplest limit diagrams such as kernels we might find ourselves in the first situation, where the limit is not even dualizable, see Example 2.7.13.

The idea to understand limits in $\mathrm{Pr}^L_{\mathrm{ca}}$ is due to Clausen and exhibits $\mathrm{Pr}^L_{\mathrm{ca}}$ as a right Bousfield localization of a simpler category. We can then make use of the standard strategy to compute limits in right Bousfield localizations; include the diagram into the larger category, take the limit there, and then localize. We begin with some preliminaries to define this larger category.

Definition 2.7.1.

1. We call a class of morphisms S in an ∞-category $\mathcal{C}$ an *ideal* if for f, g, h composable in $\mathcal{C}$ and $g \in S$, we have $fgh \in S$.

2. For an ideal S, we write $S^{\mathbb{Q}}$ for the sub-ideal of those morphisms $f : X_0 \to X_1$ which extend over a $[0,1] \cap \mathbb{Q}$-indexed diagram all of whose nonidentity morphisms are in S, and call $S^{\mathbb{Q}}$ the *factorizable* morphisms in S. If $S = S^{\mathbb{Q}}$ we call S factorizable.

3. We call an ideal in a presentable category $\mathcal{C}$ *accessible* if there exists a cardinal κ such that each morphism in $S^{\mathbb{Q}}$ factors over a κ-compact object of $\mathcal{C}$.

4. We call S a *precompact ideal* if it is accessible, contains the identity on the initial object, and we have the following pushout condition: Given a diagram $F_0 \leftarrow F_1 \to F_2$ of functors $[0,1] \cap \mathbb{Q} \to \mathcal{C}$ with all positive-length morphisms in S, the pushout $F_0 \amalg_{F_1} F_2$ takes $0 \to 1$ to a morphism in S.[9]

Observe that compact morphisms in a compactly assembled category form a precompact ideal (which is factorizable). In analogy with compactly exhaustible objects, we call an object of $\mathrm{Ind}(\mathcal{C})$ *S-exhaustible* if it can be written as colimit

$$\mathrm{colim}_{\alpha \in \mathbb{Q}}\, jX_\alpha$$

of a $\mathbb{Q}$-indexed diagram where all "positive length" morphisms are in S. In fact, each such morphism is automatically in $S^{\mathbb{Q}}$, which shows that the notion of an S-exhaustible object only depends on $S^{\mathbb{Q}}$. Observe also that if S is factorizable, these agree with the objects that can be written as colimit of $\mathbb{N}$-indexed diagrams where the morphisms are in S. In particular, if $\mathcal{C}$ is compactly assembled and S is the (factorizable) ideal of compact maps, then S-exhaustible objects are precisely the objects $\hat{\jmath}X$ where X is compactly exhaustible.

Lemma 2.7.2. *Let $f : X \to Y$ be a morphism of S-exhaustible* Ind-*objects. Then we can realize this morphism as a map of $\mathbb{Q}$-indexed diagrams $jX_\alpha \to jY_\alpha$.*

[9] Of course, this directly implies that all positive-length morphisms lie in S, since the restriction to $[a,b] \cap \mathbb{Q}$ is again the same situation.

Proof. Write $X = \mathrm{colim}_{\alpha\in\mathbb{Q}}\, jX_\alpha$ and $Y = \mathrm{colim}_{\beta\in\mathbb{Q}}\, jY_\beta$. Then

$$\begin{aligned}\mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(X,Y) &= \lim_{\alpha\in\mathbb{Q}} \mathrm{colim}_{\beta\in\mathbb{Q}} \mathrm{Map}_{\mathcal{C}}(X_\alpha, Y_\beta)\\ &= \mathrm{colim}_{k:\mathbb{Q}\to\mathbb{Q}} \lim_{\alpha\in\mathbb{Q}} \mathrm{Map}_{\mathcal{C}}(X_\alpha, Y_{k(\alpha)}),\end{aligned}$$

where the last colimit is indexed on the poset of order-preserving maps $\mathbb{Q} \to \mathbb{Q}$. The subposet of increasing maps sits cofinally in here, so we can lift the given map $f : X \to Y$ to a natural transformation of $\mathbb{Q}$-indexed diagrams $X_\bullet \Rightarrow Y_{k(\bullet)}$ for some increasing $k : \mathbb{Q} \to \mathbb{Q}$. Reindexing $Y_\bullet$ by k (which does not change the colimit Y, we may assume $k = \mathrm{id}$, giving the desired representation of f. □

Lemma 2.7.3. *For a precompact ideal S, S-exhaustible objects are closed under countable colimits.*

Proof. In view of the above Lemma, we can apply the same arguments as in 2.3.14. □

We can now define the larger category mentioned above which will give us a right bousfield localization onto $\mathrm{Pr}^L_{\mathrm{ca}}$.

Definition 2.7.4. Denote by $\mathrm{Pr}^L_{\mathrm{ideal}}$ the category whose objects are pairs $(\mathcal{C}, S)$ of presentable categories and precompact ideals, and morphisms $(\mathcal{C}, S) \to (\mathcal{D}, T)$ are left adjoint functors $F : \mathcal{C} \to \mathcal{D}$ with $F(S) \subset T$. Denote by

$$\mathrm{Fun}^L_{\mathrm{ideal}}((\mathcal{C}, S), (\mathcal{D}, T)) \subseteq \mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$$

the full subcategory on those left adjoints which send S into T. Moreover, we consider the fully faithful inclusion

$$\Lambda : \mathrm{Pr}^L_{\mathrm{ca}} \to \mathrm{Pr}^L_{\mathrm{ideal}},\ \mathcal{C} \mapsto (\mathcal{C}, S_{\mathcal{C}})$$

which equips a compactly assembled category $\mathcal{C}$ with its precompact ideal of compact morphisms. Note that Λ induces equivalences of mapping categories

$$\mathrm{Fun}^{\mathrm{ca}}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}^L_{\mathrm{ideal}}(\Lambda\mathcal{C}, \Lambda\mathcal{D}).$$

Theorem 2.7.5 (Clausen)**.** *The fully faithful inclusion $\Lambda : \mathrm{Pr}^L_{\mathrm{ca}} \hookrightarrow \mathrm{Pr}^L_{\mathrm{ideal}}$ constructed above admits a right adjoint*

$$\mathrm{Pr}^L_{\mathrm{ideal}} \to \mathrm{Pr}^L_{\mathrm{ca}},\ (\mathcal{C}, S) \mapsto (\mathcal{C}, S)^{\mathrm{ca}}.$$

We will refer to $(\mathcal{C}, S)^{\mathrm{ca}}$ as the compactly assembled core of $\mathcal{C}$ with respect to S.

Proof. By the local existence criterion for adjunctions, it suffices to fix some $(\mathcal{C}, S) \in \mathrm{Pr}^L_{\mathrm{ideal}}$, construct a functor $\varepsilon : (\mathcal{C}, S)^{\mathrm{ca}} \to \mathcal{C}$ which sends the ideal C of compact morphisms in $(\mathcal{C}, S)^{\mathrm{ca}}$ into S (i.e. $\varepsilon : \Lambda(\mathcal{C}, S)^{\mathrm{ca}} \to (\mathcal{C}, S)$), and prove that the following composite is an equivalence for every $\mathcal{D} \in \mathrm{Pr}^L_{\mathrm{ca}}$:

$$\mathrm{Fun}^{\mathrm{ca}}(\mathcal{D}, (\mathcal{C}, S)^{\mathrm{ca}}) \xrightarrow[\simeq]{\Lambda} \mathrm{Fun}^L_{\mathrm{ideal}}(\Lambda\mathcal{D}, \Lambda(\mathcal{C}, S)^{\mathrm{ca}}) \xrightarrow{\varepsilon_*} \mathrm{Fun}^L_{\mathrm{ideal}}(\Lambda\mathcal{D}, (\mathcal{C}, S)). \qquad (\star)$$

We first define $(\mathcal{C}, S)^{\mathrm{ca}}$ as the full subcategory of $\mathrm{Ind}(\mathcal{C})$ generated under colimits by S-exhaustible objects, and $\varepsilon : (\mathcal{C}, S)^{\mathrm{ca}} \to \mathcal{C}$ as the restriction of $k : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$.

Note the collection of S-exhaustible objects is small: Since S is accessible, we have a κ such that every morphism in $S^{\mathbb{Q}}$ factors over a κ-compact object of $\mathcal{C}$. Thus every S-exhaustible object is also a colimit of a sequential diagram of κ-compact objects of $\mathcal{C}$. Since $\mathcal{C}^{\kappa}$ is small by presentability, this shows that the collection of S-exhaustible objects is small. It also shows that the S-exhaustible objects lie in $\mathrm{Ind}(\mathcal{C}^{\kappa}) \subseteq \mathrm{Ind}(\mathcal{C})$. Since this inclusion is closed under colimits this also shows that $(\mathcal{C}, S)^{\mathrm{ca}} \subseteq \mathrm{Ind}(\mathcal{C}^{\kappa})$ and so it is a full subcategory of a presentable ∞-category generated by a set of objects under colimits. As a result $(\mathcal{C}, S)^{\mathrm{ca}}$ is itself presentable.

Next we claim that $(\mathcal{C}, S)^{\mathrm{ca}}$ is compactly assembled. To do this, it suffices to prove that the generators $\mathrm{colim}_{\mathbb{Q}}\, jX_\alpha$ are compactly exhaustible. Indeed, we claim that for each $\beta \in \mathbb{Q}$, the canonical map

$$\mathrm{colim}_{\mathbb{Q}_{<\beta}}\, jX_\alpha \to \mathrm{colim}_{\mathbb{Q}_{<\beta+1}}\, jX_\alpha$$

is compact in $(\mathcal{C}, S)^{\mathrm{ca}}$. It lies in $(\mathcal{C}, S)^{\mathrm{ca}}$ since the posets $\mathbb{Q}_{<\beta}$ are isomorphic to $\mathbb{Q}$ (and so both source and target are themselves S-exhaustible). Moreover, it is compact since it factors through the compact object jX_β in the ambient category $\mathrm{Ind}(\mathcal{C})$, and fully faithful colimit-preserving functors reflect compact morphisms.

Now we need to prove that $\varepsilon : (\mathcal{C}, S)^{\mathrm{ca}} \to \mathcal{C}$ takes compact morphisms into S. So let $X \to Z$ be a compact morphism in $(\mathcal{C}, S)^{\mathrm{ca}}$. We may factor it into two compact morphisms $X \to Y \to Z$, and write $Z = \mathrm{colim}_{i \in I} Z_i$ as filtered colimit of S-exhaustible objects, using that S-exhaustibles are closed under finite colimits by Lemma 2.7.3. Compactness now gives us factorisations as follows:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \mathrm{colim}_{\alpha \in \mathbb{Q}_{<\beta}}\, jZ_{i,\alpha} \arrow[r] \arrow[d] & jZ_{i,\beta} \arrow[d, "\in j(S)"] \\
Y \arrow[r] \arrow[d] & Z_i = \mathrm{colim}_{\alpha \in \mathbb{Q}}\, jZ_{i,\alpha} \arrow[d] & jZ_{i,\beta+1} \arrow[l] \\
Z \arrow[r, "\simeq"'] & \mathrm{colim}_{i \in I} Z_i &
\end{tikzcd}$$

The top rightmost vertical morphism is in (the image of j of) S, and hence is sent by $\varepsilon = k|_{(\mathcal{C},S)^{\mathrm{ca}}}$ into S. By the ideal property, we conclude that also the composite $\varepsilon(X \to Z)$ lies in S.

Finally, we need to argue that the composite $(\star)$ is an equivalence. For a functor $F : \Lambda\mathcal{D} \to (\mathcal{C}, S)$, the composite $\mathrm{Ind}(F) \circ \widehat{\jmath} : \mathcal{D} \to \mathrm{Ind}(\mathcal{C})$ preserves compact morphisms, and we claim that its image lies in $(\mathcal{C}, S)^{\mathrm{ca}}$. Indeed, since $\mathrm{Ind}(F)j \simeq jF$ and F sends compact morphisms of $\mathcal{D}$ into S, it follows that $\mathrm{Ind}(F) \circ \widehat{\jmath}$ sends strongly exhaustible objects in $\mathcal{D}$ into S-exhaustible objects in $\mathrm{Ind}(\mathcal{C})$. The claim follows from this. This defines a functor $\Phi : \mathrm{Fun}^L_{\mathrm{ideal}}(\Lambda\mathcal{D}, (\mathcal{C}, S)) \to \mathrm{Fun}^{\mathrm{ca}}(\mathcal{D}, (\mathcal{C}, S)^{\mathrm{ca}})$ making the right square in the following diagram commute:

$$\begin{tikzcd}
\mathrm{Fun}^L_{\mathrm{ideal}}(\Lambda, \mathcal{D}, \Lambda(\mathcal{C}, S)^{\mathrm{ca}}) \arrow[r, "\varepsilon_*"] & \mathrm{Fun}^L_{\mathrm{ideal}}(\Lambda\mathcal{D}, (\mathcal{C}, S)) \arrow[r, "\mathrm{forget}"] \arrow[d, "\Phi"] & \mathrm{Fun}^L(\mathcal{D}, \mathcal{C}) \arrow[d, "(\hat{\jmath})^* \circ \mathrm{Ind}"] \\
\mathrm{Fun}^{\mathrm{ca}}(\mathcal{D}, (\mathcal{C}, S)^{\mathrm{ca}}) \arrow[u, "\Lambda"] \arrow[r, equal] & \mathrm{Fun}^{\mathrm{ca}}(\mathcal{D}, (\mathcal{C}, S)^{\mathrm{ca}}) \arrow[r, hook] & \mathrm{Fun}^L(\mathcal{D}, \mathrm{Ind}(\mathcal{C}))
\end{tikzcd}$$

To see that also the left square commutes, consider a compactly assembled $F : \mathcal{D} \to (\mathcal{C}, S)^{\mathrm{ca}}$, and note that the left-top commposite in the following commutative diagram is precisely $\Phi(\varepsilon\Lambda F)$:

$$\begin{tikzcd}
\mathrm{Ind}(\mathcal{D}) \arrow[r, "\mathrm{Ind}(F)"] & \mathrm{Ind}((\mathcal{C}, S)^{\mathrm{ca}}) \arrow[r, "\mathrm{Ind}(\varepsilon)"] & \mathrm{Ind}(\mathcal{C}) \\
\mathcal{D} \arrow[u, hook, "\hat{\jmath}"] \arrow[r, "F"] & (\mathcal{C}, S)^{\mathrm{ca}} \arrow[u, hook, "\hat{\jmath}"] \arrow[ur, hook] &
\end{tikzcd}$$

That the other composite is also the identity then follows similarly by considering $F : \Lambda\mathcal{D} \to (\mathcal{C}, S)$ and the following commutative diagram:

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "\hat{\jmath}"'] \arrow[rr, bend left, equal] \arrow[d, "\Phi F"'] & \mathrm{Ind}(\mathcal{D}) \arrow[r, "k"'] \arrow[d, "\mathrm{Ind}(F)"'] & \mathcal{D} \arrow[d, "F"] \\
(\mathcal{C}, S)^{\mathrm{ca}} \arrow[r, hook] \arrow[rr, bend right, "\varepsilon"'] & \mathrm{Ind}(\mathcal{C}) \arrow[r, "k"] & \mathcal{C}
\end{tikzcd}$$

□

Addendum 2.7.6. *If all morphism in $S^{\mathbb{Q}}$ are strongly compact in $\mathcal{C}$, then $\varepsilon : (\mathcal{C}, S)^{\mathrm{ca}} \to \mathcal{C}$ is fully faithful. For example, this happens if $\mathcal{C}$ and ε are already compactly assembled.*

Proof. In this case we see from Lemma 2.3.1 that for any pair of S-exhaustible objects in $\mathrm{Ind}(C)$, the map induced by $k : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ on mapping spaces is an equivalence. Since S-exhaustibles are ω_1-compact generators and closed under countable colimits, we see that $((\mathcal{C}, S)^{\mathrm{ca}})^{\omega_1}$ is exactly the full subcategory on S-exhaustible objects. It follows that $((\mathcal{C}, S)^{\mathrm{ca}})^{\omega_1} \to \mathcal{C}$ is fully faithful, and since it lands in $\mathcal{C}^{\omega_1}$, the claim follows from the fact that $(\mathcal{C}, S)^{\mathrm{ca}}$ is compactly ω_1-compactly generated.

For the final remark, suppose that $\mathcal{C}$ and ε are compactly assembled. Given a map $f : X \to Y$ in $S^{\mathbb{Q}}$, by definition we can extend it to an $[0, 1] \cap \mathbb{Q}$-indexed diagram X_α with $X_0 = X$ and $X_1 = Y$. Now note that in $\mathrm{Ind}(\mathcal{C})$ we have a factorization of jf as follows:

$$\begin{tikzcd}
\mathrm{colim}_{\alpha<1/2}\, jX_\alpha \arrow[r] & \mathrm{colim}_{\alpha<1}\, jX_\alpha \arrow[d] \\
jX_0 \arrow[u] \arrow[r, "jf"'] & jX_1
\end{tikzcd}$$

Note that the top map is a compact map between S-exhaustible object, hence is sent by to a compact map by k (recall that ε is just the restriction of k). So upon applying k, this proves that f factors through a compact map, hence is itself compact, as desired. $\square$

Addendum 2.7.7. *If $\mathcal{C}$ is already compactly assembled and $S^{\mathbb{Q}}$ contains all compact morphisms in $\mathcal{C}$, then $(\mathcal{C}, S)^{\mathrm{ca}} \to \mathcal{C}$ also admits a left adjoint and is a (left* and *right) Bousfield localization.*

Proof. If every compact map lies in S, then $\widehat{\jmath}X$ is S-exhaustible for any $X \in \mathcal{C}$. Since $\widehat{\jmath}$ is even left adjoint to $k : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$, it is left adjoint to $(\mathcal{C}, S)^{\mathrm{ca}} \to \mathcal{C}$. So $(\mathcal{C}, S)^{\mathrm{ca}} \to \mathcal{C}$ admits a fully faithful left adjoint. It also admits a right adjoint, which is then also fully faithful. $\square$

Observe that the situation of Addendum 2.7.7 applies in particular to the class S of all morphisms which factor over an ω_1-compact object. In that case, $(\mathcal{C}, S)^{\mathrm{ca}}$ agrees with $\mathrm{Ind}(\mathcal{C}^{\omega_1})$, the functor to $\mathcal{C}$ is the colimit functor, and the fully faithful adjoints are our $\widehat{\jmath}$ and j. One may think of Addendum 2.7.7 as describing a smaller version of this situation.

Lemma 2.7.8. *The functor $(\mathcal{C}, S)^{\mathrm{ca}} \to \mathcal{C}$ is an equivalence if and only if $\mathcal{C}$ is compactly assembled and $S^{\mathbb{Q}}$ consists precisely of all compact morphisms in $\mathcal{C}$.*

Proof. If the two classes of morphisms agree and $\mathcal{C}$ is compactly assembled, $(\mathcal{C}, S)^{\mathrm{ca}} \to \mathcal{C}$ is fully faithful and essentially surjective since S-exhaustibles and compactly exhaustibles agree. Conversely, denote the two classes of morphisms by C and $S^{\mathbb{Q}}$, and assume that $(\mathcal{C}, S)^{\mathrm{ca}} \to \mathcal{C}$ is an equivalence. Then $\mathcal{C}$ is compactly assembled, since $(\mathcal{C}, S)^{\mathrm{ca}}$ is. Also, compact morphisms are taken into S, and so $C \subseteq S$, and since compact morphisms can be factored, also $C \subseteq S^{\mathbb{Q}}$.

This shows that every compactly exhaustible object is also S-exhaustible. So $\widehat{\jmath} : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ takes values in $(\mathcal{C}, S)^{\mathrm{ca}}$. As it is left adjoint to the colimit functor, which is an equivalence by assumption, we get that $\widehat{\jmath}$ is the inverse equivalence. For a diagram X_α with $\alpha \in \mathbb{Q}$ and all nonidentity morphisms in S, we have

$$\mathrm{colim}_{\alpha \in \mathbb{Q}}\, jX_\alpha \in (\mathcal{C}, S)^{\mathrm{ca}},$$

and this is taken to $\mathrm{colim}\, X_\alpha$ in $\mathcal{C}$ by the colimit functor. Applying the inverse equivalence, we learn

$$\widehat{\jmath}\,\mathrm{colim}\, X_\alpha \simeq \mathrm{colim}\, jX_\alpha$$

for any $\mathbb{Q}$-indexed diagram with nonidentity morphisms in S. Now let $X_0 \to X_1$ be an arbitrary morphism from $S^{\mathbb{Q}}$, and X_α with $\alpha \in [0,1] \cap \mathbb{Q}$ an extension to a diagram with nonidentity morphisms in S. In $\mathrm{Ind}(\mathcal{C})$, we have that $jX_0 \to jX_1$ factors over

$$\mathrm{colim}_{\alpha < 1}\, jX_\alpha \simeq \widehat{\jmath}\,\mathrm{colim}_{\alpha < 1}\, X_\alpha,$$

which shows that $X_0 \to X_1$ is compact, finishing the proof. $\square$

Remark 2.7.9. Given a functor $F : (\mathcal{C}, S) \to (\mathcal{D}, T)$ in $\mathrm{Pr}^L_{\mathrm{ideal}}$, the right adjoint $(F^{\mathrm{ca}})^R$ of $F^{\mathrm{ca}} : (\mathcal{C}, S)^{\mathrm{ca}} \to (\mathcal{D}, T)^{\mathrm{ca}}$ is in general given by the composition

$$(\mathcal{D}, T)^{\mathrm{ca}} \subseteq \mathrm{Ind}(\mathcal{D}) \xrightarrow{\mathrm{Ind}(F^R)} \mathrm{Ind}(\mathcal{C}) \to (\mathcal{C}, S)^{\mathrm{ca}}$$

where the last map is right adjoint to the inclusion (which exists by the adjoint functor theorem). In the case that F^R also sends T into S, we see that $(F^{\mathrm{ca}})^R$ is actually just the restriction of $\mathrm{Ind}(F^R) = \mathrm{Ind}(F)^R$.

Now to compute limits in $\mathrm{Pr}^L_{\mathrm{ca}}$, we can employ the usual strategy for right bousfield localizations; given any diagram $\mathcal{C}_\bullet : I \to \mathrm{Pr}^L_{\mathrm{ca}}$, we see that $\lim^{\mathrm{ca}}_i \mathcal{C}_i = (\lim_i \Lambda \mathcal{C}_i)^{\mathrm{ca}}$. For this to be useful, we need to understand limits in $\mathrm{Pr}^L_{\mathrm{ideal}}$.

Proposition 2.7.10. *Let $(\mathcal{C}_\bullet, S_\bullet) : I \to \mathrm{Pr}^L_{\mathrm{ideal}}$ be a diagram in $\mathrm{Pr}^L_{\mathrm{ideal}}$. Its limit is computed "pointwise", concretely given by $(\mathcal{C}, S)$, where $\mathcal{C} = \lim_i \mathcal{C}_i$ is computed in Pr^L and*

$$S = \{f : c \to c' \mid p_i(f) \in S_i \text{ for all } i\}$$

forms a precompact ideal in the $\mathcal{C}$.

Proof. We first check that S forms a precompact ideal in $\mathcal{C}$. The ideal condition is clear. Since colimits in $\lim_i \mathcal{C}_i$ are also computed pointwise, we can check the pushout condition pointwise. For the accessibility condition, we need to show that there exists κ such that for any diagram $X : [0, 1] \cap \mathbb{Q} \to \lim^L \mathcal{C}_i$ where all $p_i(X_\alpha \to X_{\alpha'}$ for $\alpha < \alpha'$ are compact, $X_0 \to X_1$ factors through κ-compact Y.

We take $Y = \mathrm{colim}_{\alpha<1} X_\alpha$. As mentioned above, this colimit is formed pointwise. Since a sequential colimit along compact maps is ω^1-compact, all $p_i(Y)$ are ω_1-compact. The pointwise ω_1-compact objects are contained in the κ-compact objects of $\lim_i \mathcal{C}_i$ for some κ depending only on the diagram, finishing the proof that S is a precompact ideal.

By construction, the limit cone in Pr^L now lifts to a cone $\mathrm{const}(\mathcal{C}, S) \Rightarrow (\mathcal{C}_\bullet, S_\bullet)$ in $\mathrm{Pr}^L_{\mathrm{ideal}}$, which induces a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}^L_{\mathrm{ideal}}((\mathcal{D}, T), (\mathcal{C}, S)) \arrow[r, hook] \arrow[d] & \mathrm{Fun}^L(\mathcal{D}, \mathcal{C}) \arrow[d, "\simeq"] \\
\lim_i \mathrm{Fun}^L_{\mathrm{ideal}}((\mathcal{D}, T), (\mathcal{C}_\bullet, S_\bullet)) \arrow[r, hook] & \lim_i \mathrm{Fun}^L(\mathcal{D}, \mathcal{C}_i)
\end{tikzcd}$$

So the left vertical functor is automatically fully faithful, and easily seen to be essentially surjective by construction of S. □

Corollary 2.7.11. *Given a diagram $\mathcal{C}_\bullet : I \to \mathrm{Pr}^L_{\mathrm{ca}}$, its limit is given by*

$$\lim^{\mathrm{ca}} \mathcal{C}_i = (\lim^{\mathrm{ideal}}_i (\mathcal{C}_i, S_i))^{\mathrm{ca}} = (\lim_i \mathcal{C}_i, S)^{\mathrm{ca}}.$$

where S_i denotes the precompact ideal of compact morphisms in $\mathcal{C}_i$, and S is as in Proposition 2.7.10.

Note that this in particular shows that $\lim^{\mathrm{ca}} \mathcal{C}_i$ is a full subcategory of $\mathrm{Ind}(\lim \mathcal{C}_i)$. This will be important later on.

Proposition 2.7.12. *The forgetful functor $\mathrm{Pr}^L_{\mathrm{ca}} \to \mathrm{Pr}^L$ reflects finite limits, and creates finite products. In particular, $\mathrm{Pr}^L_{\mathrm{ca}}$ is also semiadditive.*

Proof. Clearly $*$ is terminal in both Pr^L and $\mathrm{Pr}^L_{\mathrm{ca}}$, so it remains to check reflection of pullbacks and creation of binary products.

1. Consider a square in $\mathrm{Pr}^L_{\mathrm{ca}}$

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{B} \arrow[d, "g"] \\ \mathcal{C} \arrow[r, "f"'] & \mathcal{D} \end{tikzcd}$$

   and suppose that it is a pullback in Pr^L. By the formula for limits it will suffices to show that the "limit precompact ideal" in $\mathcal{A}$ coincides with the compact morphisms in $\mathcal{A}$, i.e. that a morphism $\phi : a \to a'$ in $\mathcal{A}$ is compact if and only if $F\phi$ and $G\phi$ (and $fG\phi$) are compact. Since the square is in $\mathrm{Pr}^L_{\mathrm{ca}}$, the "only if" direction is clear. Conversely, suppose $F\phi$ and $G\phi$ and hence $fG\phi = gF\phi$ are compact. Given a map to a filtered colimit $a' \to \mathrm{colim}_j A_j$, we find factorizations of $F(a \to \mathrm{colim}_j A_j)$ through $F(A_k)$ and $G(a \to \mathrm{colim}_j A_j)$ through $G(A_\ell)$. Since J is filtered, there is some $m \in J$ with maps $k, \ell \to m$. This shows that the projection of $a \to \mathrm{colim}_j A_j$ to each of the categories factors through the m-th stage, hence also does so already in $\mathcal{A}$ since mapping spaces in limits of categories are computes as limits of the mapping spaces.

2. Consider $\mathcal{C}, \mathcal{D} \in \mathrm{Pr}^L_{\mathrm{ca}}$ and let $\mathcal{C} \times \mathcal{D}$ be the usual product taken in Pr^L. Then $\mathcal{C} \times \mathcal{D}$ is compactly assembled, and compact morphisms in $\mathcal{C} \times \mathcal{D}$ are precisely pairs of compact morphisms. Hence also the projections are compactly assembled, i.e. we get a cone in $\mathrm{Pr}^L_{\mathrm{ca}}$ which is a limit cone in Pr^L, so the previous point shows it is also a limit cone in $\mathrm{Pr}^L_{\mathrm{ca}}$. □

However, the creation of finite limits fails rather drastically in general:

Example 2.7.13 ([Efi24, Example 3.11])**.** Let k be a field and consider the morphism $k[x, y] \to k[x^{\pm 1}]$ sending $x \mapsto x$ and $y \mapsto 0$. The basechange

$$k[x^{\pm 1}] \otimes_{k[x,y]} - : D(k[x, y]) \to D(k[x^{\pm 1}])$$

is a strongly left adjoint functor between compactly generated categories, however Efimov showed that its kernel (computed in Pr^L) is not even dualizable!

Nevertheless, in good cases we can compute pullbacks of compactly assembled categories in Pr^L.

Proposition 2.7.14. *Consider a cospan in* $\mathrm{Pr}^L_{\mathrm{ca}}$

$$\begin{tikzcd} & \mathcal{C} \arrow[d, "p"] \\ \mathcal{D} \arrow[r, "q"'] & \mathcal{E} \end{tikzcd}$$

If p is a left Bousfield localization, then the pullback square computed in Pr^L *lies inside* $\mathrm{Pr}^L_{\mathrm{ca}}$ *and is also a pullback there. Moreover, the induced map* $\mathcal{C} \times_{\mathcal{E}} \mathcal{D} \to \mathcal{D}$ *is also a left Bousfield localization.*

Proof. Denote the fully faithful right adjoint of p by $p^R : \mathcal{E} \subseteq \mathcal{E}$, and consider the pullback $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$ in Pr^L. This can be explicitly described as the ∞-category of pairs of an object $d \in \mathcal{D}$ together with a morphism $c \to p^R q(d)$ in $\mathcal{C}$ which is inverted by p. One easily checks that the right adjoint to the projection $\mathrm{pr}_{\mathcal{D}} : \mathcal{C} \times_{\mathcal{E}} \mathcal{D} \to \mathcal{D}$ is given by sending $d \in \mathcal{D}$ to the pair $(p^R qd, d) \in \mathcal{C} \times_{\mathcal{E}} \mathcal{D}$ (with the identity morphism on $p^R qd$). This right adjoint is clearly fully faithful, so the functor $\mathrm{pr}_{\mathcal{D}}$ is also a Bousfield localization. Note that so far we have not used that either the categories or the functors are compactly assembled.

Since the forgetful $\mathrm{Pr}^L_{\mathrm{ca}} \to \mathrm{Pr}^L$ reflects pullbacks by Proposition 2.7.12, it now suffices to check that $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$ and the projections $\mathrm{pr}_{\mathcal{C}}, \mathrm{pr}_{\mathcal{D}}$ are compactly assembled. To this end, note that we have a fully faithful inclusion into the lax pullback

$$i : \mathcal{C} \times_{\mathcal{E}} \mathcal{D} \subseteq \mathcal{C} \overrightarrow{\times}_{\mathcal{E}} \mathcal{D}$$

This is the ∞-category of all triples $(c, d, q(d) \to p(c))$ consisting of objects $d \in \mathcal{D}, c \in \mathcal{C}$ and a map $q(d) \to p(c)$. The image of the inclusion consist of those object where the map is an equivalence. Now the inclusion i admits a right adjoint

$$i^R : \mathcal{C} \overrightarrow{\times}_{\mathcal{E}} \mathcal{D} \to \mathcal{C} \times_{\mathcal{E}} \mathcal{D},\ (c, d, q(d) \to p(c)) \mapsto (c \times_{p(c)} q(d), d).$$

Since $\mathcal{C}$ is compactly assembled, filtered colimits in it are exact, and we see that i^R preserves filtered colimits. Because of this, to prove that $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$ is compactly assembled, it then suffices to check that $\mathcal{C} \overrightarrow{\times}_{\mathcal{E}} \mathcal{D}$ is compactly assembled. Using a retract we can reduce to the case that all three categories are compactly generated. But then it is easy to see that the lax arrow category is also compactly generated.

Finally, we check that $\mathrm{pr}_{\mathcal{C}}$ and $\mathrm{pr}_{\mathcal{D}}$ are compactly assembled, so that the entire pullback square lies in $\mathrm{Pr}^L_{\mathrm{ca}}$. But this is immediate from the fact that they factor through the compactly assembled inclusion $\mathcal{C} \times_{\mathcal{E}} \mathcal{D} \subseteq \mathcal{C} \overrightarrow{\times}_{\mathcal{E}} \mathcal{D} \to \mathcal{C}$, and that a morphism in the lax pullback is compact if and only if it is pointwise so, which is easy to check. □

An example of the second situation mentioned in the introduction, where the limit computed in Pr^L lives in $\mathrm{Pr}^L_{\mathrm{ca}}$, but the comparison map does not, is that of infinite products:

Example 2.7.15. For infinite products the map

$$\prod^{\mathrm{ca}} \mathcal{C}_i \to \prod \mathcal{C}_i$$

is not an equivalence in general, even though the target is compactly assembled, since it is also the coproduct in Pr^L and therefore also in $\mathrm{Pr}^L_{\mathrm{ca}}$. We claim that the compact morphisms in $\prod \mathcal{C}_i$ are given by those morphisms which are levelwise compact and finitely supported, that is given by a morphism $X \to \emptyset \to Y$ almost everywhere. To see this we test against levelwise colimits and the colimit

$$(Y_0, \emptyset, \emptyset, \emptyset, ...) \to (Y_0, Y_1, \emptyset, \emptyset, ...) \to (Y_0, Y_1, Y_2, \emptyset, ...) \ .$$

There are in general certainly a lot of levelwise compact maps, which are not finitely supported. Thus we are in the situation of Addendum 2.7.7 and we see that the functor $\prod^{\mathrm{ca}} \mathcal{C}_i \to \prod \mathcal{C}_i$ is a left and right Bousfield localization. The class of levelwise compact morphisms is also factorizable, thus by Lemma 2.7.8 we see that the map

$$\prod{}^{\mathrm{ca}} \mathcal{C}_i \to \prod \mathcal{C}_i$$

is an equivalence precisely if every levelwise compact morphism is already compact. This is only the case if the product is finite (i.e. almost all of the $\mathcal{C}_i$ are given by the point).

In fact, this distinction may be more clear when working with compactly generated categories. For $\mathcal{D}_i \in \mathrm{Cat}_\infty^{\mathrm{rex,idem}}$, taking the product of $\mathrm{Ind}(\mathcal{D}_i)$ in $\mathrm{Pr}^L_{\mathrm{ideal}}$ gives $(\prod_i \mathrm{Ind}(\mathcal{D}_i), S)$ where S consists precisely of those morphisms factoring through $\prod_i \mathcal{D}_i \subseteq \prod_i \mathrm{Ind}(\mathcal{D}_i)$. Therefore $\prod_i^{\mathrm{ca}} \mathrm{Ind}(\mathcal{D}_i) = \mathrm{Ind}(\prod_i \mathcal{D}_i)$, and the comparison functor is an Ind-extension of $\prod_i j$:

$$\begin{tikzcd}
\mathrm{Ind}(\prod_i \mathcal{D}_i) \arrow[r, "\Phi"] & \prod_i \mathrm{Ind}(\mathcal{D}_i) \\
\prod_i \mathcal{D}_i \arrow[u, hook, "j"] \arrow[ur, hook, "\prod_i j"'] &
\end{tikzcd}$$

In particular, we have $(\prod_i^{\mathrm{ca}} \mathrm{Ind}(\mathcal{D}_i))^\omega = \prod_i \mathcal{D}_i$ whereas $(\prod_i \mathrm{Ind}(\mathcal{D}_i))^\omega = \bigoplus_i \mathcal{D}_i$.

We have shown the following:

Lemma 2.7.16. *The functor* $\mathrm{Ind} : \mathrm{Cat}_\infty^{\mathrm{rex,idem}} \simeq \mathrm{Pr}^L_\omega \subseteq \mathrm{Pr}^L_{\mathrm{ca}}$ *preserves products.*

Finally we give an example of the last situation, where the comparison functor from the compactly assembled limit to the usual limit lives in $\mathrm{Pr}^L_{\mathrm{ca}}$, but is not an equivalence.

Example 2.7.17. We compute the limit of

$$\ldots \to \mathcal{D}(\mathbb{Z}/p^n) \to \mathcal{D}(\mathbb{Z}/p^{n-1}) \to \ldots$$

in $\mathrm{Pr}^L_{\mathrm{ca}}$ which is called the ∞-category of nuclear modules over the analytic ring $\mathbb{Z}_p$ and denoted $\widetilde{\mathrm{Nuc}}(\mathbb{Z}_p)$. [10]

[10] There is a slight difference to the original nuclear category of Clausen-Scholze, which is why we include the tilde in the notation. We will explain this subtlety later.

In Pr^L, the limit is $\mathcal{D}(\mathbb{Z})_p^\wedge$, the category of p-complete derived abelian groups (an object $X \in \mathcal{D}(\mathbb{Z})$ is p-complete if $X \simeq \lim_{\mathbb{N}} X/p^n$. Equivalently, if $\lim(\ldots X \xrightarrow{p} X \ldots) \simeq 0$). Note that this category is compactly generated: Shifts of $\mathbb{Z}/p$ are compact, and generate, since if $\mathrm{Map}(\mathbb{Z}/p[n], X) = 0$ for all n, $p : X \to X$ is an equivalence, but under completeness, this means $X = 0$. Also, the right adjoint functors $\mathcal{D}(\mathbb{Z}/p^n) \to \mathcal{D}(\mathbb{Z})_p^\wedge$ are just restriction, and so commute with filtered colimits (since anything hit by restriction is already p-complete). So the limit cone is in $\mathrm{Pr}^L_{\mathrm{ca}}$.

However, $\mathcal{D}(\mathbb{Z})_p^\wedge$ is not the limit in $\mathrm{Pr}^L_{\mathrm{ca}}$ (universality fails). Instead, the limit is given by $(\mathcal{D}(\mathbb{Z})_p^\wedge, S)^{\mathrm{ca}}$, where S is the class of morphisms $X \to Y$ such that all $X/p^n \to Y/p^n$ are compact (over $\mathbb{Z}/p^n$).

This S contains identities on compact objects in $\mathcal{D}(\mathbb{Z})_p^\wedge$ (such as $\mathbb{Z}/p^n$), but it also contains the identity on $\mathbb{Z}_p$. So $j\mathbb{Z}_p$ is among the S-exhaustibles. We also have more surprising nontrivial objects in $\lim^{\mathrm{ca}} \mathcal{D}(\mathbb{Z}/p^n)$, for example the colimit

$$\mathbb{Q}_p = \mathrm{colim}(j\mathbb{Z}_p \xrightarrow{p} j\mathbb{Z}_p \xrightarrow{p} \ldots)$$

Note that in this case, we do actually have that

$$\lim\nolimits^{\mathrm{ca}} \mathcal{D}(\mathbb{Z}/p^n) \to \lim\nolimits^{L} \mathcal{D}(\mathbb{Z}/p^n)$$

is a Bousfield localization, related to the fact that the original limit cone was already in $\mathrm{Pr}^L_{\mathrm{ca}}$. This localization for example kills $\mathbb{Q}_p$.

Finally, we want to mention an application of the notion of precompact ideals and the compactly assembled core functor which will be useful later on:

Proposition 2.7.18. *Let X be a locally compact Hausdorff space, and let M denote the compact maps in $\mathrm{Shv}(X)$ in the image of $j : \mathrm{Open}(X) \subseteq \mathrm{Shv}(X)$. Concretely, we have*

$$M = \{\underline{U} \to \underline{V} \mid U \subseteq K \subseteq V \textit{ for some compact } K\}$$

by Lemma 2.2.8. Then:

1. *The compact maps in $\mathrm{Open}(X)$ generate the compact maps in $\mathrm{Shv}(X)$ as a precompact ideal. In other words, the ideal of compact maps in $\mathrm{Shv}(X)$ is the smallest precompact ideal containing M.*

2. *If $\mathcal{C}$ is compactly assembled, then a left-adjoint functor $F : \mathrm{Shv}(X) \to \mathcal{C}$ is compactly assembled if and only if it sends the maps in M to compact maps in $\mathcal{C}$.*

To prove this, we will need the following Lemma:

Lemma 2.7.19. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and $(f_i)_i$ a small collection of morphisms in $\mathcal{C}$. Let $\langle f_i \rangle_{\mathrm{ideal}}$ respectively $\langle f_i \rangle_{\mathrm{p-ideal}}$ denote the ideal respectively precompact ideal generated by the f_i, i.e. the smallest one containing all the f_i. Then:*

1. *$F(\langle f_i \rangle_{\mathrm{ideal}}) \subseteq \langle F f_i \rangle_{\mathrm{ideal}}$*

2. *If F preserves finite colimits, then $F(\langle f_i\rangle_{\mathrm{p-ideal}}) \subseteq \langle Ff_i\rangle_{\mathrm{p-ideal}}$.*

Proof. The first claim is easy and is left as an exercise to the reader. For the second, we build $\langle f_i\rangle_{\mathrm{p-ideal}}$ via transfinite induction. Let $S_0 = \{\mathrm{id}_\emptyset\} \cup \langle f_i\rangle_{\mathrm{ideal}}$. Given a limit ordinal λ, if we have defined S_α for $\alpha < \lambda$, we let $S_\lambda = \bigcup_{\alpha<\lambda} S_\alpha$. Given S_α, defined $S_{\alpha+1}$ as follows: Let M be the collection of all positive-length morphisms of $F_0 \amalg_{F_1} F_2$ for all possible diagrams $F_0 \leftarrow F_1 \rightarrow F_2$ of functors $[0,1] \cap \mathbb{Q} \to \mathcal{C}$ that send all positive-length morphisms into S_α. Then let $S_{\alpha+1} = \langle S_\alpha \cup M\rangle_{\mathrm{ideal}}$.

We now have $\langle f_i\rangle_{\mathrm{p-ideal}} = S_{\omega_1}$ since all choices of F_0, F_1, F_2 land in some stage $\alpha < \omega_1$. Moreover, it is clear by transfinite induction that $F(S_\alpha) \subseteq \langle Ff_i\rangle_{\mathrm{p-ideal}}$ for each α. $\square$

Proof of Proposition 2.7.18. By the above Lemma, it remains to see the first point. Since each map in M is compact in $\mathrm{Shv}(X)$, one inclusion is clear. For the other, consider the fully faithful inclusions

$$\mathrm{Shv}(X) \xrightarrow{\hat{\jmath}} \mathrm{Ind}(\mathrm{Shv}(X)) \quad \text{and} \quad \mathcal{C} \coloneqq (\mathrm{Shv}(X), \langle M\rangle_{\mathrm{p-ideal}})^{\mathrm{ca}} \subseteq \mathrm{Ind}(\mathrm{Shv}(X)).$$

By definition $\mathcal{C}$ is the full subcategory generated under colimits by $\langle M\rangle_{\mathrm{p-ideal}}$-exhaustible Ind-objects. Now $\mathrm{Shv}(X)$ itself is generated under colimits by the representables $\underline{U}$ for $U = \bigcup_n U_n$ compactly exhausted, so $U_n \subseteq U_{n+1}$ is a compact map in $\mathrm{Open}(X)$ for each n. But then $\underline{U_n} \to \underline{U_{n+1}}$ lies in M, hence $\hat{\jmath}(\underline{U}) \in \mathcal{C}$. Thus $\hat{\jmath}(\mathrm{Shv}(X)) \subseteq \mathcal{C}$, and we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Shv}(X) \arrow[r, hook, "\hat{\jmath}"] \arrow[dr, hook, "i"'] & \mathrm{Ind}(\mathrm{Shv}(X)) \arrow[r, "k"] & \mathrm{Shv}(X) \\
& (\mathrm{Shv}(X), \langle M\rangle_{\mathrm{p-ideal}})^{\mathrm{ca}} \arrow[u, hook] \arrow[ur, "\varepsilon"'] &
\end{tikzcd}$$

Since i is compactly assembled and ε sends compact maps into $\langle M\rangle_{\mathrm{p-ideal}}$, we see that all compact maps in $\mathrm{Shv}(X)$ lie in $\langle M\rangle_{\mathrm{p-ideal}}$, as desired. $\square$

2.8 Tensor product on Pr^L

As promised in the introduction, compactly assembled categories admit a further characterization as *dualizable* categories, with respect to a symmetric-monoidal structure on Pr^L. The correct generality for this is in the stable setting, where the examples we care about take place. In this section, we want to discuss the required tensor product on Pr^L and the notion of stable ∞-categories. This tensor product is due to Lurie [Lur17a] and generally all results in this section are due to him.

Definition 2.8.1. For $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Pr}^L$ write $\mathrm{Fun}^{\mathrm{biL}}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ for the full subcategory of the functor category consisting of all functors which are colimit-preserving in both variables separately.

Definition 2.8.2. The tensor product of two presentable ∞-categories $\mathcal{C}, \mathcal{D}$ is a presentable ∞-category $\mathcal{C} \otimes \mathcal{D}$ together with a functor

$$\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$$

preserving colimits in each variable separately, such that precomposition induces an equivalence

$$\mathrm{Fun}^L(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{biL}}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$$

for each presentable $\mathcal{E}$.

As usual, the universal property makes this essentially unique once it exists. We will see shortly that this is the case. Note also that

$$\mathrm{Fun}^L(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \simeq \mathrm{Fun}^{\mathrm{biL}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \simeq \mathrm{Fun}^L(\mathcal{C}, \mathrm{Fun}^L(\mathcal{D}, \mathcal{E})),$$

so that the tensor product exhibits Fun^L as corresponding internal Hom. Since $\mathrm{Fun}^L(\mathrm{An}, \mathcal{C}) \simeq \mathcal{C}$, we see that An is the neutral element.

Example 2.8.3. If $\mathcal{C} = \mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathrm{An})$ and $\mathcal{D} = \mathrm{Fun}(\mathcal{D}_0^{\mathrm{op}}, \mathrm{An})$, then

$$\mathrm{Fun}^{\mathrm{biL}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) = \mathrm{Fun}^L(\mathcal{C}, \mathrm{Fun}^L(\mathcal{D}, \mathcal{E})) = \mathrm{Fun}(\mathcal{C}_0, \mathrm{Fun}(\mathcal{D}_0, \mathcal{E})) = \mathrm{Fun}(\mathcal{C}_0 \times \mathcal{D}_0, \mathcal{E}),$$

which agrees with $\mathrm{Fun}^L(\mathrm{Fun}((\mathcal{C}_0 \times \mathcal{D}_0)^{\mathrm{op}}, \mathrm{An}), \mathcal{E})$. So we see that

$$\mathcal{C} \otimes \mathcal{D} = \mathrm{Fun}((\mathcal{C}_0 \times \mathcal{D}_0)^{\mathrm{op}}, \mathrm{An}).$$

Example 2.8.4. Given a left Bousfield localization $\mathcal{C} \to \mathcal{C}'$ we can identify $\mathcal{C}' \subseteq \mathcal{C}$ as the full subcategory on the $W_{\mathcal{C}}$-local objects, where $W_{\mathcal{C}}$ is some small set of morphisms in $\mathcal{C}$. Doing the same for $\mathcal{D} \to \mathcal{D}'$, we claim that the induced functor

$$\mathcal{C} \otimes \mathcal{D} \to \mathcal{C}' \otimes \mathcal{D}'$$

is also a left Bousfield localization, where $\mathcal{C}' \otimes \mathcal{D}'$ can explicitly be described as the full subcategory on the object local with respect to $W_{\mathcal{C}} \otimes \mathcal{D}^{\kappa} \cup \mathcal{C}^{\kappa} \otimes W_{\mathcal{D}}$, where κ is large enough so that $\mathcal{C}$ and $\mathcal{D}$ are κ-compactly generated. Indeed, we have

$$\begin{aligned}
\mathrm{Fun}^L(\mathcal{C}' \otimes \mathcal{D}', \mathcal{E}) &= \mathrm{Fun}^{\mathrm{biL}}(\mathcal{C}' \times \mathcal{D}', \mathcal{E}) \\
&= \mathrm{Fun}^{\mathrm{biL}, W_{\mathcal{C} \otimes \mathcal{D}}-\mathrm{loc}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \\
&= \mathrm{Fun}^{L, W_{\mathcal{C} \otimes \mathcal{D}}-\mathrm{loc}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}).
\end{aligned}$$

Since we have seen that every presentable ∞-category can be written as Bousfield localization of $\mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathrm{An})$, these two examples combine to give existence of tensor products generally. In fact, we can provide a more useful formula:

Lemma 2.8.5. *Let $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Pr}^L$ and $L : \mathcal{C} \to \mathcal{D}$ be a left adjoint. Then we have an equivalence $\mathcal{C} \otimes \mathcal{E} \simeq \mathrm{Fun}^{\lim}(\mathcal{C}^{\mathrm{op}}, \mathcal{E})$. Moreover, under this equivalence the functor $L \otimes \mathcal{E} : \mathcal{C} \otimes \mathcal{E} \to \mathcal{D} \otimes \mathcal{E}$ is identified with the left adjoint to the functor $(L^{\mathrm{op}})^* : \mathrm{Fun}^{\lim}(\mathcal{D}^{\mathrm{op}}, \mathcal{E}) \to \mathrm{Fun}^{\lim}(\mathcal{C}^{\mathrm{op}}, \mathcal{E})$.*

Proof. Writing $\mathcal{C}$ and $\mathcal{E}$ as left Bousfield localizations of $\mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathrm{An})$ and $\mathrm{Fun}(\mathcal{E}_0^{\mathrm{op}}, \mathrm{An})$ with respect to some generating equivalences, we have

$$\begin{aligned}\mathrm{Fun}^{\lim}(\mathcal{C}^{\mathrm{op}}, \mathcal{E}) &\subseteq \mathrm{Fun}^{\lim}(\mathrm{Fun}(\mathcal{E}_0^{\mathrm{op}}, \mathrm{An})^{\mathrm{op}}, \mathcal{E}) \\ &= \mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathcal{E}) \\ &\subseteq \mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathrm{Fun}(\mathcal{E}_0^{\mathrm{op}}, \mathrm{An})) \\ &= \mathrm{Fun}((\mathcal{C}_0 \times \mathcal{E}_0)^{\mathrm{op}}, \mathrm{An}) = \mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathrm{An}) \otimes \mathrm{Fun}(\mathcal{E}_0^{\mathrm{op}}, \mathrm{An}),\end{aligned}$$

where both inclusions are characterized by locality conditions. Tracing these through to the last term, one sees that $\mathrm{Fun}^{\lim}(\mathcal{C}^{\mathrm{op}}, \mathcal{E})$ agrees with the full subcategory of $\mathrm{Fun}((\mathcal{C}_0 \times \mathcal{E}_0)^{\mathrm{op}}, \mathrm{An}) = \mathrm{PShv}(\mathcal{C}_0) \otimes \mathrm{PShv}(\mathcal{E}_0)$ on local objects, i.e. agrees with the left Bousfield localization $\mathcal{C} \otimes \mathcal{E}$ as identified in the above example.

Note also that the above proves that $\mathcal{C} \otimes \mathrm{Fun}(\mathcal{A}, \mathrm{An}) \simeq \mathrm{Fun}(\mathcal{A}, \mathcal{C})$ for any small category $\mathcal{A}$. Now the second claim is clear for the unit $\mathcal{E} = \mathrm{An}$, from the fact that left Kan extension preserves representable functors. The case $\mathcal{E} = \mathrm{Fun}(\mathcal{E}_0^{\mathrm{op}}, \mathrm{An})$ then follows from the identification we just mentioned, and the general case by a similar argument as above using the inclusion $\mathcal{C} \otimes \mathcal{E} \subseteq \mathcal{C} \otimes \mathrm{Fun}(\mathcal{E}_0^{\mathrm{op}}, \mathrm{An})$. □

Proposition 2.8.6. *For topological spaces X and Y we have* $\mathrm{Shv}(X) \otimes \mathrm{Shv}(Y) \simeq \mathrm{Shv}(X \times Y)$.

Proof. $\mathrm{Shv}(X)$ arises as Bousfield localization of $\mathrm{PShv}(X) = \mathrm{Fun}(\mathrm{Open}(X)^{\mathrm{op}}, \mathrm{An})$, analogously for Y. So $\mathrm{Shv}(X) \otimes \mathrm{Shv}(Y)$ arises as Bousfield localization of

$$\mathrm{Fun}((\mathrm{Open}(X) \times \mathrm{Open}(Y))^{\mathrm{op}}, \mathrm{An}).$$

The locality condition is precisely descent in each variable. Since "boxes" $U \times V$ provide a basis of the topology of $X \times Y$, and the coordinatewise descent condition generates the same Grothendieck topology, this agrees with $\mathrm{Shv}(X \times Y)$. □

Proposition 2.8.7. *For $\mathcal{C} \in \mathrm{Pr}^L$, we have an equivalence* $\mathrm{Shv}(X; \mathcal{C}) \simeq \mathrm{Shv}(X; \mathrm{An}) \otimes \mathcal{C}$, *which is compatible with the restriction functoriality of sheaves in that for all continuous maps $f : X \to Y$ we have commutative squares:*

$$\begin{tikzcd} \mathrm{Shv}(Y; \mathcal{C}) \arrow[r, "f^*_{\mathcal{C}}"] & \mathrm{Shv}(X; \mathcal{C}) \\ \mathrm{Shv}(Y) \otimes \mathcal{C} \arrow[u, "\simeq"] \arrow[r, "f^*_{\mathrm{An}} \otimes \mathcal{C}"] & \mathrm{Shv}(X) \otimes \mathcal{C} \arrow[u, "\simeq"'] \end{tikzcd}$$

Proof. Using Lemma 2.8.5 we will identify

$$\mathrm{Shv}(X) \otimes \mathcal{C} = \mathrm{Fun}^{\lim}(\mathrm{Shv}(X)^{\mathrm{op}}, \mathcal{C}).$$

We now obtain a commutative diagram with vertical full subcategories

$$\begin{tikzcd} \mathrm{Fun}^{\lim}(\mathrm{Shv}(X)^{\mathrm{op}}, \mathcal{C}) \arrow[d, hook, "(((-)^{\mathrm{shv}})^{\mathrm{op}})^*"'] & \mathrm{Shv}(X; \mathcal{C}) \arrow[d, hook] \\ \mathrm{Fun}^{\lim}(\mathrm{PShv}(X)^{\mathrm{op}}, \mathcal{C}) \arrow[r, "\simeq", "(j^{\mathrm{op}})^*"'] & \mathrm{Fun}(\mathrm{Open}(X)^{\mathrm{op}}, \mathcal{C}) \end{tikzcd}$$

and it is not hard to verify that these full subcategories are identified under this equivalence.

Now let $f : X \to Y$ be a continuous map. Recall that one defines the pushforward $f_*^{\mathcal{C}} : \mathrm{Shv}(X;\mathcal{C}) \to \mathrm{Shv}(Y;\mathcal{C})$ by simply precomposing with $(f^{-1})^{\mathrm{op}} : \mathrm{Open}(Y)^{\mathrm{op}} \to \mathrm{Open}(X)^{\mathrm{op}}$. This preserves limits, hence admits a left adjoint $f_{\mathcal{C}}^*$ by the adjoint functor theorem. To check commutativity of the square in the statement, we may replace $f_{\mathcal{C}}^*$ respectively $f^* \otimes \mathcal{C}$ by their right adjoints $f_*^{\mathcal{C}}$ respectively precomposition with $(f_{\mathrm{An}}^*)^{\mathrm{op}}$ (by Lemma 2.8.5), giving the left rectangle:

$$\begin{tikzcd}
\mathrm{Shv}(Y;\mathcal{C}) \arrow[d, equal] & & \mathrm{Shv}(X;\mathcal{C}) \arrow[ll, "f_*^{\mathcal{C}}"'] \arrow[d, equal] & & & \\
\mathrm{Fun}^{\mathrm{shv}}(\mathrm{Open}(Y)^{\mathrm{op}},\mathcal{C}) & & \mathrm{Fun}^{\mathrm{shv}}(\mathrm{Open}(X)^{\mathrm{op}},\mathcal{C}) \arrow[ll, "((f^{-1})^{\mathrm{op}})^*"'] & & \mathrm{Open}(Y) \arrow[r, "f^{-1}"] \arrow[d, hook, "j"'] & \mathrm{Open}(X) \arrow[d, hook, "j"] \\
\mathrm{Fun}^{\mathrm{lim}}(\mathrm{Shv}(Y)^{\mathrm{op}},\mathcal{C}) \arrow[u, "(j^{\mathrm{op}})^*", "\simeq"'] & & \mathrm{Fun}^{\mathrm{lim}}(\mathrm{Shv}(X),\mathcal{C}) \arrow[ll, "((f_{\mathrm{An}}^*)^{\mathrm{op}})^*"] \arrow[u, "\simeq", "(j^{\mathrm{op}})^*"'] & & \mathrm{Shv}(Y) \arrow[r, "f_{\mathrm{An}}^*"'] & \mathrm{Shv}(X)
\end{tikzcd}$$

Now f_{An}^* is given by left Kan extension along $(f^{-1})^{\mathrm{op}}$ followed by sheafification. However, left Kan extension preserves representables and representables are sheaves, so the right square and hence also the left rectangle commute, as desired. □

Example 2.8.8. Set is a left Bousfield localization of An at the class W of π_0-isomorphisms. This exhibits Set $\otimes$ Set as localization of An $\otimes$ An $\simeq$ An, and inspection shows that it is again at the same class of morphisms, so Set $\otimes$ Set $\simeq$ Set. Consequently,

$$\mathrm{Shv}(X;\mathrm{Set}) \otimes \mathrm{Shv}(Y;\mathrm{Set}) = \mathrm{Shv}(X) \otimes \mathrm{Set} \otimes \mathrm{Shv}(Y) \otimes \mathrm{Set} = \mathrm{Shv}(X \times Y;\mathrm{Set}).$$

Lemma 2.8.9. *Writing* An_* *for the category of pointed objects* $\mathrm{An}_{*/}$*, we have*

$$\mathrm{An}_* \otimes \mathcal{C} = \mathcal{C}_*,$$

the category of pointed objects in $\mathcal{C}$ *(slice under the terminal object). In particular,* $\mathrm{An}_* \otimes \mathrm{An}_* \simeq \mathrm{An}_*$.

Proof. We may write An_* as Bousfield localization of An^{Δ^1}, consisting of the full subcategory of An^{Δ^1} on all arrows whose first entry is $*$. (The left adjoint $\mathrm{An}^{\Delta^1} \to \mathrm{An}_*$ takes $A \to X$ to X/A). This exhibits $\mathrm{Fun}^{\mathrm{lim}}((\mathrm{An}_*)^{\mathrm{op}},\mathcal{C})$ as full subcategory of $\mathrm{Fun}(\Delta^1,\mathcal{C})$ with objects characterized by some locality condition. Unwrapping things, we exactly find the full subcategory of arrows where the first entry is $*$, i.e. $\mathcal{C}_*$. □

Definition 2.8.10. An ∞-category $\mathcal{C}$ is called *stable* if it has finite limits and colimits, a zero object, and a square

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}$$

is a pullback diagram if and only if it is a pushout diagram.

For example, this holds in derived categories such as $D(\mathbb{Z})$. Conversely, stable ∞-categories behave a lot like derived (or triangulated) categories; they are additive, we can write pullbacks as fibers, etc. It can be seen that the condition on squares is equivalent to the suspension (or shift) functor $\Sigma : \mathcal{C} \to \mathcal{C}$ being an equivalence, where ΣX is defined by the pushout

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma X. \end{tikzcd}$$

A kind of universal example (in presentable stable categories) is therefore given as follows:

Definition 2.8.11. The category of spectra Sp is defined as the following colimit in Pr^L

$$\operatorname{colim}(\mathrm{An}_* \xrightarrow{\Sigma} \mathrm{An}_* \xrightarrow{\Sigma} \mathrm{An}_* \to \cdots).$$

Note that this colimit can be computed as limit in Pr^R of the right adjoint functors

$$\lim(\cdots \to \mathrm{An}_* \xrightarrow{\Omega} \mathrm{An}_* \xrightarrow{\Omega} \mathrm{An}_*),$$

so explicitly a spectrum consists of a sequence of pointed anima X_n with equivalences $X_n \simeq \Omega X_{n+1}$.

Lemma 2.8.12.

1. *The canonical map* $\mathrm{An} \to \mathrm{Sp}$ *induces an equivalence* $\mathrm{Sp} \simeq \mathrm{Sp} \otimes \mathrm{Sp}$. *The inverse equivalence makes* Sp *into a commutative algebra in* Pr^L.

2. *A presentable category* $\mathcal{C}$ *is stable if and only if the canonical map*

$$\mathcal{C} \to \mathrm{Sp} \otimes \mathcal{C}$$

is an equivalence. This makes $\mathcal{C}$ *into a module over* Sp *in* Pr^L *(with symmetric-monoidal structure given by* $\otimes$*).*

Proof. Note that

$$\mathrm{Sp} \otimes \mathcal{C} \simeq \operatorname{colim}(\mathrm{An}_* \xrightarrow{\Sigma} \cdots) \otimes \mathcal{C} \simeq \operatorname{colim}(\mathcal{C}_* \xrightarrow{\Sigma} \cdots).$$

If $\mathcal{C}$ is stable, we have $\mathcal{C}_* \simeq \mathcal{C}$, and Σ is an equivalence, so in that case the colimit is just $\mathcal{C}$ again. Conversely, $\mathrm{Sp} \otimes \mathcal{C}$ is stable, since the suspension functor on it can be written as $\Sigma \otimes \mathrm{id}_{\mathcal{C}}$, but $\Sigma : \mathrm{Sp} \to \mathrm{Sp}$ is an equivalence. □

Definition 2.8.13. We denote the full subcategory of Pr^L on the stable presentable categories by $\mathrm{Pr}^L_{\mathrm{st}}$.

By the above, these are exactly the modules over the idempotent algebra Sp. In particular, $\mathrm{Pr}^L_{\mathrm{st}}$ inherits a symmetric monoidal structure from Pr^L, with the same tensor product and unit Sp.

Note that if $\mathcal{C}$ is stable, then $\mathcal{C}^\kappa$ is also stable: The κ-compact objects are always closed under κ-small colimits. In a stable category, they are also closed under finite limits, for which it suffices to check fiber sequences. But these agree with cofiber sequences, so by rotating the sequence one concludes. Conversely, if $\mathcal{C}$ is small stable, $\mathrm{Ind}_\kappa(\mathcal{C})$ is stable.

Examples of presentable stable ∞-categories include derived categories of rings. We may also think of Sp in such a way: Since $\mathrm{Sp} \otimes \mathrm{Sp} \simeq \mathrm{Sp}$, Sp is an idempotent algebra object in Pr^L. This gives Sp a symmetric-monoidal structure, whose unit is the sphere spectrum $\mathbb{S} \in \mathrm{Sp}$. Explicitly, it is the image of S^0 under the canonical left adjoint $\mathrm{An}_* \to \mathrm{Sp}$. Since every object in a symmetric-monoidal category is canonically a module over the unit, we have $\mathrm{Sp} \simeq \mathrm{Mod}(\mathbb{S})$.

One can prove that Sp is the free presentable stable category on one generator, which concretely means that evaluating at $\mathbb{S}$ induces an equivalence

$$\mathrm{Fun}^L(\mathrm{Sp}, \mathcal{C}) \xrightarrow{\simeq} \mathcal{C}$$

for any $\mathcal{C} \in \mathrm{Pr}^L_{\mathrm{st}}$. In other words, giving a left adjoint functor $\mathrm{Sp} \to \mathcal{C}$ is the same as picking an object of $c \in \mathcal{C}$; the functor then sends $X \mapsto X \otimes c$, using the module structure of $\mathcal{C}$ over Sp in Pr^L.

For a ring R, the unit $R[0] \in \mathcal{D}(R)$ induces a symmetric monoidal left adjoint $\mathrm{Sp} \to D(R)$, whose right adjoint takes $R[0]$ to an algebra object in Sp, known as Eilenberg-MacLane spectrum HR. In fact, this right adjoint induces an equivalence $\mathcal{D}(R) \simeq \mathrm{Mod}_{\mathrm{Sp}}(HR)$.

Example 2.8.14. For commutative ring spectra R, S, one has $\mathrm{Mod}(R) \otimes \mathrm{Mod}(S) \simeq \mathrm{Mod}(R \otimes_{\mathbb{S}} S)$. In particular, for ordinary rings R, S, we have $\mathcal{D}(R) \otimes \mathcal{D}(S) \simeq \mathrm{Mod}(HR \otimes_{\mathbb{S}} HS)$.

The tensor product of two Eilenberg-MacLane spectra is rarely itself an Eilenberg-MacLane spectrum (except in the rational case). So even if we only start with "ordinary derived categories", the tensor product in Pr^L leads us to more general ∞-categories. However, the functor $\mathrm{CRing} \to \mathrm{CAlg}(\mathrm{Sp})$ taking an ordinary ring R to the Eilenberg-MacLane spectrum HR is fully faithful (Eilenberg-MacLane spectra are in a sense discrete), and so we will sometimes drop the H and view ordinary rings as ring spectra, i.e. as algebras over $\mathbb{S}$.

Definition 2.8.15. If $\mathcal{C}$ is a commutative algebra in Pr^L, and $\mathcal{D}, \mathcal{E}$ are modules over it, the relative tensor product is defined as the geometric realization

$$\mathcal{D} \otimes_{\mathcal{C}} \mathcal{E} = \mathrm{colim}\big(\ \mathcal{D} \otimes \mathcal{E} \leftleftarrows \mathcal{D} \otimes \mathcal{C} \otimes \mathcal{E} \Lleftarrow \mathcal{D} \otimes \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{E} \dots \ \big)$$

Lemma 2.8.16. *For ordinary rings R, S, we have*

$$\mathcal{D}(R) \otimes_{\mathcal{D}(\mathbb{Z})} \mathcal{D}(S) = \mathrm{Mod}_{\mathcal{D}(\mathbb{Z})}(R \otimes^L_{\mathbb{Z}} S),$$

in particular, if R, S are Tor-independent, this simplifies to $\mathcal{D}(R \otimes_{\mathbb{Z}} S)$.

Note that $\mathcal{D}(\mathbb{Z})$, as opposed to Sp, is not idempotent in Pr^L, as $\mathcal{D}(\mathbb{Z}) \otimes \mathcal{D}(\mathbb{Z}) \simeq \mathrm{Mod}(H\mathbb{Z} \otimes_{\mathbb{S}} H\mathbb{Z})$, but $H\mathbb{Z} \otimes_{\mathbb{S}} H\mathbb{Z}$ is very different from $H\mathbb{Z}$! So the $\mathcal{D}(\mathbb{Z})$-linear tensor product almost always differs from the underlying tensor product, and $\mathcal{D}(\mathbb{Z})$-linearity is really additionally structure on a category, as opposed to stability.

$\mathcal{D}(\mathbb{Z})$-linear categories often arise from so-called *dg-categories*, which are a version of higher categories enriched in chain complexes rather than anima. Let us explain this enriched perspective a bit. For an algebra $\mathcal{C}$ in Pr^L, the unit $\mathrm{An} \to \mathcal{C}$ has a right adjoint $U : \mathcal{C} \to \mathrm{An}$ that is lax symmetric monoidal.

Proposition 2.8.17. *For a $\mathcal{C}$-module $\mathcal{D}$ in Pr^L there is an extension*

$$\begin{tikzcd} & \mathcal{C} \arrow[d, "U"] \\ \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \arrow[r, "\mathrm{Map}_{\mathcal{D}}"'] \arrow[ur, dashed, "\mathrm{Map}_{\mathcal{D}}^{\mathcal{C}}"] & \mathrm{An} \end{tikzcd} \tag{2.2}$$

characterised by the universal property that

$$\mathrm{Map}_{\mathcal{C}}(C, \mathrm{Map}_{\mathcal{D}}^{\mathcal{C}}(A, B)) \simeq \mathrm{Map}_{\mathcal{D}}(C \otimes A, B).$$

for $C \in \mathcal{C}$ and $A, B \in \mathcal{D}$.

Proof. We define a functor

$$\mathcal{D}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An}), \qquad (A, B) \mapsto (C \mapsto \mathrm{Map}_{\mathcal{D}}(C \otimes A, B))$$

and note that in fact it lands in $\mathrm{Fun}^{\lim}(\mathcal{C}^{\mathrm{op}}, \mathrm{An}) \simeq \mathcal{C}$. This defines the desired functor with the universal property. The commutativity of the triangle 2.2 follows since the functor U is given by evaluation at the tensor unit of $\mathcal{C}$ under the equivalence $\mathrm{Fun}^{\lim}(\mathcal{C}^{\mathrm{op}}, \mathrm{An}) \simeq \mathcal{C}$. $\square$

One can in fact also lift the composition of maps

$$\mathrm{Map}_{\mathcal{C}}(b, c) \times \mathrm{Map}_{\mathcal{C}}(a, b) \to \mathrm{Map}_{\mathcal{C}}(a, c)$$

to maps

$$\mathrm{Map}_{\mathcal{D}}^{\mathcal{C}}(b, c) \otimes \mathrm{Map}_{\mathcal{D}}^{\mathcal{C}}(a, b) \to \mathrm{Map}_{\mathcal{D}}^{\mathcal{C}}(a, c)\ .$$

Details left for the reader, also see [GH15] for a highly coherent statement in the sense of enriched categories.

In the case that $\mathcal{C} = \mathrm{Sp}$ we get mapping spectra, and for simplicity we write $\mathrm{map}_{\mathcal{C}}(A, B)$ for $\mathrm{Map}_{\mathcal{C}}^{\mathrm{Sp}}(A, B)$. In fact, for any small stable ∞-category $\mathcal{C}$ we get a mapping spectrum functor

$$\mathrm{map}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Sp},$$

for example by restricting the mapping spectrum functor from $\mathrm{Ind}(\mathcal{C})$.

2.9 Dualizable stable ∞-categories

In this section we would like to analyse when objects in $\mathrm{Pr}^L_{\mathrm{st}}$ are dualizable. These will exactly be the compactly assembled stable ∞-categories. Recall that an object X in a symmetric monoidal ∞-category category $\mathcal{C}$ is called dualizable if there exists another object $\mathcal{C}^\vee$ and maps

$$\mathrm{ev} : X^\vee \otimes X \to \mathbb{1} \qquad \mathrm{coev} : \mathbb{1} \to X \otimes X^\vee$$

such that the compositions

$$X \xrightarrow{\mathrm{coev}\otimes \mathrm{id}} X \otimes X^\vee \otimes X \xrightarrow{\mathrm{id}\otimes \mathrm{ev}} X$$
$$X^\vee \xrightarrow{\mathrm{id}\otimes \mathrm{coev}} X^\vee \otimes X \otimes X^\vee \xrightarrow{\mathrm{ev}\otimes \mathrm{id}} X^\vee$$

are both homotopic to the identity. These are often called "zig-zag identities", "triangle identities" or "snake identities". We denote the full subcategory on the dualizable objects by $\mathcal{C}^{\mathrm{dbl}}$. This might look familiar to the identities for unit and counit of an adjunction. And indeed, it is not hard to verify that ev and coev induce natural transformations exhibiting $X \otimes - : \mathcal{C} \to \mathcal{C}$ as both left and right adjoint (since $\mathcal{C}$ is *symmetric* monoidal) to $X^\vee \otimes -$. Beware that the converse generally fails, i.e. one can have $(X \otimes -) \dashv (Y \otimes -)$ without X and Y being duals.

Recall that $\mathcal{C}$ is closed symmetric monoidal if it admits an internal hom object

$$[-,-] = \underline{\mathrm{Hom}}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$$

such that we have tensor-hom adjunctions, i.e. equivalences

$$\mathrm{Map}_{\mathcal{C}}(X \otimes Y, Z) \simeq \mathrm{Map}_{\mathcal{C}}(X, [Y, Z])$$

natural in $X, Y, Z \in \mathcal{C}$. The canonical example of our interest is Pr^L which admits the internal homs Fun^L.

Lemma 2.9.1. *Suppose that $\mathcal{C}$ is a closed symmetric monoidal category. For $X \in \mathcal{C}$ the following are equivalent:*

1. *X is dualizable.*
2. *There exists a map $c : \mathbb{1} \to X \otimes [X, \mathbb{1}]$ such that the following diagram commutes*

$$\begin{tikzcd} & X \otimes [X,\mathbb{1}] \otimes X \arrow[dr, "X\otimes\varepsilon_{\mathbb{1}}"] & \\ X \arrow[ur, "c\otimes X"] \arrow[rr, equal] & & X \end{tikzcd}$$

   Here $\varepsilon_{\mathbb{1}} : [X, \mathbb{1}] \otimes X \to \mathbb{1}$ is the counit of the adjunction $- \otimes X \dashv [X, -]$ at $\mathbb{1}$.

3. *For every $Y \in \mathcal{C}$, the canonical map $[X, \mathbb{1}] \otimes Y \to [X, Y]$ is an equivalence. This map is adjoint to $[X, \mathbb{1}] \otimes X \otimes Y \xrightarrow{\varepsilon_{\mathbb{1}}\otimes Y} Y$.*

In particular, if X is dualizable, then necessarily $X^\vee = [X, \mathbb{1}]$ with evaluation $\mathrm{ev} = \varepsilon_{\mathbb{1}}$.

Proof. Clearly (1) implies (2). For the converse, we have to prove the remaining triangle identity. Using the adjunction $- \otimes X \dashv [X, -]$, this is equivalent to the commutativity of the following square, which follows from the given triangle identity:

$$\begin{tikzcd}
{[X,\mathbb{1}] \otimes X} \arrow[r, "{[X,\mathbb{1}]\otimes c \otimes X}"] \arrow[d, "\varepsilon_{\mathbb{1}}"'] \arrow[dr, equal] & {[X,\mathbb{1}] \otimes X \otimes [X,\mathbb{1}] \otimes X} \arrow[d, "{[X,\mathbb{1}]\otimes X \otimes \varepsilon_{\mathbb{1}}}"] \\
\mathbb{1} & {[X,\mathbb{1}] \otimes X} \arrow[l, "\varepsilon_{\mathbb{1}}"]
\end{tikzcd}$$

For the last point let $\phi_Y : [X, \mathbb{1}] \otimes Y \to [X, Y]$ denote the map mentioned in the statement. One has a natural commutative diagram

$$\begin{tikzcd}
\mathcal{C}(Z, [X,\mathbb{1}] \otimes Y) \arrow[r, "X\otimes -"] \arrow[d, "(\phi_Y)_*"] & \mathcal{C}(X \otimes Z, X \otimes [X,\mathbb{1}] \otimes Y) \arrow[r, "(\varepsilon_{\mathbb{1}}\otimes Y)_*"] \arrow[d, "(X\otimes\phi_Y)_*"] & \mathcal{C}(X \otimes Z, Y) \arrow[d, equal] \\
\mathcal{C}(Z, [X,Y]) \arrow[r, "X\otimes -"] & \mathcal{C}(X \otimes Z, X \otimes [X,Y]) \arrow[r, "(\varepsilon_Y)_*"] & \mathcal{C}(X \otimes Z, Y)
\end{tikzcd}$$

The bottom composite is always an equivalence by the adjunction $X \otimes - \dashv [X, -]$. Given (2), the top composite admits an inverse by the triangle identities, hence ϕ is an equivalence. Conversely, if ϕ is an equivalence, then also the top composite is one. Setting $Y = X$ and $Z = \mathbb{1}$ one sees that $c : X \to X \otimes [X, \mathbb{1}] \simeq [X, \mathbb{1}] \otimes X$ satisfies the triangle identity in (2). $\square$

In particular, if $\mathcal{C} \in \mathrm{Pr}^L_{\mathrm{st}}$ is dualizable, then its dual must be $\mathrm{Fun}^L(\mathcal{C}, \mathrm{Sp})$.

Theorem 2.9.2 (Lurie)**.** *If $\mathcal{C}$ is stable and presentable, then $\mathcal{C}$ is dualizable in $\mathrm{Pr}^L_{\mathrm{st}}$ if and only if it is compactly assembled.*

Before we give the proof of this theorem we note the following specialisation of Theorem 2.2.15. If $\mathcal{C}$ is stable, then the following conditions on $\mathcal{C}$ are equivalent:

1. $\mathcal{C}$ is compactly assembled.
2. $\mathcal{C}$ is generated under colimits by weakly (strongly) compactly exhaustible objects.
3. The colimit functor $k : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint.
4. The colimit functor $\mathrm{Ind}(\mathcal{C}^{\omega_1}) \to \mathcal{C}$ admits a fully faithful left adjoint.
5. $\mathcal{C}$ is a retract in $\mathrm{Pr}^L_{\mathrm{st}}$ of a compactly generated, stable ∞-category.
6. Filtered colimits in $\mathcal{C}$ *distribute* over small products, i.e. for filtered I we have

$$\prod_K \mathrm{colim}_I F \simeq \mathrm{colim}_{K^I} \prod_K F.$$

The slight changes follow since in stable ∞-categories colimits are automatically exact and since Ind of a stable ∞-category is also stable.

Remark 2.9.3. In particular, we see that if $\mathcal{C} \in \mathrm{Pr}^L$ is compactly assembled, then $\mathrm{Sp}(\mathcal{C}) = \mathcal{C} \otimes \mathrm{Sp}$ is dualizable. However, note that the converse fails: Set is even compactly generated, but $\mathrm{Sp}(\mathrm{Set}) = 0$ because $\Omega \simeq \mathrm{const}_* : \mathrm{Set}_* \to \mathrm{Set}_*$.

Lemma 2.9.4. *For a small stable category $\mathcal{C}_0 \in \mathrm{Cat}_\infty^{\mathrm{ex}}$ and presentable $\mathcal{D} \in \mathrm{Pr}^L$ we have:*

1. $\mathrm{Ind}(\mathcal{C}_0) \otimes \mathcal{D} \simeq \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}_0^{\mathrm{op}}, \mathcal{D})$.
2. *If $\mathcal{D}$ is moreover stable, we have* $\mathrm{Ind}(\mathcal{C}_0) \otimes \mathcal{D} \simeq \mathrm{Fun}^L(\mathrm{Ind}(\mathcal{C}_0^{\mathrm{op}}), \mathcal{D})$.

Proof. For the first statement, we have

$$\mathrm{Ind}(\mathcal{C}_0) \otimes \mathcal{D} \simeq \mathrm{Fun}^{\lim}(\mathrm{Ind}(\mathcal{C}_0)^{\mathrm{op}}, \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}_0^{\mathrm{op}}, \mathcal{D})$$

by the universal property of Ind. If additionally $\mathcal{D}$ is stable,

$$\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}_0^{\mathrm{op}}, \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{rex}}(\mathcal{C}_0^{\mathrm{op}}, \mathcal{D}) \simeq \mathrm{Fun}^L(\mathrm{Ind}(\mathcal{C}_0^{\mathrm{op}}), \mathcal{D}) \qquad \square$$

Note that this implies

$$\mathrm{Fun}^L(\mathcal{D}, \mathrm{Ind}(\mathcal{C}_0) \otimes \mathcal{E}) \simeq \mathrm{Fun}^L(\mathcal{D}, \mathrm{Fun}^L(\mathrm{Ind}(\mathcal{C}_0^{\mathrm{op}}), \mathcal{E})) \simeq \mathrm{Fun}^L(\mathcal{D} \otimes \mathrm{Ind}(\mathcal{C}_0^{\mathrm{op}}), \mathcal{E}).$$

This suggests that $\mathrm{Ind}(\mathcal{C}_0)$ is in fact dual to $\mathrm{Ind}(\mathcal{C}_0^{\mathrm{op}})$:

Lemma 2.9.5. *If $\mathcal{C}_0$ is small stable,* $\mathrm{Ind}(\mathcal{C}_0)$ *is a dualizable object in* $\mathrm{Pr}^L_{\mathrm{st}}$, *with dual* $\mathrm{Ind}(\mathcal{C}_0^{\mathrm{op}})$.

Proof. We have an evaluation map, obtained as the functor

$$\mathrm{ev} : \mathrm{Ind}(\mathcal{C}_0^{\mathrm{op}}) \otimes \mathrm{Ind}(\mathcal{C}_0) \simeq \mathrm{Ind}(\mathcal{C}_0^{\mathrm{op}} \otimes^{\mathrm{ex}} \mathcal{C}_0) \to \mathrm{Sp}$$

which is the Ind-extension of the functor $\mathrm{map}_{\mathcal{C}_0} : \mathcal{C}_0^{\mathrm{op}} \times \mathcal{C}_0 \to \mathrm{Sp}$ (using that it is exact in both arguments). Here $\otimes^{\mathrm{ex}}$ denotes the tensor product in $\mathrm{Cat}_\infty^{\mathrm{ex}}$. We also have a coevaluation

$$\mathrm{coev} : \mathrm{Sp} \to \mathrm{Ind}(\mathcal{C}_0) \otimes \mathrm{Ind}(\mathcal{C}_0^{\mathrm{op}}) \simeq \mathrm{Ind}(\mathcal{C}_0 \otimes^{\mathrm{ex}} \mathcal{C}_0^{\mathrm{op}}).$$

Since the target is stable, such a morphism is given by a single object. As objects are given by finite-limit preserving functors $(\mathcal{C}_0 \otimes^{\mathrm{ex}} \mathcal{C}_0^{\mathrm{op}})^{\mathrm{op}} \to \mathrm{An}$, i.e. functors $\mathcal{C}_0^{\mathrm{op}} \times \mathcal{C}_0 \to \mathrm{An}$ which preserve finite limits in both arguments, this can again be given by $\mathrm{map}_{\mathcal{C}_0}$. One may then check that the "snake identities" are satisfied, which we leave to the reader. $\square$

Proof of Theorem 2.9.2. By the above, compactly generated categories are dualizable in $\mathrm{Pr}^L_{\mathrm{st}}$. Since compactly assembled categories are retracts of compactly generated categories, they are also dualizable.[11]

[11] In general if the ambient category is idempotent complete or closed symmetric monoidal, then retracts of dualizable objects are dualizable. This follows by simply using the 'restricted' evaluation and coevaluation maps.

Conversely, assume $\mathcal{C}$ is dualizable with dual $\mathcal{C}^\vee$. With κ large enough so that $\mathcal{C}$ is κ-compactly generated, we have that

$$\mathrm{Ind}(\mathcal{C}^\kappa) \to \mathcal{C}$$

is a left Bousfield localization, see Corollary 2.1.27. So also

$$\mathrm{Ind}(\mathcal{C}^\kappa) \otimes \mathcal{C}^\vee \to \mathcal{C} \otimes \mathcal{C}^\vee$$

is a left Bousfield localization (see Example 2.8.4), in particular it is essentially surjective. Since in $\mathrm{Pr}^L_{\mathrm{st}}$, a functor out of Sp is given precisely by an object, this means the counit $\mathrm{Sp} \to \mathcal{C} \otimes \mathcal{C}^\vee$ lifts to $\mathrm{Sp} \to \mathrm{Ind}(\mathcal{C}^\kappa) \otimes \mathcal{C}^\vee$. Under duality, this means that the identity $\mathcal{C} \to \mathcal{C}$ lifts to a functor $\mathcal{C} \to \mathrm{Ind}(\mathcal{C}^\kappa)$, i.e. that $\mathcal{C}$ is a retract of a compactly generated category, finishing the proof. □

Definition 2.9.6. We denote the category of dualizable stable presentable ∞-categories and compactly assembled functors by $\mathrm{Pr}^L_{\mathrm{dual}}$.

Note that for a left adjoint functor between stable ∞-categories being compactly assembled is equivalent to being strongly left adjoint, that is the right adjoint admits a further right adjoint. This follows by Proposition 2.6.1 and the adjoint functor theorem since right adjoint functors between stable ∞-categories automatically preserve finite colimits.

We recall that a *Verdier sequence* of stable ∞-categories

$$\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{E}$$

is a sequence that is a fiber and cofiber sequence in the category $\mathrm{Cat}^{\mathrm{ex}}_\infty$ of stable ∞-categories and exact functors. Concretely, this means:

1. i is fully faithful and its image is closed under retracts.

2. p exhibits $\mathcal{E}$ as the Dwyer-Kan localization at the "mod-$\mathcal{C}$-equivalences", i.e. those morphisms with (co)fiber in $\mathcal{C}$. (I.e. it is universal among functors taking those morphisms to equivalences).

Often one only starts with a given $i : \mathcal{C} \hookrightarrow \mathcal{D}$, and denotes the cofiber by $\mathcal{D}/\mathcal{C}$. This is also called the Verdier quotient. Conversely, one has:

- Every exact Dwyer-Kan localization $p : \mathcal{D} \to \mathcal{E}$ between stable ∞-categories is a Verdier quotient, i.e. sits in a Verdier sequence with $\mathcal{C} = \ker(p)$.

- Every exact, fully faithful functor $i : \mathcal{C} \hookrightarrow \mathcal{D}$ of stable ∞-categories whose image is closed under retracts in $\mathcal{D}$ is a Verdier kernel, i.e. participates in a Verdier sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ with $\mathcal{E} = \mathcal{D}/\mathcal{C}$.

We denote the ∞-category of small idempotent complete stable ∞-categories and exact functors by $\mathrm{Cat}_\infty^{\mathrm{perf}} \subseteq \mathrm{Cat}_\infty^{\mathrm{ex}}$. Then a Karoubi sequence is a fiber and cofiber sequence in $\mathrm{Cat}_\infty^{\mathrm{perf}}$. Concretely, this is a sequence

$$\mathcal{C} \to \mathcal{D} \to \mathcal{E}$$

where $\mathcal{C} \to \mathcal{D}$ is fully faithful and the morphism $\mathcal{D}/\mathcal{C} \to \mathcal{E}$ is an idempotent completion. Note that generally a Karoubi sequence is not a Verdier sequence since $\mathcal{D}/\mathcal{C} \to \mathcal{E}$ doesn't have to be essentially surjective (equivalently $\mathcal{D} \to \mathcal{E}$ is not essentially surjective.)

However, if we have a sequence in $\mathrm{Pr}^L_{\mathrm{st}}$

$$\mathcal{C} \to \mathcal{D} \to \mathcal{E}$$

then it is Verdier (in the large version $\widehat{\mathrm{Cat}}_\infty^{\mathrm{ex}}$) if and only if it is Karoubi (in $\widehat{\mathrm{Cat}}_\infty^{\mathrm{perf}}$) if and only if $\mathcal{D} \to \mathcal{E}$ is a left Bousfield localization with kernel $\mathcal{C}$, i.e. it is a fiber and cofiber sequence in $\mathrm{Pr}^L_{\mathrm{st}}$.[12]

Example 2.9.7. A sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ of small idempotent complete stable ∞-categories is a Karoubi sequence iff

$$\mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D}) \to \mathrm{Ind}(\mathcal{E})$$

is a Verdier sequence, i.e. $\mathrm{Ind}(\mathcal{D}) \to \mathrm{Ind}(\mathcal{E})$ is a left Bousfield localization with kernel $\mathrm{Ind}(\mathcal{C})$. This is sometimes called the Thomason-Neeman localization theorem. For a modern reference, see [CDH$^+$23, Theorem A.3.11].

Note also that in this case all the functors are strongly left adjoint, since they are compactly assembled, so the right adjoint commutes with filtered colimits, but by exactness also with finite colimits.

Definition 2.9.8. A functor of small ∞-categories $p : \mathcal{D} \to \mathcal{E}$ is called a *homological epimorphism* if the induced functor

$$\mathrm{Ind}(p) : \mathrm{Ind}(\mathcal{D}) \to \mathrm{Ind}(\mathcal{E})$$

is a left Bousfield localization. A map of ring spectra $R \to S$ is called homological epimorphism if the base-change functor $\mathrm{Perf}(R) \to \mathrm{Perf}(S)$ is a homological epimorphism, equivalently if the base-change functor $\mathrm{Mod}(R) \to \mathrm{Mod}(S)$ is a left Bousfield localization.

Note that by the last example we have that Karoubi quotients are homological epis.

Lemma 2.9.9. *For a map $R \to S$ of ring spectra with fiber I the following are equivalent:*

1. *$R \to S$ is a homological epimorphism.*

[12] If $\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{E}$ is a cofiber sequence in $\mathrm{Pr}^L_{\mathrm{st}}$, then $\mathcal{E} \xrightarrow{p^R} \mathcal{D} \xrightarrow{i^R} \mathcal{C}$ is a fiber sequence in $\widehat{\mathrm{Cat}}_\infty^{\mathrm{ex}}$. Hence p^R is fully faithful, so p is a Bousfield localization. Conversely, if p is a Bousfield localization with kernel $\mathcal{C}$, then it is also a cofiber sequence in $\mathrm{Pr}^L_{\mathrm{st}}$; the sequence of right adjoints composes to 0, and $\ker i^R = \mathrm{im}(i)^\perp = \ker(p)^\perp = \mathrm{im}(p^R)$. The last equality follows because $\ker(p)^\perp$ consists by stability precisely of those object which are local with respect to p-equivalences, i.e. the p-local objects.

2. *The multiplication $S \otimes_R S \to S$ is an equivalence.*

3. *We have $I \otimes_R S \simeq 0$.*

4. *The map $I \otimes_R I \to I$ induced by the multiplication is an equivalence.*

Proof. The right adjoint to the base-change functor $\mathrm{Mod}(R) \to \mathrm{Mod}(S)$ is the restriction functor. This is fully faithful if and only if the counit of the adjunction is an equivalence, i.e. $S \otimes_R M \to M$ is an equivalence for any S-module M. As we may rewrite the left hand side as $(S \otimes_R S) \otimes_S M$, it suffices to check this for $M = S$, i.e. that $S \otimes_R S \to S$ is an equivalence. This proves that (1) implies (2).

To see (2) $\Leftrightarrow$ (3), we consider the fiber sequence

$$I \otimes_R S \to R \otimes_R S \to S \otimes_R S,$$

noting that $S \simeq R \otimes_R S \to S \otimes_R S$ splits the multiplication $S \otimes_R S \to S$. Finally, for (3) $\Leftrightarrow$ (4), we consider the fiber sequence

$$I \otimes_R I \to I \otimes_R R \to I \otimes_R S. \qquad \square$$

We say that I is an idempotent ideal in this case. Note that (by viewing ordinary rings as Eilenberg MacLane spectra), this contains the "almost mathematics" situation of a surjective map of rings $R \to S$ with kernel a flat ideal with $I^2 = I$, but also localizations of R, where the fiber will typically be a derived object with π_{-1}, see the Examples below.

Example 2.9.10. There are homological epimorphisms $p : \mathcal{D} \to \mathcal{E}$ between small stable ∞-categories, in fact between ring spectra, such that $\ker(p) \to \mathcal{D} \to \mathcal{E}$ is not a Karoubi sequence. In fact, $\ker(p)$ might be zero. For example, if R is a local ring with flat maximal ideal $\mathfrak{m}$ with $\mathfrak{m}^2 = \mathfrak{m}$, the kernel of $\mathrm{Perf}(R) \to \mathrm{Perf}(R/\mathfrak{m})$ consists of perfect complexes over R which have zero base-change to $R/\mathfrak{m}$, but all of these are zero by Nakayama (cf. Example 2.2.14). Nevertheless, this is a homological epimorphism.

Lemma 2.9.11. *Given a Verdier sequence*

$$\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$$

where either F or G admits a right adjoint, the other does too, and

$$\mathcal{C} \xrightarrow{G^R} \mathcal{B} \xrightarrow{F^R} \mathcal{A}$$

is again a Verdier sequence. The analogous statement holds for left adjoints.

Proof. If G admits a right adjoint G^R, it is fully faithful since G is a localization. We define a functor $F^R : \mathcal{B} \to \mathcal{A}$ by

$$F^R(b) = \mathrm{fib}(b \xrightarrow{\eta_G} G^R G b),$$

noting that this lies in the kernel of G. One checks directly that this is right adjoint to F, since $\mathrm{Map}(Fa, G^R G b) \simeq \mathrm{Map}(GFa, Gb) \simeq 0$.

Conversely, assume we have a right adjoint F^R to F. We define an endofunctor $\mathcal{B} \to \mathcal{B}$ by taking $b \mapsto \mathrm{cofib}(FF^Rb \xrightarrow{\varepsilon_F} b)$. This functor annihilates $\mathcal{A}$, so factors uniquely through a functor $G^R : \mathcal{C} \to \mathcal{B}$. We now have that $GG^R \simeq \mathrm{id}$, and a cofiber sequence

$$FF^R \to \mathrm{id} \to G^RG$$

from which we directly get that $G \to GG^RG$ and $G^R \to G^RGG^R$ are equivalences, proving the adjunction.

Finally, F^R is a right Bousfield localization since F is fully faithful, and G^R is the inclusion of its kernel, as the above cofiber sequence proves that an object is in the image of G^R if and only if it is in the kernel of F^R. □

Definition 2.9.12. Let $R \to S$ be a map of ring spectra with fiber I, and assume the map $S \otimes_R S \to S$ is an equivalence. Following the logic of Lemma 2.9.9, we may think of I as an idempotent ideal in R, in some higher-algebraic sense[13]. Then we define $\mathrm{Mod}(R, I) := \mathrm{Ker}(S \otimes_R - : \mathrm{Mod}(R) \to \mathrm{Mod}(S))$.

We shall see soon in that in fact every dualizable, stable ∞-category is of this form. Note that every map of pairs $R \to S$ to $R' \to S'$ with ideals I, J induces a map $\mathrm{Mod}(R, I) \to \mathrm{Mod}(S, J)$ which is strongly left adjoint. The latter fact can be seen using that compact morphisms in $\mathrm{Mod}(R)$ are precisely given by those morphisms that factor in $\mathrm{Mod}(R)$ through a compact object and the functor $\mathrm{Mod}(R) \to \mathrm{Mod}(S)$ preserves compact objects.

Example 2.9.13. Let R be a ring spectrum with an element $x \in \pi_*(R)$. Then there is a localization $R \to R[x^{-1}]$ which universally inverts x. Note that in the absence of commutativity (or at least an Ore condition) this is a bit hard to describe explicitly. Nevertheless we claim that $R \to R[x^{-1}]$ is a homological epimorphism.

To see this we consider the functor

$$\mathrm{Mod}(R) \to \mathrm{Mod}(R[x^{-1}])$$

induced from base change along $R \to R[x^{-1}]$. The target can be identified with those R-modules M on which x acts invertibly which follows from the universal property.[14] This functor is the Bousfield localization at the maps $R \xrightarrow{x} R$ given by right multiplication with x (which are left R-module maps).

It follows that the kernel is generated by R/x, hence compactly generated. In the commutative case, this subcategory can be explicitly described as those R-modules M with x-power torsion homotopy groups. There are a number of classical notations for this category like $\mathrm{Mod}(R)^{x\text{-nil}}$ or $\mathrm{Mod}(R \text{ on } R/x)$.

Example 2.9.14. Let $A \to B$ a surjective map of perfect $\mathbb{F}_p$-algebras. Then $A \to B$ is a homological epimorphism. This follows since $B \otimes_A B$ is itself a perfect animated ring

[13] One can make this structure precise, but we won't need it here, since we may equivalently characterize it in terms of S

[14] Note that in general $R \to R[x^{-1}]$ is not flat, even for ordinary rings [?]. Thus we have to take the derived base change.

(the Frobenius is the tensor product of the Frobenii). But the higher homotopy groups of a perfect animated $\mathbb{F}_p$-algebra are always trivial since the Frobenius acts by zero there. So we only need to check that the map out of the *underived* tensor product $B \otimes_A B \to B$ is an isomorphism which is clear from surjectivity of $A \to B$. This also works of A is only perfectoid (B still is a perfect $\mathbb{F}_p$-algebra) since again the tensor product $B \otimes_A B \simeq B \widehat{\otimes}_A B$ is perfect which follows since it is perfectoid and characteristic p.

Example 2.9.15. If R is an ordinary commutative ring with flat ideal I with $I^2 = I$, then $\mathcal{D}(R) \to \mathcal{D}(R/I)$ is a homological epimorphism. It follows that the kernel

$$\mathcal{D}(R, I) = \ker(\mathcal{D}(R) \to \mathcal{D}(R/I))$$

is compactly assembled. This is the derived version of almost mathematics.

Example 2.9.16 (Wodzicki)**.** Let A be a C^*-algebra with a closed ideal $I \subseteq A$. Then I is idempotent.

Lemma 2.9.17. *For every homological epimorphism $\mathcal{D} \to \mathcal{E}$ between small stable ∞-categories the kernel*

$$K = \ker(\mathrm{Ind}(\mathcal{D}) \to \mathrm{Ind}(\mathcal{E}))$$

is dualizable. Conversely, every dualizable stable ∞-category arises in this way.

Proof. In view of Proposition 2.7.14, it remains to see that every dualizable stable category K is such a kernel. Indeed, we can write K as the kernel of the projection

$$\mathrm{Ind}(K^{\omega_1}) \to \mathrm{Ind}(K^{\omega_1})/K$$

where $K \to \mathrm{Ind}(K^{\omega_1})$ is given by $\widehat{\jmath}$. Since $\widehat{\jmath}$ is strongly left adjoint, so is the projection by Lemma 2.9.11. Hence it preserves compact objects, and since its right adjoint is conservative, sends generators to generators. This proves that the quotient is again compactly generated, as desired. We will systematically investigate this quotient in Section 3.3. $\square$

Warning 2.9.18. *If we have a Verdier sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ with strongly left adjoint functors and $\mathcal{D}$ is dualizable, then so are $\mathcal{C}$ and $\mathcal{E}$ since both are retracts of $\mathcal{D}$ in Pr^L. But the converse fails: even if $\mathcal{C}$ and $\mathcal{E}$ are compactly generated, $\mathcal{D}$ need not be dualizable. The question of when such an extension $\mathcal{D}$ is dualizable has recently been answered by Efimov, see [Efi24, Proposition 3.3].*

We have the fully faithful functor

$$\mathrm{Ind} : \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathrm{Pr}^L_{\mathrm{dual}}$$

Every object in the $\mathrm{Pr}^L_{\mathrm{dual}}$ is the kernel of a Bousfield localization in $\mathrm{Pr}^L_{\mathrm{dual}}$. In other words: everyone in $\mathrm{Pr}^L_{\mathrm{dual}}$ admits a 'canonical' resolution by objects in the essential image.

Construction 2.9.19. *There is a functor* $(-)^\vee : \mathrm{Pr}^L_{\mathrm{dual}} \to \mathrm{Pr}^L_{\mathrm{dual}}$ *such that the diagram*

$$\begin{tikzcd} \mathrm{Cat}_\infty^{\mathrm{perf}} \arrow[r] \arrow[d, "\mathrm{op}"] & \mathrm{Pr}^L_{\mathrm{dual}} \arrow[d, "(-)^\vee"] \\ \mathrm{Cat}_\infty^{\mathrm{perf}} \arrow[r] & \mathrm{Pr}^L_{\mathrm{dual}} \end{tikzcd}$$

commutes. On objects this functor sends $\mathcal{C}$ *to the dual* $\mathcal{C}^\vee = \mathrm{Fun}^L(\mathcal{C}, \mathrm{Sp})$*, and an internally left adjoint functor* $L : C \to D$ *with right adjoint* R *is sent to the dual* $R^\vee = R^* : \mathrm{Fun}^L(\mathcal{D}, \mathrm{Sp}) \to \mathrm{Fun}^L(\mathcal{C}, \mathrm{Sp})$ *of its right adjoint. This indeed gives a strongly left adjoint functor, as* $R^* \dashv L^*$.

For objects, the square commutes since $\mathrm{Ind}(\mathcal{C})^\vee \simeq \mathrm{Ind}(\mathcal{C}^{\mathrm{op}})$ *by Lemma 2.9.5). For morphisms, let* $f : \mathcal{C} \to \mathcal{D}$ *be in* $\mathrm{Cat}_\infty^{\mathrm{perf}}$*. Then* $\mathrm{Ind}(f)$ *admits the right adjoint* $R = f^*$*, and under the identification* $\mathrm{Ind}(\mathcal{C})^\vee = \mathrm{Fun}^L(\mathrm{Ind}(\mathcal{C}), \mathrm{Sp}) \simeq \mathrm{Ind}(\mathcal{C}^{\mathrm{op}})$ *from Lemma 2.9.4 the functor* $R^\vee = R^*$ *gets identified with* $(f^{\mathrm{op}})^*$.

Finally, to see that this square commutes as a square of functors of ∞*-categories, we see that* $\mathrm{Pr}^L_{\mathrm{dual}} \to \mathrm{Pr}^L_{\mathrm{dual}}$ *is an equivalence, since it is idempotent (one can skip the passing to the left adjoint step to see this) and by the previous claim it restricts to an equivalence of the full subcategories* $\mathrm{Cat}_\infty^{\mathrm{perf}}$*. But the only non-trivial self equivalence of* $\mathrm{Cat}_\infty^{\mathrm{perf}}$ *is given by opping.*

2.10 H-unital ring spectra

Recall from Lemma 2.9.9 that a map of ring spectra $R \to S$ is a homological epimorphism precisely if the corresponding fiber I is idempotent. In this section we shall highlight some of the properties that this fiber has. First note that the fiber I is a non-unital ring spectrum. Formally, a non-unital algebra is an algebra over a non-unital version of the $\mathbb{E}_1$ operad. This has been investigated in-depth by Lurie in [Lur17a, Section 5.4.3, 5.4.4]. We will black-box the following important statements.

Theorem 2.10.1. *Let* $\mathcal{C}$ *be a symmetric monoidal category and* $1 \leq k \leq \infty$.

1. *The forgetful functor* $\theta_k : \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C}) \to \mathrm{Alg}^{\mathrm{nu}}_{\mathbb{E}_k}(\mathcal{C})$ *from* $\mathbb{E}_k$*-algebras to non-unital* $\mathbb{E}_k$*-algebras induces an equivalence onto the subcategory of quasi-unital* $\mathbb{E}_k$*-algebras and quasi-unital morphisms. A non-unital* $\mathbb{E}_k$*-algebra* A *is quasi-unital if there exists a map* $e : \mathbb{1} \to A$ *such that the composites*

$$A \xrightarrow{e \otimes A} A \otimes A \xrightarrow{\mu} A \qquad \text{and} \qquad A \xrightarrow{A \otimes e} A \otimes A \xrightarrow{\mu} A$$

are homotopic to the identitiy. If $k > 1$*, it suffices to check one of these. A morphism* $f : A \to B$ *is quasi-unital if* A *admits a quasi-unit* $e : \mathbb{1} \to A$ *such that* $f \circ e$ *is a quasi-unit for* B.

2. *The forgetful functor* θ_k *admits a left adjoint* $(-)^+ : \mathrm{Alg}^{\mathrm{nu}}_{\mathbb{E}_k}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$ *which we call unitalization. Given a non-unital* $\mathbb{E}_k$*-algebra* A*, the unit map* $e : \mathbb{1} \to A^+$ *of*

its unitalization, and the unit of the adjunction $\eta : A \to A^+$ together exhibit A^+ as coproduct in $\mathcal{C}$:

$$A \sqcup \mathbb{1} \xrightarrow[\simeq]{\eta \sqcup e} A^+.$$

3. *If $\mathcal{C}$ is stable and its tensor product is exact in both variables, we have an equivalence*

$$\begin{aligned} \mathrm{Alg}^{\mathrm{nu}}_{\mathbb{E}_k}(\mathcal{C}) &\simeq \mathrm{Alg}^{\mathrm{aug}}_{\mathbb{E}_k}(\mathcal{C}) := \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})_{/\mathbb{1}} \\ A &\mapsto (A^+ \simeq A \oplus \mathbb{1} \xrightarrow{0 \oplus \mathrm{id}} \mathbb{1}) \\ \mathrm{fib}(B \to \mathbb{1}) &\leftarrow\!\shortmid (B \to \mathbb{1}) \end{aligned}$$

between the categories of non-unital and augmented $\mathbb{E}_k$-algebras.

Proof. The first claim follows from [Lur17a, Theorem 5.4.4.5, Corollary 5.4.4.7]. The second claim is [Lur17a, Proposition 5.4.4.8], and the last claim is [Lur17a, Proposition 5.4.4.10]. □

We will mostly be interested in non-unital ring spectra, which is the case $\mathcal{C} = \mathrm{Sp}$ and $k = 1$ in the above theorem.

Definition 2.10.2. A non-unital ring spectrum A is called H-unital (homologically unital) if the multiplication map

$$A \otimes_{A^+} A \to A$$

is an equivalence or equivalently if $A^+ \to \mathbb{S}$ is a homological epimorphism, see Lemma 2.9.9.

Example 2.10.3. Any ring spectrum A, viewed it as a non-unital ring spectrum, is H-unital. Indeed, in this case $A^+ = A \times \mathbb{S}$ in $\mathrm{Alg}(\mathrm{Sp})$, hence $\mathrm{Mod}(A^+) \simeq \mathrm{Mod}(A) \times \mathrm{Mod}(\mathbb{S})$[15], and the basechange along $A^+ \to \mathbb{S}$ is just the projection, which is clearly a Bousfield localization.

Example 2.10.4. Assume that we have a filtered diagram $i \mapsto A_i$ of ring spectra that are unital, but the transition maps might be non-unital. Then we call the colimit $A = \mathrm{colim}\, A_i$ in non-unital ring spectra a locally unital ring spectrum. We claim that A is H-unital. To see this, observe that $A^+ \simeq \mathrm{colim}_i\, A_i^+$ as unital rings (as unitalization is a left adjoint) and hence

$$A \otimes_{A^+} A \simeq \mathrm{colim}\, A_i \otimes_{A_i^+} A_i \simeq \mathrm{colim}\, A_i \simeq A.$$[16]

For example, we can consider the ring

$$M_\infty(R) = \mathrm{colim}_{n\to\infty} M_n(R)$$

of $(\infty \times \infty)$-matrices over a given ring R in which almost all entries are zero.

[15] More generally, just as in ordinary land, if $R, S \in \mathrm{Alg}(\mathrm{Sp})$, then the idempotents $(1,0),(0,1) \in \pi_0(R \times S) = \pi_0(R) \times \pi_0(S)$ induce a splitting $\mathrm{Mod}(R \times S) \simeq \mathrm{Mod}(R) \times \mathrm{Mod}(S)$.

[16] The same argument works for sifted colimits. We thank Claudius Heyer for pointing this out.

Proposition 2.10.5 (Tamme)**.** *Consider a pullback diagram in* $\mathrm{Alg}(\mathrm{Sp})$

$$\begin{tikzcd} R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d] \\ R' \arrow[r] & S' \end{tikzcd}$$

and suppose that $R \to S$ *is a homological epimorphism. Then*

1. $S' \simeq S \otimes_R R'$.
2. $R' \to S'$ *is a homological epimorphism.*
3. *We have a pullback square of module categories*

$$\begin{tikzcd} \mathrm{Mod}(R) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Mod}(S) \arrow[d] \\ \mathrm{Mod}(R') \arrow[r] & \mathrm{Mod}(S') \end{tikzcd}$$

and the induced functor on fibers

$$\mathrm{Mod}(R, I) \to \mathrm{Mod}(R', I)$$

is an equivalence, where $I = \mathrm{fib}(R \to S) = \mathrm{fib}(R' \to S')$.

Proof. Since $R \to S$ is a homological epimorphism, we have $I \otimes_R S \simeq 0$ and thus

$$I \otimes_R S' \simeq (I \otimes_R S) \otimes_S S' \simeq 0.$$

From the fiber sequence

$$I \otimes_R I \to I \otimes_R R' \to I \otimes_R S'$$

we thus learn $I \otimes_R R' \simeq I$. Now the fiber sequence

$$I \otimes_R R' \to R \otimes_R R' \to S \otimes_R R'$$

tells us that $S \otimes_R R' \simeq S'$. For the second statement, we observe

$$S' \otimes_{R'} I \simeq (S \otimes_R R') \otimes_{R'} I \simeq S \otimes_R I \simeq 0.$$

For the last statement, the functor $\mathrm{Mod}(R) \to \mathrm{Mod}(R') \times_{\mathrm{Mod}(S')} \mathrm{Mod}(S)$ is fully faithful for any pullback of rings, and admits a right adjoint taking a triple of $M \in \mathrm{Mod}(R')$, $N \in \mathrm{Mod}(S)$ and equivalence $\varphi : S' \otimes_{R'} M \simeq S' \otimes_S N$ to the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d] & N \arrow[d] \\ M \arrow[r] & S' \otimes_{R'} M, \end{tikzcd}$$

viewed as R-module. To prove the claim it suffices to check that this right adjoint is conservative. So assume $P = 0$, and base-change the above pullback diagram along $R \to S$. We have $S \otimes_R N \simeq N$, $S \otimes_R M \simeq S' \otimes_{R'} M$, and $S \otimes_R S' \otimes_{R'} M \simeq S' \otimes_{R'} M \simeq S' \otimes_{R'} M$. So we see that

$$\begin{tikzcd} S \otimes_R P \arrow[r] \arrow[d] & N \arrow[d] \\ S' \otimes_{R'} M \arrow[r] & S' \otimes_{R'} M \end{tikzcd}$$

is a pullback, hence $N = 0$. Thus $S' \otimes_{R'} M = 0$ and the above pullback also proves $M = 0$. □

Corollary 2.10.6. *If $R \to S$ is a morphism of ring spectra whose fiber I is H-unital, then $R \to S$ is a homological epimorphism and* $\mathrm{Mod}(R, I) = \mathrm{Mod}(I^+, I)$. *In particular, the ∞-category* $\mathrm{Mod}(R, I)$ *is independent of R.*

Proof. Apply the above Proposition to the pullback square

$$\begin{tikzcd} I^+ \arrow[r] \arrow[d] & \mathbb{S} \arrow[d] \\ R \arrow[r] & S. \end{tikzcd}$$

□

There is a sort of converse to the last proposition.

Proposition 2.10.7 (Tamme)**.** *Consider a pullback diagram in* $\mathrm{Alg}(\mathrm{Sp})$

$$\begin{tikzcd} R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d] \\ R' \arrow[r] & S' \end{tikzcd}$$

and suppose that all rings and the horizontal fibers are connective. If $R' \to S'$ is a homological epimorphism, then $R \to S$ is too. In particular, if $R \to S$ is a homological epimorphism between connective rings with $\pi_0(R) \to \pi_0(S)$ surjective, the fiber I of $R \to S$ is H-unital.

Proof. By [Lur18, Proposition 16.2.2.1], the diagram

$$\begin{tikzcd} \mathrm{Mod}(R)_{\geq 0} \arrow[r, "F"] \arrow[d] & \mathrm{Mod}(S)_{\geq 0} \arrow[d] \\ \mathrm{Mod}(R')_{\geq 0} \arrow[r, "F'"] & \mathrm{Mod}(S')_{\geq 0} \end{tikzcd}$$

is a pullback diagram[17]. It follows then by abstract nonsense (as in the proof of Proposition 2.7.14) that the upper map is also a Bousfield localization, thus by unfolding what this means,

[17]This is wrong without the connectivity conditions on the rings and the fiber! Think of $k = k[x] \times_{k[x^{\pm 1}]} k[x^{-1}]$, but the pullback on module categories is given by quasicoherent sheaves on $\mathbb{P}^1$.

that $S \otimes_R S \simeq S$. Note also that a posteriori the square of module categories without passing to connective objects is a pullback by Proposition 2.10.5. $\square$

Example 2.10.8. The previous statement is wrong without connectivity assumptions. A counterexample is $k[x] \to k[x^{\pm 1}]$: if the fiber I, which has π_{-1}, was H-unital then we claim that the square

$$\begin{tikzcd} \mathrm{Mod}(k) \arrow[r] \arrow[d] & \mathrm{Mod}(k[x^{-1}]) \arrow[d] \\ \mathrm{Mod}(k[x]) \arrow[r] & \mathrm{Mod}(k[x^{\pm}]) \end{tikzcd}$$

would have to be a pullback (which it isn't as argued in Footnote 17). To see this note that if I was H-unital then the map $k \to k[x^{-1}]$, which has the same fiber would be a homological epimorphism and thus Proposition 2.10.5 the square would be a pullback.

One can also give a concrete argument for the case $k = \mathbb{Q}$, i.e. consider $R = \mathbb{Q}[x]$ and $S = \mathbb{Q}[x^{\pm 1}]$. The fiber I has homotopy groups given by $\mathbb{Q}[x^{\pm 1}]/\mathbb{Q}[x]$ in degree -1. I is H-unital if and only if $I \otimes_{I^+} \mathbb{S} = 0$, which we may equivalently compute in the rational world (i.e. $I \otimes_{I \oplus \mathbb{Q}} \mathbb{Q}$). This is computed as colimit of the semi-simplicial diagram

$$\cdots \qquad I \otimes I \otimes I \; \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \; I \otimes I \rightrightarrows I$$

i.e. it has a filtration with n-th associated graded given by $I^{\otimes(n+1)}[n]$, which is concentrated in degree -1. So the resulting spectral sequence degenerates and leads to a countably infinitely dimensional π_{-1}. In particular, $I \otimes_{I^+} \mathbb{S} \neq 0$ and I is not H-unital.

Definition 2.10.9. For a non-unital ring spectrum A we define the category of H-unital modules over A as

$$\mathrm{Mod}_H(A) := \mathrm{Mod}(A^+, A) = \ker(\mathrm{Mod}(A^+) \to \mathrm{Mod}(\mathbb{S}))$$

If A is H-unital then $A \in \mathrm{Mod}_H(A)$ and $\mathrm{Mod}_H(A)$ is a stable, dualizable ∞-category. A map $A \to B$ of H-unital ring spectra induces a functor $\mathrm{Mod}_H(A) \to \mathrm{Mod}_H(B)$ which is strongly left adjoint. The map $A \to B$ is called Morita equivalence if the induced functor is an equivalence.

An H-unital module is the same as a module M over A^+ such that

$$A \otimes_{A^+} M \simeq M$$

One can define a tensor product for non-unital modules using semi-simplicial realisations. Then this even reads as

$$A \otimes_A M \simeq M \; .$$

However, we warn the reader that one should be very careful with this semi-simplicial Bar resolution since tensoring over a non-unital ring can behave quite pathological and unexpected. Therefore we prefer to write $\otimes_{A_+}$ instead.

Example 2.10.10. Assume that A is a unital ring spectrum. Then

$$\mathrm{Mod}_H(A) = \mathrm{Mod}(A)$$

since $\mathrm{Mod}(A^+) = \mathrm{Mod}(A) \times \mathrm{Mod}(\mathbb{S})$ and the map $A^+ \to \mathbb{S}$ induces projection to the second summand, so the fiber is simply $\mathrm{Mod}(A)$.

But note that $\mathrm{Mod}_H(A)$ only depends on the underlying non-unital ring A. In particular, for a non-unital map $\varphi : A \to B$ between unital rings we get an induced strongly left adjoint functor

$$\mathrm{Mod}(A) \to \mathrm{Mod}(B)$$

which is somewhat surprising. One can explicitly describe this functor also without reference to H-unitality of course, namely this functor is induced by a $B - A$-bimodule by Morita theory, see Proposition 2.11.1 in the next section for a quick recap. This bimodule is given by the idempotent $e = \varphi(1)$ in B. That is,

$$Be \coloneqq \mathrm{colim}(B \xrightarrow{\cdot e} B \xrightarrow{\cdot e} B \xrightarrow{\cdot e} ...)$$

where this is a colimit in $B-A$-bimodules. Note that Be is a retract of B as a B-A-bimodule.

Example 2.10.11. Let R be a ring (spectrum) and consider the map

$$R \to M_n(R)$$

where $M_n(R)$ is the ring spectrum of $n \times n$-matrices, i.e. $\mathrm{end}_R(R^n)$. The map is given by sending R to matrices where all terms are zero except the upper left corner. This is a non-unital map. The corresponding idempotent e is given by the matrix with a 1 on the upper left element and zero's everywhere else. The corresponding submodule is given by R^n as an $M_n(R)$-R-bimodule. We therefore see that the induced functor

$$\mathrm{Mod}(R) \to \mathrm{Mod}(M_n(R))$$

is an equivalence, i.e. that the map φ is a Morita equivalence.

Lemma 2.10.12. *Let A be a locally unital ring, i.e. a filtered colimit $A = \mathrm{colim}_i A_i$ of unital rings along non-unital maps. Then the canonical functor*

$$\varinjlim^{\mathrm{Pr}^L} \mathrm{Mod}(A_i) \to \mathrm{Mod}_H(A)$$

is an equivalence.

Proof. This canonical functor sits in a commutative diagram

$$\begin{tikzcd}
\varinjlim^{\mathrm{Pr}^L} \mathrm{Mod}(A_i) \arrow[r] \arrow[d] & \mathrm{Mod}_H(A) \arrow[d] \\
\varinjlim^{\mathrm{Pr}^L} \mathrm{Mod}(A_i^+) \arrow[r] \arrow[d] & \mathrm{Mod}(A^+) \arrow[d] \\
\varinjlim^{\mathrm{Pr}^L} \mathrm{Sp} \arrow[r] & \mathrm{Sp}
\end{tikzcd}$$

We have that
$$A^+ = \operatorname{colim} A_i^+$$
and thus that $\mathrm{Mod}(A^+) = \underrightarrow{\mathrm{colim}}^{\mathrm{Pr}^L} \mathrm{Mod}(A_i^+)$. Indeed, generally if $R_\bullet : I \to \mathrm{Alg}(\mathrm{Sp})$ is a filtered diagram of ring spectra, then we claim that $\mathrm{colim}_i \mathrm{Mod}(R_i) \simeq \mathrm{Mod}(R)$. Since the diagram is entirely in $\mathrm{Pr}^L_{\mathrm{st},\omega} \simeq \mathrm{Cat}_\infty^{\mathrm{perf}}$ we have $\mathrm{colim}_i \mathrm{Mod}(R) \simeq \mathrm{Ind}(\mathrm{colim}_i \mathrm{Perf}(R_i))$, where the colimit is now compute din $\mathrm{Cat}_\infty^{\mathrm{perf}}$, hence in Cat_∞. We claim the comparison functor $F : \mathrm{colim}_i \mathrm{Perf}(R_i) \to \mathrm{Perf}(R)$ induced by the basechanges is an equivalence. It is clearly essentially surjective, since already each $\mathrm{Perf}(R_i) \to \mathrm{Perf}(R)$ is. To see it is fully faithful, pick $M, N \in \mathrm{Perf}(R_i)$ and note that we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{colim}_j \mathrm{Perf}(R_j)}(\mathrm{inc}_i M, \mathrm{inc}_i N) \arrow[d, equal] \arrow[dr, "F"] & \\
\mathrm{colim}_{j>i} \mathrm{Map}_{\mathrm{Perf}(R_j)}(R_j \otimes_{R_i} M, R_j \otimes_{R_i} N) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{Map}_{\mathrm{Perf}(R)}(R \otimes_{R_i} M, R \otimes_{R_i} N) \arrow[d, "\simeq"'] \\
\mathrm{colim}_{j>i} \mathrm{Map}_{\mathrm{Mod}(R_i)}(M, R_j \otimes_{R_i} N) \arrow[r, "\simeq"] & \mathrm{Map}_{\mathrm{Mod}(R_i)}(M, R \otimes_{R_i} N)
\end{tikzcd}$$

where $\mathrm{inc}_i : \mathrm{Perf}(R_i) \to \mathrm{colim}_j \mathrm{Perf}(R_j)$ is the inclusion functor. Thus $\mathrm{colim}_i \mathrm{Perf}(R_i) \simeq \mathrm{Perf}(R)$, as desired.

Coming back to the original proof, we see that the lower two horizontal functors in the first diagram are now equivalences. The claim now follows from the assertion that the vertical sequences are fibers sequences. Since this is clear for the right hand sequences we need to argue why the left hand sequence is a fiber sequence, i.e. why taking the kernel commutes with the filtered colimit. This however follows since the kernel of the left vertical map is the cokernel of the right adjoint, since the map is an strongly left adjoint Bousfield localization (seen by the fact that the lower maps are equivalences). Cokernels clearly commute with colimits. □

Example 2.10.13. We find that $\mathrm{Mod}(M_\infty(R)) \simeq \mathrm{Mod}(R)$, that is the map $R \to M_\infty(R)$ is a Morita equivalence.

Proposition 2.10.14. *If A is locally unital (filtered colimit of unital rings along non-unital maps) then $\mathrm{Mod}_H(A)$ is compactly generated. Conversely every compactly generated, stable ∞-category is equivalent to $\mathrm{Mod}_H(A)$ for some locally unital ring spectrum A.*

Proof. For $A = \operatorname{colim} A_i$ we get by Lemma 2.10.12 that
$$\mathrm{Mod}_H(A) = \underrightarrow{\mathrm{colim}}^{\mathrm{Pr}^L} \mathrm{Mod}(A_i)\ .$$
All the transition maps are strongly left adjoint, thus the colimit is also compactly generated, see Lemma 2.5.7. Now for a general compactly generated, stable ∞-category $\mathcal{C}$ we choose a set of compact generators $\{X_i\}_{i \in I}$, that is
$$\mathcal{C} = \langle X_i \mid i \in I \rangle$$

where the brackets denote the stable subcategory generated by the elements under colimits. Now let $\mathcal{P}_0(I)$ denote the poset of finite subsets of I, and for $F \in \mathcal{P}_0(I)$ let $\mathcal{C}_F = \langle X_i \mid i \in F \rangle$. We claim that we then also have

$$\operatorname{colim}^{\mathrm{Pr}^L}_{F \in \mathcal{P}_0(I)} \mathcal{C}_F \simeq \mathcal{C}.$$

Indeed, as in the proof of Lemma 2.10.12, we note that we can check this on compact objects, i.e. that $\operatorname{colim}_{F \in \mathcal{P}_0(I)} \mathcal{C}_F^\omega \simeq \mathcal{C}^\omega$, which is clear by construction. Note that each $\mathcal{C}_F$ admits a single compact generator $X_F \coloneqq \bigoplus_{i \in F} X_i$ and is therefore equivalent to $\mathrm{Mod}(A_F)$, where

$$A_F = \mathrm{end}_{\mathcal{C}}(X_F)$$

is the endomorphism spectrum of X_F. The maps $\mathcal{C}_F \to \mathcal{C}_{F'}$ for $F \subseteq F'$ are induced from the non-unital maps

$$A_F \to A_{F'}$$

which send an endomorphism f of X_F to the endomorphism $f \oplus 0$ of $X_{F'} = X_F \oplus X_{F' \setminus F}$. In particular we find that

$$\mathcal{C} = \operatorname{colim}^{\mathrm{Pr}^L} \mathrm{Mod}(A_F)$$

and so that

$$\mathcal{C} = \mathrm{Mod}_H(A)$$

where $A = \operatorname{colim} A_F$. $\square$

Remark 2.10.15. The non-unital ring spectrum A from the last proof can also be described somewhat explicitly: the underlying spectrum is given by

$$A = \bigoplus_{i,j} \mathrm{map}_{\mathcal{C}}(X_i, X_j)$$

The multiplication map $A \otimes A \to A$ is induced by the maps

$$\mathrm{map}_{\mathcal{C}}(X_i, X_j) \otimes \mathrm{map}_{\mathcal{C}}(X_k, X_l) \to A$$

which for $j = k$ are given by composition (considered as an element of $\mathrm{map}_{\mathcal{C}}(X_i, X_l)$) and for $j \neq k$ by the zero map.

Theorem 2.10.16. *Every dualizable, stable ∞-category $\mathcal{C}$ is equivalent to $\mathrm{Mod}_H(A)$ for some H-unital ring spectrum A.*

Proof. We choose a set $(X_i)_{i \in I}$ of ω_1-compact objects of $\mathcal{C}$ with the property that they represent all equivalence classes of objects of $\mathcal{C}^{\omega_1}$, and consider

$$A \coloneqq \bigoplus_{i,j} \mathrm{Map}^{\mathrm{ca}}(X_i, X_j)$$

(recall that $\mathrm{Map}^{\mathrm{ca}}_{\mathcal{C}}(X,Y) = \mathrm{map}_{\mathrm{Ind}(\mathcal{C})}(jX, \widehat{\jmath}Y)$). We claim this is a non-unital ring. Indeed, consider the localization sequence

$$\mathcal{C} \xrightarrow{\widehat{\jmath}} \mathrm{Ind}(\mathcal{C}^{\omega_1}) \to \mathrm{Ind}(\mathcal{C}^{\omega_1})/\mathcal{C}.$$

By construction, the objects jX_i are compact generators of $\mathrm{Ind}(\mathcal{C}^{\omega_1})$. By Lemma 2.9.11 the projection is strongly left adjoint with fully faithful right adjoint, and hence sends compact generators to compact generators. Using Proposition 2.10.14, this yields equivalences $\mathrm{Ind}(\mathcal{C}^{\omega_1}) \simeq \mathrm{Mod}_H(A')$ and $\mathrm{Ind}(\mathcal{C}^{\omega_1})/\mathcal{C} \simeq \mathrm{Mod}_H(A'')$, where

$$A' = \bigoplus_{i,j\in I} \mathrm{map}_{\mathrm{Ind}(\mathcal{C}^{\omega_1})}(jX_i, jX_j)$$
$$A'' = \bigoplus_{i,j\in I} \mathrm{map}_{\mathrm{Ind}(\mathcal{C}^{\omega_1})/\mathcal{C}}(jX_i, jX_j),$$

as well as a non-unital ring homomorphism $A' \to A''$ (since the A' and A'' arose as colimit of endomorphisms of $\bigoplus_{i\in I} X_i$).

Using the adjoints in the above Verdier sequence, one sees

$$\mathrm{map}_{\mathrm{Ind}(\mathcal{C}^{\omega_1})/\mathcal{C}}(jX_i, jX_j) \simeq \mathrm{map}_{\mathrm{Ind}(\mathcal{C}^{\omega_1})}(jX_i, jX_j/\widehat{\jmath}X_j),$$

and so the fiber of $A' \to A''$ is the A defined above. In particular, A inherits a non-unital ring structure.

We now claim that A is H-unital. Given this, it follows from the following left pullback square of rings and Proposition 2.10.5 that also $A'^{+} \to A''^{+}$ is an H-epi and we get the right pullback square of module categories:

$$\begin{tikzcd} A^+ \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{S} \arrow[d] \\ A'^+ \arrow[r] & A''^+ \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Mod}(A^+) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Mod}(\mathbb{S}) \arrow[d] \\ \mathrm{Mod}(A'^+) \arrow[r] & \mathrm{Mod}(A''^+) \end{tikzcd}$$

Taking horizontal fibers shows that $\mathrm{Mod}_H(A) = \mathrm{fib}(\mathrm{Mod}(A'^+) \to \mathrm{Mod}(A''^+))$. Finally, we obtain $\mathcal{C} \simeq \mathrm{Mod}_H(A)$ by commuting fibers; consider the following diagram where all horizontal and vertical sequences are fiber sequences:

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d, equal] & \mathrm{Mod}_H(A') \arrow[r] \arrow[d] & \mathrm{Mod}_H(A'') \arrow[d] \\ \mathrm{Mod}_H(A) \arrow[r] \arrow[d] & \mathrm{Mod}(A'^+) \arrow[r] \arrow[d] & \mathrm{Mod}(A''^+) \arrow[d] \\ 0 \arrow[r] & \mathrm{Mod}(\mathbb{S}) \arrow[r, equal] & \mathrm{Mod}(\mathbb{S}) \end{tikzcd}$$

To see that A is H-unital, we need to check that the augmented semi-simplicial object

$$A \longleftarrow A \otimes A \leftleftarrows A \otimes A \otimes A \qquad \ldots$$

is a colimit cone (as the left Kan extension of this diagram to Δ^{op} is exactly the bar construction computing $A \otimes_{A^+} A$, and the colimit of the left Kan extension is the colimit of the original diagram by transitivity of Kan extensions). For any object Y, we have a left A-module $M(Y) = \bigoplus_i \mathrm{Map}^{\mathrm{ca}}(X_i, Y)$. We will show indeed that

$$M(Y) \longleftarrow A \otimes M(Y) \Longleftarrow A \otimes A \otimes M(Y) \qquad \ldots$$

is a colimit diagram. Assume first that Y is ω_1-compact, and write $Y = \mathrm{colim}_n Y_n$ along compact maps. We may assume Y_n to be ω_1-compact, and hence $Y_n = X_{i_n}$ for some sequence i_n. We choose witnesses $jX_{i_n} \to \widehat{\jmath}X_{i_{n+1}}$ representing these compact maps. We have a map

$$M(X_{i_n}) \to A \otimes M(X_{i_{n+1}})$$

induced by $\mathbb{S} \to \mathrm{Map}^{\mathrm{ca}}(X_{i_n}, X_{i_{n+1}})$. These satisfy the identities of an "extra degeneracy" up to postcomposing with $M(X_{i_n}) \to M(X_{i_{n+1}})$, i.e. we get a dashed lift in

$$\begin{tikzcd}
{|A^{\otimes \bullet} \otimes M(X_{i_n})|} \arrow[r] \arrow[d] & {|A^{\otimes \bullet} \otimes M(X_{i_{n+1}})|} \arrow[d] \\
M(X_{i_n}) \arrow[r] \arrow[ur, dashed] & M(X_{i_{n+1}})
\end{tikzcd}$$

In the colimit, that means we have an equivalence

$$M(Y) \simeq |A^{\otimes \bullet} \otimes M(Y)|.$$

Since both sides commute with ω_1-filtered colimits and Y was an arbitrary ω_1-compact object, this more generally follows for arbitrary Y, in particular for $Y = \bigoplus X_i$, where we have $M(Y) \simeq A$. $\square$

In particular we see that every dualizable category is an almost category $\mathrm{Mod}(R, I)$ with I H-unital (so that the category doesn't even depend on R).

2.11 H-unital Morita Theory

In this section we will analyse functors between categories of the form $\mathrm{Mod}_H(A)$ in terms of non-unital ring spectra. Let us first recall and extend usual Morita theory for ring spectra. For an ordinary land analogue see [GV98].

Proposition 2.11.1. *1. For unital rings A, B, we have*

$$\mathrm{Fun}^L(\mathrm{Mod}(A), \mathrm{Mod}(B)) \simeq \mathrm{BiMod}(B, A).$$

2. *For unital rings A, B, we have that $\mathrm{Map}_{\mathrm{Ring}}(A,B)$ agrees with the space of pairs consisting of a left adjoint functor $\mathrm{Mod}(A) \to \mathrm{Mod}(B)$ together with an equivalence $F(A) \simeq B$. Equivalently a commutative diagram*

$$\begin{tikzcd} \mathrm{An} \arrow[d] \arrow[dr] & \\ \mathrm{Mod}(A) \arrow[r] & \mathrm{Mod}(B) \end{tikzcd}$$

in Pr^L.

3. *For H-unital rings A, B, we have that*

$$\mathrm{Fun}^L_H(\mathrm{Mod}(A), \mathrm{Mod}(B)) \simeq \mathrm{BiMod}_H(B, A),$$

where the right hand side is given by the full subcategory of B^+-A^+-bimodules which lie in $\mathrm{Mod}_H(A)$ and $\mathrm{Mod}_H(B)$ when viewed as left or right modules.

4. *For H-unital rings A, B where A admits a unit, $\mathrm{Map}_{\mathrm{Ring}^{\mathrm{nu}}}(A,B)$ agrees with the space of left adjoint functors $F : \mathrm{Mod}(A) \to \mathrm{Mod}(B)$ together with maps $F(A) \xrightarrow{i} B \xrightarrow{r} F(A)$ exhibiting $F(A)$ as retract of B.*

Proof. For the first statement, observe that $F(A)$ is a left B-module, but also has $\mathrm{end}(A)$ acting by functoriality. As $\mathrm{end}(A) = A$ acting from the right, we have a B-A-bimodule structure on $F(A)$. As every map $A \to X$ gives a map $F(A) \to F(X)$, we have a natural transformation

$$F(A) \otimes_A X \to F(X),$$

which is an equivalence if F preserves colimits.

For the second statement, observe that for a ring homomorphism $A \to B$, the associated functor $\mathrm{Mod}(A) \to \mathrm{Mod}(B)$ is given by

$$B \otimes_A -,$$

i.e. corresponds to the B-A bimodule B. Conversely, given a functor F with $F(A) \simeq B$ as left module, the functor provides a ring homomorphism

$$A \to \mathrm{end}_{\mathrm{Mod}(A)}(A) \to \mathrm{end}_{\mathrm{Mod}(B)}(B) \simeq B$$

which describes the right A-module structure on $F(A)$. So ring homomorphisms correspond precisely to functors with an isomorphism $F(A) \simeq B$.

For the third statement, observe that the Verdier sequence

$$\mathrm{Mod}_H(A) \rightleftarrows \mathrm{Mod}(A^+) \rightleftarrows \mathrm{Mod}(\mathbb{S})$$

exhibits $\mathrm{Mod}_H(A)$ as Bousfield localization of $\mathrm{Mod}(A^+)$ with kernel the modules restricted along $A^+ \to \mathbb{S}$. This means that exact functors $\mathrm{Mod}_H(A) \to \mathcal{C}$ correspond to exact functors $\mathrm{Mod}(A^+) \to \mathcal{C}$ which annihilate modules restricted from $\mathbb{S}$. So left adjoint functors

$$\mathrm{Mod}_H(A) \to \mathrm{Mod}_H(B)$$

correspond to left adjoint functors

$$\mathrm{Mod}(A) \to \mathrm{Mod}(B)$$

which take values in $\mathrm{Mod}(B^+)$ and annihilate modules restricted from $\mathbb{S}$. Translated to bimodules, the first condition just means $\mathbb{S} \otimes_{B^+} M \simeq 0$. For the second it is necessary that $M \otimes_{A^+} \mathbb{S} = 0$, but also sufficient, since if N is restricted from $\mathbb{S}$, $M \otimes_{A^+} N \simeq M \otimes_{A^+} \mathbb{S} \otimes_{\mathbb{S}} N$.

For the final statement observe that for a non-unital ring homomorphism $A \to B$, the restriction of the base-change $B^+ \otimes_{A^+} -$ to $\mathrm{Mod}_H(A)$ and $\mathrm{Mod}_H(B)$ is computed as the composite

$$\mathrm{Mod}_H(A) \to \mathrm{Mod}(A^+) \to \mathrm{Mod}(B^+) \to \mathrm{Mod}_H(B),$$

i.e. is given by the B^+-A^+-bimodule $B \otimes_{A^+} A$. If A is unital, we have an A^+-module homomorphism $A^+ \to A$ exhibiting A as retract of A^+, so $B \otimes_{A^+} A$ as retract of B. Conversely, if $F(A)$ is a retract of B, in particular of B^+, we get a non-unital ring map

$$A \to \mathrm{end}_{B^+}(F(A)) \to \mathrm{end}_{B^+}(B^+) = B^+$$

which lifts to the fiber of $B^+ \to \mathbb{S}$. (The first map is unital, the second one is non-unital and arises from the retraction). $\square$

Now we would also like to understand functors between dualizable stable ∞-categories using zig-zag's of maps of non-unital rings. As a warm up, we first prove a version for unital rings.

Proposition 2.11.2. *For A and B unital rings every strongly left adjoint functor F : $\mathrm{Mod}(A) \to \mathrm{Mod}(B)$ is induced by a zig-zag $A \to C \overset{\simeq}{\leftarrow} B$ of non-unital maps, where C is also unital and $B \to C$ is a Morita equivalence.*

Proof. For a functor F we define

$$C \coloneqq \mathrm{end}_B(F(A) \oplus B)$$

Since $F(A) \oplus B$ is a compact generator of $\mathrm{Mod}(B)$ we have that

$$\mathrm{Mod}(B) \simeq \mathrm{Mod}(C)$$

induced by the map $B \to C$ induced by the split inclusion $B \to F(A) \oplus B$. Similarly we have a map

$$A = \mathrm{end}_A(A) \to \mathrm{end}_B(F(A)) \to \mathrm{end}_B(F(A) \oplus B)$$

of non-unital rings which induces the functor F. $\square$

We would like to show a converse to the latter statement. To this end we note that there is a natural notion of 2-morphisms between non-unital morphisms $f, g : A \to B$ where A and B are unital. For simplicity lets first assume that A and B are discrete. Then such a 2-morphism is given by given by an element $b \in f(1)Bg(1)$ such that

$$f \cdot b \simeq b \cdot g \ .$$

We claim that such a 2-morphism is the same as a natural transformation of the functors

$$\mathrm{Mod}(A) \to \mathrm{Mod}(B)$$

induced by f and g. This follows since by Proposition 2.11.1 and Example 2.10.10 such a natural transformation is given by a B-A-bimodule map $Bf(1) \to Bg(1)$. Every left B-module map $Bf(1) \to Bg(1)$ is given by right multiplication with an element $b \in f(1)Bg(1)$ and this is a right A-module map precisely if for every $a \in A$ we have that $f(a)b = bg(a)$. Now if A and B are not discrete anymore we find similar that the space of natural transformations between the induced functors can be expressed as elements [18] $b \in f(1)Bg(1)$ together with an equivalence $f \cdot b \simeq b \cdot g$.

Using this notion of 2-morphisms we can define an $(\infty, 2)$-category of unital algebras, non-unital maps and 2-morphisms. We denote the ∞-categorical core, i.e. the largest $(\infty, 1)$-category contained in this $(\infty, 2)$-category by $\mathrm{Alg}_2^{\mathrm{u}}$. Concretely the 2-morphisms in $\mathrm{Alg}_2^{\mathrm{u}}$ are given by elements $b \in f(1)Bg(1)$ that are units in the sense that there exists a $b' \in g(1)Bf(1)$ such that $bb' \simeq f(1)$ and $bb \simeq g(1)$. Now we have a functor of ∞-categories

$$\mathrm{Alg}_2^{\mathrm{u}} \to \mathrm{Pr}^L_\omega \qquad A \mapsto \mathrm{Mod}(A)$$

which sends the class of Morita equivalences W to equivalences of ∞-categories. Thus we get an induced functor

$$\mathrm{Alg}_2^{\mathrm{u}}[W^{-1}] \to \mathrm{Pr}^L_{\mathrm{st},\omega} \ .$$

This functor lands in the full subcategory of $\mathrm{Pr}^L_{\mathrm{st},\omega}$ consisting of those compactly generated ∞-categories that admit a compact generator, aka monogenic ones.

Proposition 2.11.3. *The functor $\mathrm{Alg}_2^{\mathrm{u}}[W^{-1}] \to \mathrm{Pr}^L_{\mathrm{st},\omega}$ is fully faithful with essential image the monogenic stable ∞-categories.*

Note that this is a statement about ∞-categories. Both categories extend in fact naturally to $(\infty, 2)$-categories and one could also make a $(\infty, 2)$-categorical statement here. But we will not attempt to formulate or proof such a statement here to avoid the use of DK-localizations for $(\infty, 2)$-categories.

Our proof of Proposition 2.11.3 relies on the following statement and will be given below.

Lemma 2.11.4. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and W the class of morphisms send to equivalences by F. Assume that for every pair of objects $X, Y \in \mathcal{C}$ the induced functor*

$$\mathrm{colim}_{Y \xrightarrow{\simeq} \hat{Y}} \mathrm{Map}_{\mathcal{C}}(X, \hat{Y}) \to \mathrm{Map}_{\mathcal{D}}(F(X), F(Y))$$

is an equivalence. Then the induced functor $\mathcal{C}[W^{-1}] \to \mathcal{D}$ is fully faithful.

[18] By an element in a spectrum A we mean a map $\mathbb{S} \to A$ or equivalently a point in $\Omega^\infty A$.

Proof. Consider the left Kan extension functor $L : \mathrm{Fun}(\mathcal{C}, \mathrm{An}) \to \mathrm{Fun}(\mathcal{C}[W^{-1}], \mathrm{An})$ which is left adjoint to the fully faithful restriction functor $\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathrm{An}) \to \mathrm{Fun}(\mathcal{C}, \mathrm{An})$. By general nonsense we have that for every fixed object X the corepresentable functor $\underline{X} \in \mathrm{Fun}(\mathcal{C}, \mathrm{An})$ is send by L to the corepresentable $\underline{X} \in \mathrm{Fun}(\mathcal{C}[W^{-1}], \mathrm{An})$. For an arbitrary object $F \in \mathrm{Fun}(\mathcal{C}, \mathrm{An})$ we construct $\widehat{F} \in \mathrm{Fun}(\mathcal{C}, \mathrm{An})$

$$\widehat{F}(Y) = \mathrm{colim}_{Y \xrightarrow{\simeq} \hat{Y}} F(\hat{Y})$$

and maps from $\widehat{F}$ into any functor G which lies in the image of the restriction

$$\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathrm{An}) \to \mathrm{Fun}(\mathcal{C}, \mathrm{An})$$

(i.e. G sends W to equivalences) are equivalent to maps from F to G. Thus if $\widehat{F}$ has the property that it sends W to equivalences, then $\widehat{F} = L(F)$.

We now apply this construction to $\underline{X} \in \mathrm{Fun}(\mathcal{C}, \mathrm{An})$ and the assumption of the statement implies that

$$\widehat{\underline{X}} \simeq \mathrm{Map}_{\mathcal{D}}(F(X), F(-))$$

indeed sends W to equivalences. Therefore we have that $\widehat{\underline{X}}$ is given by the left Kan extension, that the mapping space in $\mathcal{C}[W^{-1}]$ which finishes the proof. □

Remark 2.11.5. A more explicit description of the colimit $\mathrm{colim}_{Y \xrightarrow{\simeq} \hat{Y}} \mathrm{Map}_{\mathcal{C}}(X, \hat{Y})$ from the previous statement is given as the geometric realization of the ∞-category of spans

$$X \to \hat{Y} \xleftarrow{\simeq} Y \ .$$

This follows using that the ∞-category of such spans is the unstraightening of the functor

$$\left(Y \xrightarrow{\simeq} \hat{Y}\right) \mapsto \mathrm{Map}_{\mathcal{C}}(X, \hat{Y}) \ .$$

Proof of Proposition 2.11.3. We want to apply Lemma 2.11.4. To this end we have to show that the functor

$$\mathrm{ZigZag}(A, B) \to \mathrm{Fun}^{\mathrm{sL}}(\mathrm{Mod}(A), \mathrm{Mod}(B))^{\simeq} \tag{2.3}$$

is an equivalence after realizing the source, which is the zigzag-category whose objects are unital algebras. Morphisms are zigzags of the form $A \to C \xleftarrow{\simeq} B\}$ with $\simeq$ indicating that the morphism is a Morita equivalence. A 2-morphism in this category is a diagram

$$\begin{tikzcd}
 & C \arrow[dd] & \\
A \arrow[ur] \arrow[dr] & & B \arrow[ul, "\simeq"'] \arrow[dl, "\simeq"] \\
 & C' &
\end{tikzcd}$$

where the triangles can are filled by 2-morphisms in $\mathrm{Alg}_2^{\mathrm{u}}$.

To see that (2.5) is an equivalence we use the construction from Proposition 2.11.2 to produce a functor in the opposite direction:

$$\mathrm{Fun}^{\mathrm{sL}}(\mathrm{Mod}(A), \mathrm{Mod}(B))^{\simeq} \to \mathrm{ZigZag}(A, B) \quad F \mapsto \Big(A \to C(F) \overset{\simeq}{\leftarrow} B\Big)$$

where $C(F) = \mathrm{end}_B(F(A) \oplus B)$, which is clearly functorial in natural equivalences. [19]

Now by construction (see the proof of Proposition 2.11.2) we see that the composition

$$\mathrm{Fun}^{\mathrm{sL}}(\mathrm{Mod}(A), \mathrm{Mod}(B))^{\simeq} \to \mathrm{ZigZag}(A, B) \to \mathrm{Fun}^{\mathrm{sL}}(\mathrm{Mod}(A), \mathrm{Mod}(B))^{\simeq}$$

is equivalent to the identity. It therefore remains to also show that the composition

$$|\mathrm{ZigZag}(A, B)| \to \mathrm{Fun}^{\mathrm{sL}}(\mathrm{Mod}(A), \mathrm{Mod}(B))^{\simeq} \to |\mathrm{ZigZag}(A, B)|$$

is homotopic to the identity. To this end, it suffices to construct a zigzag of natural morphisms in $\mathrm{ZigZag}(A, B)$ from any span $A \overset{\varphi}{\to} C \overset{\psi}{\leftarrow} B$ to the induced span $A \to C(F) \overset{\simeq}{\leftarrow} B$ with F the induced functor from the span. Let us first work out what $C(F)$ is: by definition it is given by $\mathrm{end}_B(F(A) \oplus B)$ where F is the functor

$$\mathrm{Mod}(A) \xrightarrow{\varphi^*} \mathrm{Mod}(C) \xleftarrow{\psi^*} \mathrm{Mod}(B)$$

induced from the span. Since the right hand functor is an equivalence we have

$$C(F) = \mathrm{end}_B(F(A) \oplus B) \simeq \mathrm{end}_C(\varphi^*(A) \oplus \psi^*(B))$$

with the maps $A \to C(F)$ and $B \to C(F)$ given by the maps $A \to \mathrm{end}_C(\varphi^*(A)) \to C(F)$ and $B \to \mathrm{end}_C(\psi^*(B)) \to C(F)$. Recall that $\varphi^*(A)$ is a retract of C and $\psi^*(B)$ is a retract of C as well (see Example 2.10.10). Thus we have maps of non-unital rings $\mathrm{end}_C(\psi^*(B)) \to C$ and $\mathrm{end}_C(\varphi^*(A)) \to C$ as well as $C(F) = \mathrm{end}_C(\varphi^*(A) \oplus \psi^*(B)) \to \mathrm{end}_C(C \oplus C)$ we now consider the diagram in $\mathrm{Alg}_2^{\mathrm{u}}$ given as

$$\begin{tikzcd}
& \mathrm{end}_C(\varphi^*(A) \oplus \psi^*(B)) \arrow[d] & \\
A \arrow[ur, "i_0"] \arrow[r, "i_0"] \arrow[dr, "\varphi"'] & \mathrm{end}_C(C \oplus C) & B \arrow[ul, "i_1"'] \arrow[l, "i_1"] \arrow[dl, "\psi"] \\
& C \arrow[u, "i_1"] &
\end{tikzcd}$$

where i_0 always denotes an 'inclusion' into the first summand (i.e. the map on endomorphisms obtained by the inclusion) and i_1 the inclusion into the second summand. Note that this a diagram of unital rings and non-unital maps except for the lower left triangle. This

[19] Note that this functor in fact doesn't use the 2-morphisms in Alg^u, so it lands in in the zigzag category associated with the smaller ∞-category of unital algebras and non-unital maps.

does not commute, but we claim that there is a 2-morphism in $\mathrm{Alg}_2^{\mathrm{u}}$ filing it. Concretely this 2-morphism is given by the element

$$\begin{pmatrix} 0 & \mathrm{id}_C \\ 0 & 0 \end{pmatrix} \in \mathrm{end}_C(C \oplus C)$$

which conjugates one map into the other (and in fact lies in the correct summand of $\mathrm{end}_C(C \oplus C)$). Alternatively one can also consider the induced diagram on module categories and see that it commutes (almost all the non-unital rings are generators of $\mathrm{Mod}(C)$). The left lower triangle commutes since both maps $C \to \mathrm{end}_C(C \oplus C)$ induced the same functor, as the corresponding bimodules are both given by C^2 with left $\mathrm{end}_C(C \oplus C)$-action and right C-action. This finishes the proof. □

Remark 2.11.6. Given the previous statement, one might ask to which extend the category of spans $A \to C \xleftarrow{\simeq} B$ without the conjugation 2-morphisms already models the homotopy type of the space

$$\mathrm{Fun}^{\mathrm{sL}}(\mathrm{Mod}_H(A), \mathrm{Mod}_H(B))^{\simeq},$$

that is whether the map from the realization of the category of these smaller spans to the space of strongly left adjoint functors is an equivalence. The proof of the previous proposition shows that this map admits a section. We however believe that it is not an equivalence, since we believe that for $A = B$ the two spans

$$A \xrightarrow{\mathrm{id}} A \xleftarrow{\mathrm{id}} A \qquad A \xrightarrow{i_1} M_2(A) \xleftarrow{i_2} A$$

are not equivalent in the realization of the smaller span category. Here i_1 and i_2 are the inclusions into the upper left and lower right corner of matrices. But these spans both induce the identity functor $\mathrm{Mod}_H(A) \to \mathrm{Mod}_H(A)$.

More generally let us denote category of unital algebras and non-unital maps by $\mathrm{Alg}_1^{\mathrm{u}}$. We believe that the functor $\mathrm{Alg}_1^{\mathrm{u}} \to \mathrm{Pr}^L_{\mathrm{st,mono}}$ which takes the ∞-category of modules is not a Dwyer-Kan localization since we see no reason that the two maps

$$A \to M_2(A)$$

in $\mathrm{Alg}_1^{\mathrm{u}}$ become equivalent in the DK localization at the Morita equivalences (we also can't prove the opposite though). However, our previous proof shows that the functor

$$\mathrm{Alg}_1^{\mathrm{u}}[W^{-1}] \to \mathrm{Alg}_2^{\mathrm{u}}[W^{-1}] \simeq \mathrm{Pr}^L_{\mathrm{st,mono}}$$

admits a section.

Now we would like to turn to the case of locally unital rings.

Construction 2.11.7. *For a locally unital ring spectrum $A = \operatorname{colim} A_i$ we consider the system of idempotents $e_i = f(1)$ in A. These define retract diagrams*

$$Ae_i \hookrightarrow A \twoheadrightarrow Ae_i$$

as left A-modules. Similarly we have retract diagrams

$$Ae_i \hookrightarrow Ae_j \twoheadrightarrow A_i \tag{2.4}$$

for $i \to j$ in I. The system $i \mapsto Ae_i$ in fact forms a functor from I to the ∞-category of retracts (where the morphisms are retract diagrams). Then we have that

$$e_i A e_i = \mathrm{end}_{A^+}(Ae_i)$$

are unital rings and the map $e_iAe_i \to e_jAe_j$ for $i \to j$ in I extends to non-unital ring map using the retract diagram (2.4). We then have that $A = \operatorname{colim} e_iAe_i$, that is we may replace the diagram A_i by the new diagram e_iAe_i to obtain A as a colimit.

Now from an external perspective what we have done is the following: in the category $\mathrm{Mod}_H(A)$ the object A is a generator, but not compact in general. However, we can write the A-module A as a filtered colimit $A = \operatorname{colim} Ae_i$ where all the maps are part of retract diagrams and then we get that the non-unital ring A is given by

$$A = \varinjlim_i \mathrm{end}_{A^+}(Ae_i)\ .$$

But note that $\mathrm{end}_{A^+}(A) \neq A$. So this gives a way of recovering A from the category $\mathrm{Mod}_H(A)$ together with the filtered diagram $i \mapsto Ae_i$.

Finally note that if $I = \mathbb{N}$, so that the diagram is sequential we can in fact write A as an A-module as the direct sum

$$A = \bigoplus_{i \in \mathbb{N}} A(e_i - e_{i-1}) \qquad e_{-1} \coloneqq 0$$

and so we see that A is then the countable sum of generators and we are exactly in the situation of the proof of Proposition 2.10.14.

Proposition 2.11.8. *For A and B locally unital every strongly left adjoint functor $F : \mathrm{Mod}_H(A) \to \mathrm{Mod}_H(B)$ is induced by a zig-zag $A \to C \xleftarrow{\simeq} B$ of locally unital rings where $B \to C$ is a Morita equivalence.*

Proof. We consider the filtered diagrams

$$A = \operatorname{colim}_{i \in I} Ae_i \qquad B = \operatorname{colim}_{j \in J} Be_j$$

in $\mathrm{Mod}_H(A)$ and $\mathrm{Mod}_H(B)$ as in Construction 2.11.7. We can assume without loss of generality that $I = J$, e.g. by passing to the product. Now we define

$$C \coloneqq \operatorname{colim}_I \mathrm{end}_{B^+}(F(Ae_i) \oplus Be_i)$$

where $i \mapsto F(Ae_i) \oplus Be_i$ also forms a retract style diagram in $\mathrm{Mod}_H(B)$. Again the elements $F(Ae_i) \oplus Be_i$ are compact generators since F is strongly left adjoint and the Be_i already form a generating set. Thus we have that

$$\mathrm{Mod}_H(B) \simeq \mathrm{Mod}_H(C)$$

induced by the map $B \to C$ induced by the split inclusion $Be_i \to F(Ae_i) \oplus Be_i$. Similarly we have a map

$$\mathrm{end}_{A^+}(Ae_i) \to \mathrm{end}_{B^+}(F(Ae_i)) \to \mathrm{end}_{B^+}(F(Ae_i) \oplus Be_i)$$

which induces the functor F. □

We would like to combine Propositions 2.10.14 and 2.11.8 into DK localization statement similar to Proposition 2.11.3. To this end we define a 2-morphism between non-unital maps $f, g : A \to B$ of non-unital ring spectra generally as a natural transformation of induced functors

$$f^*, g^* : \mathrm{Mod}_H(A) \to \mathrm{Mod}_H(B) \ .$$

We will use this for H-unital and for locally unital ring spectra and using the invertible 2-morphisms define ∞-categories $\mathrm{Alg}_2^{\mathrm{lu}}$ and $\mathrm{Alg}_2^{\mathrm{H}}$. Again, similar to the case Alg_2^{u} these are the $(\infty, 1)$-cores of very natural $(\infty, 2)$-categories which are the more canonical objects. But for simplicity we will restriction to the $\infty = (\infty, 1)$-categorical realm here.

Theorem 2.11.9. *The functors*

$$\mathrm{Alg}_2^{\mathrm{lu}} \to \mathrm{Pr}^L_\omega \qquad \textit{and} \qquad \mathrm{Alg}_2^{\mathrm{H}} \to \mathrm{Pr}^L_{\mathrm{dual}} \ .$$

given by Mod_H *are Dwyer–Kan localizations.*

In order to prove this Theorem we need the following auxiliary construction.

Construction 2.11.10. *Let R be a non-unital ring spectra. We want to define another non-unital ring spectrum $M_n(R)$ of $n \times n$-matrices over R. As a spectrum this is simply given as R^{n^2} but we would like to give it a non-unital ring structure. To this end we consider the unital ring spectra $M_n(R^+)$ of $n \times n$-matrices over the unitalizations R^+ and $M_n(\mathbb{S})$ over the sphere.*

The morphism $R^+ \to \mathbb{S}$ induced a map of ring spectra

$$M_n(R^+) \to M_n(\mathbb{S})$$

and we define $M_n(R)$ to be the fiber.

Lemma 2.11.11. *1. The map $i_k : R \to M_n(R)$ given by inclusion into the k-th diagonal entry is a Morita equivalence and all the functors*

$$i_k^* : \mathrm{Mod}_R \to M_n(R)$$

are equivalent.

2. If R is locally unital, then so is $M_n(R)$.

3. If R is H-unital, then so is $M_n(R)$.

Proof. TO BE WRITTEN □

Proof of Theorem 2.11.9. Let us first prove the case of locally unital rings. We proceed similar to the proof of Proposition 2.11.3, namely we want to apply Lemma 2.11.4. To this end we have to show that the functor

$$\mathrm{ZigZag}(A, B) \to \mathrm{Fun}^{\mathrm{sL}}(\mathrm{Mod}(A), \mathrm{Mod}(B))^{\simeq} \tag{2.5}$$

is an equivalence after realizing the source, which is the zigzag-category whose objects are locally unital algebras. We use the construction from Proposition 2.11.8 to produce a functor in the opposite direction:

$$\mathrm{Fun}^{\mathrm{sL}}(\mathrm{Mod}(A), \mathrm{Mod}(B))^{\simeq} \to \mathrm{ZigZag}(A, B) \quad F \mapsto \Big(A \to C(F) \xleftarrow{\simeq} B\Big)$$

where $C(F) = \mathrm{colim}\,\mathrm{end}_{B^+}(F(Ae_i) \oplus Be_i)$, which is clearly functorial in natural equivalences. Note that we fix $A = \mathrm{colim}_I\, Ae_i$ and $B = \mathrm{colim}_I\, Be_i$ once and for all. The composition

$$\mathrm{Fun}^{\mathrm{sL}}(\mathrm{Mod}(A), \mathrm{Mod}(B))^{\simeq} \to \mathrm{ZigZag}(A, B) \to \mathrm{Fun}^{\mathrm{sL}}(\mathrm{Mod}(A), \mathrm{Mod}(B))^{\simeq}$$

is by construction equivalent to the identity and it remains to also show that the composition

$$|\mathrm{ZigZag}(A, B)| \to \mathrm{Fun}^{\mathrm{sL}}(\mathrm{Mod}(A), \mathrm{Mod}(B))^{\simeq} \to |\mathrm{ZigZag}(A, B)|$$

is homotopic to the identity. We proceed exactly as in the proof of Proposition 2.11.3 and note that for a given zig-zag $A \xrightarrow{\varphi} C \xleftarrow{\psi} B$ we have that

$$C(F) = \mathrm{colim}\,\mathrm{end}_{B^+}(F(Ae_i) \oplus Be_i) \simeq \mathrm{colim}\,\mathrm{end}_{C^+}(\varphi^*(Ae_i) \oplus \psi^*(Be_i))$$

and we consider the natural diagram

$$\begin{tikzcd}
 & \mathrm{colim}\,\mathrm{end}_{C^+}(\varphi^*(Ae_i) \oplus \psi^*(Be_i)) \arrow[d] & \\
A \arrow[ur, "i_0"] \arrow[r, "i_0"] \arrow[dr, "\varphi"'] & M_2(C) & B \arrow[ul, "i_1"'] \arrow[l, "i_1"'] \arrow[dl, "\psi"] \\
 & C \arrow[u, "i_1"] &
\end{tikzcd}$$

□

2.12 The symmetric monoidal structure on $\mathrm{Pr}^L_{\mathrm{ca}}$.

The characterisation of stable and compactly assembled categories as dualizable objects suggests that the tensor product on Pr^L descends to one on $\mathrm{Pr}^L_{\mathrm{ca}}$, although it doesn't directly imply it (since it only works in the stable case, and doesn't say anything about morphisms). We first prove the following lemma:

Lemma 2.12.1. *1. A presentable category is compactly assembled if and only if we find compactly generated $\mathcal{C}'$ and a pair of adjoint functors*

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook, shift left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{C}' \arrow[l, shift left, "R"] \end{tikzcd}$$

where both L and R are in Pr^L, and L is fully faithful.

2. A left adjoint functor $F : \mathcal{C} \to \mathcal{D}$ between compactly assembled categories is compactly assembled if and only if we find a diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook, shift left, "L"] \arrow[r, phantom, "\perp"] \arrow[d, "F"'] & \mathcal{C}' \arrow[l, shift left, "R"] \arrow[d, "F'"] \\ \mathcal{D} \arrow[r, hook, shift left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{D}' \arrow[l, shift left, "R"] \end{tikzcd}$$

where all morphisms are in Pr^L, $\mathcal{C}'$ and $\mathcal{D}'$ are compactly generated, F' preserves compact objects, and the left adjoint functors L are fully faithful.

Proof. We essentially know the first statement already: Retracts of compactly generated categories are compactly assembled, and in the other direction any compactly assembled category comes with the adjunction $\hat{\jmath} : \mathcal{C} \rightleftarrows \mathrm{Ind}(\mathcal{C}^{\omega_1}) : k$ where $\hat{\jmath}$ is fully faithful. Naturality of this also proves one direction of the second statement, since a compactly assembled $F : \mathcal{C} \to \mathcal{D}$ commutes with $\hat{\jmath}$ and k, see Proposition 2.6.1 and Lemma 2.1.36.

For the final step, assume we have a diagram as in the statement of (2). The functors L have a filtered-colimit-preserving right adjoint, so they preserve compact morphisms. They also detect compact morphisms: Since L is fully faithful and preserves colimits, to test whether a morphism is compact in $\mathcal{C}$ (or $\mathcal{D}$), we may test this after applying L. Since F' preserves compact morphisms, this shows that F preserves compact morphisms. $\square$

Essentially, this lemma says that objects and morphisms in $\mathrm{Pr}^L_{\mathrm{ca}}$ are characterized as nicely controlled retracts of objects and morphisms in Pr^L_{ω}, formed in Pr^L, since fully faithfullness of L implies $RL \simeq \mathrm{id}$. Although we will not need this here, we have learned from Maxime Ramzi that in the stable case, compactly assembled functors (i.e. internal left adjoints to $\mathrm{Pr}^L_{\mathrm{st}}$) are even closed under arbitrary retracts in $\mathrm{Pr}^L_{\mathrm{st}}$, which follows from the more general 2-categorical statement [Ram, Lemma 1.45].

Proposition 2.12.2. *The tensor product of Pr^L restricts to a symmetric-monoidal structure on $\mathrm{Pr}^L_{\mathrm{ca}}$, characterized by corepresenting functors $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which are "bi-compactly assembled": They preserve colimits in each variable, and take $f \times g$ to compact morphisms whenever f and g are compact.*

Proof. The tensor product of compactly generated categories is compactly generated. This follows from the fact that $\mathrm{Cat}_\infty^{\mathrm{rex}(\kappa)}$ also admits a tensor product (where κ-small colimit preserving functors out of $\mathcal{C}_0 \otimes \mathcal{D}_0$ correspond to functors out of $\mathcal{C}_0 \times \mathcal{D}_0$ which preserve κ-small colimits in each argument), see [Lur17a, Section 4.8.1] for a much more general statement. By checking universal properties, one directly sees

$$\mathrm{Ind}_\kappa(\mathcal{C}_0) \otimes \mathrm{Ind}_\kappa(\mathcal{D}_0) \simeq \mathrm{Ind}_\kappa(\mathcal{C}_0 \otimes^{\mathrm{rex}(\kappa)} \mathcal{D}_0).$$

It follows that tensor products of compactly assembled ∞-categories stay compactly assembled, due to the characterisation as retracts of compactly generated categories.

For morphisms, we argue similarly: A pair of adjoint functors $L : \mathcal{C} \rightleftarrows D : R$ where the left adjoint is fully faithful stays such after tensoring with some $\mathcal{E}$, since it is characterized by natural transformations $LR \to \mathrm{id}$ and $\mathrm{id} \to RL$, the latter of which is an equivalence. Now take $F : \mathcal{C} \to \mathcal{D}$ compactly assembled and $\mathcal{E}$ compactly assembled. The previous lemma gives adjunctions

$$\begin{tikzcd} \mathcal{E} \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] & \mathcal{E}' \arrow[l, shift left] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] \arrow[d, "F"'] & \mathcal{C}' \arrow[l, shift left] \arrow[d, "F'"] \\ \mathcal{D} \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] & \mathcal{D}' \arrow[l, shift left] \end{tikzcd}$$

with $\mathcal{C}', \mathcal{D}'$ and $\mathcal{E}'$ compactly generated, and F' compact-object preserving. Tensoring and composing, we obtain a diagram

$$\begin{tikzcd} \mathcal{C} \otimes \mathcal{E} \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] \arrow[d, "F\otimes\mathrm{id}"'] & \mathcal{C}' \otimes E' \arrow[l, shift left] \arrow[d, "F'\otimes\mathrm{id}"] \\ \mathcal{D} \otimes \mathcal{E} \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] & \mathcal{D}' \otimes \mathcal{E}' \arrow[l, shift left] \end{tikzcd}$$

This shows that $F \otimes \mathrm{id}$ is also compactly assembled.

Finally, for the universal property, we consider $\mathcal{C}, \mathcal{D}$ compactly assembled and adjunctions $\mathcal{C} \rightleftarrows \mathcal{C}'$ and $\mathcal{D} \rightleftarrows \mathcal{D}'$ with fully faithful left adjoints as above. In the diagram

$$\begin{tikzcd} \mathcal{C} \times \mathcal{D} \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] \arrow[d, "\otimes"'] & \mathcal{C}' \times \mathcal{D}' \arrow[l, shift left] \arrow[d, "\otimes"] \\ \mathcal{C} \otimes \mathcal{D} \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] & \mathcal{C}' \otimes \mathcal{D}' \arrow[l, shift left] \end{tikzcd}$$

we see that the top horizontal inclusion takes a pair of compactly assembled morphisms to a morphism in $\mathcal{C}' \times \mathcal{D}'$ factoring through a pair of compact objects. Since the bottom horizontal functor detects compact morphisms, this shows that $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ takes pairs of compact morphisms to compact morphisms. This also shows that a compactly assembled functor $\mathcal{C} \otimes \mathcal{D} \to \mathcal{E}$ gives rise to a "bi-compactly assembled" functor $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$. Finally, we need to prove that, given a functor $\mathcal{C} \otimes \mathcal{D} \to \mathcal{E}$ for which $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is "bi-compactly assembled", the functor $\mathcal{C} \otimes \mathcal{D} \to \mathcal{E}$ is compactly assembled. Since such a functor in particular restricts to a functor

$$\mathcal{C}^{\omega_1} \times \mathcal{D}^{\omega_1} \to \mathcal{E}^{\omega_1},$$

we obtain a diagram

$$\begin{tikzcd}
\mathcal{C} \otimes \mathcal{D} \arrow[r, hook, shift left] \arrow[d, "\otimes"'] \arrow[r, phantom, "\perp"] & \mathrm{Ind}(\mathcal{C}^{\omega_1}) \otimes \mathrm{Ind}(\mathcal{D}^{\omega_1}) \arrow[l, shift left] \arrow[d, "\otimes"] \\
\mathcal{E} \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] & \mathrm{Ind}(\mathcal{E}^{\omega_1}) \arrow[l, shift left]
\end{tikzcd}$$

This exhibits the functor $\mathcal{C} \otimes \mathcal{D} \to \mathcal{E}$ as compactly assembled. □

Corollary 2.12.3. *For any compactly assembled ∞-category $\mathcal{C}$ and every locally compact Hausdorff space X the ∞-category $\mathrm{Shv}(X; \mathcal{C})$ is compactly assembled.*

Proof. According to Proposition 2.2.20 we find that sheaves of anima is compactly assembled. Then the claim follows from the assertion that

$$\mathrm{Shv}(X; \mathcal{C}) = \mathrm{Shv}(X; \mathrm{An}) \otimes \mathcal{C}$$

which is Proposition 2.8.7 combined with Proposition 2.12.2. □

Lemma 2.12.4. *For compactly assembled $\mathcal{C}, \mathcal{D}$, let S be the class of morphisms in $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$ consisting of all $\eta : F \to G$ with the property that for any compact morphism $X \to Y$ in $\mathcal{C}$, we have that the composite $F(X) \to G(Y)$ in the square*

$$\begin{tikzcd}
F(X) \arrow[r, "\eta_X"] \arrow[d] & G(X) \arrow[d] \\
F(Y) \arrow[r, "\eta_Y"] & G(Y)
\end{tikzcd}$$

is compact. Then S forms a precompact ideal.

Proof. S is clearly an ideal and contains the identity on the initial object. The pushout condition is also easily seen.

For the accessibility, we take a diagram of F_α, $\alpha \in [0, 1] \cap \mathbb{Q}$, such that for any compact $x \to y$ and any $\alpha < \alpha'$, the composite $F_\alpha(x) \to F_{\alpha'}(y)$ is compact. We need to prove that $F_0 \to F_1$ factors through some κ-compact G where κ is independent of the choice of F_α. We take $G = \mathrm{colim}_{\alpha<1} F_\alpha$. For X ω_1-compact, we may write $X = \mathrm{colim}\, X_n$ along compact maps. Using any sequence α_n tending to 1 from below, we see that

$$G(x) = \mathrm{colim}_n F_{\alpha_n}(X_n)$$

is a sequential colimit along compact maps, hence ω_1-compact. This proves that G takes ω_1-compact objects to ω_1-compact objects. Since $\mathcal{C}$ is ω_1-compactly generated, G is κ-compact in $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$ for some κ only depending on the size of $\mathcal{C}^{\omega_1}$. □

Definition 2.12.5. For compactly assembled $\mathcal{C}, \mathcal{D}$, we define an internal Hom by

$$\underline{\mathrm{Hom}}^{\mathrm{ca}}(\mathcal{C}, \mathcal{D}) = (\mathrm{Fun}^L(\mathcal{C}, \mathcal{D}), S)^{\mathrm{ca}},$$

with S as above.

Lemma 2.12.6. *This is actually an internal Hom, i.e.*

$$\mathrm{Fun}^{\mathrm{ca}}(\mathcal{C}, \underline{\mathrm{Hom}}^{\mathrm{ca}}(\mathcal{D}, \mathcal{E})) \simeq \mathrm{Fun}^{\mathrm{ca}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}).$$

Proof. By the universal property of compactly assembled cores, the left hand side agrees with the full subcategory of $\mathrm{Fun}^L(\mathcal{C}, \mathrm{Fun}^L(\mathcal{D}, \mathcal{E}))$ on all functors taking compact morphisms into S, i.e. under the adjunction to $\mathrm{Fun}^{\mathrm{biL}}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ to all functors taking pairs of compact morphisms to compact morphisms. But this is the same as $\mathrm{Fun}^{\mathrm{ca}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E})$ by the previous lemma. □

Example 2.12.7. The compact full subcategory of compact objects in $\underline{\mathrm{Hom}}^{\mathrm{ca}}(\mathcal{C}, \mathcal{D})$ agrees with $\mathrm{Fun}^{\mathrm{ca}}(\mathcal{C}, \mathcal{D})$. This is either seen by looking at $\mathrm{Fun}^{\mathrm{ca}}(\mathrm{An}, \underline{\mathrm{Hom}}^{\mathrm{ca}}(\mathcal{C}, \mathcal{D})) = \mathrm{Fun}^{\mathrm{ca}}(\mathrm{An} \otimes \mathcal{C}, \mathcal{D})$, or directly by observing that the compact objects of $(\mathcal{C}, S)^{\mathrm{ca}}$ are exactly given by those objects of $\mathcal{C}$ whose identity lies in S, which in the case of $\underline{\mathrm{Hom}}^{\mathrm{ca}}(\mathcal{C}, \mathcal{D})$ gives exactly those functors $F \in \mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$ which preserve compact morphisms.

Example 2.12.8. If $\mathcal{C}$ is already compactly generated, $\eta : F \to G$ in $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$ is in S if and only if $\eta : F(X) \to G(X)$ is compact for each $\mathcal{C}^\omega$, i.e. S consists exactly of the pointwise compact morphisms in $\mathrm{Fun}^{\mathrm{rex}}(\mathcal{C}^\omega, \mathcal{D})$. If $\mathcal{D}$ is also compactly generated, those are exactly the morphisms which factor pointwise through a compact object. It does still not follow that $\underline{\mathrm{Hom}}^{\mathrm{dual}}(\mathcal{C}, \mathcal{D})$ agrees with $\mathrm{Ind}(\mathrm{Fun}^{\mathrm{rex}}(\mathcal{C}^\omega, \mathcal{D}^\omega)$, since such a natural transformation does not necessarily factor through a pointwise compact functor. For $\mathcal{C} = \mathrm{Mod}(A)$ and $\mathcal{D} = \mathrm{Mod}(B)$, $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D}) = \mathrm{BiMod}(B, A)$, and this unwinds to the following: S consists of those morphisms of bimodules $M \to M'$ for which $M \otimes_A N \to M' \otimes_A N$ factors through a compact B-module for each N, which does not necessarily agree with those $M \to M'$ which factor through a B-A bimodule which is compact as B-module.

Proposition 2.12.9. *The symmetric monoidal structure on $\mathrm{Pr}^L_{\mathrm{ca}}$ induces a closed symmetric monoidal structure on $\mathrm{Pr}^L_{\mathrm{dual}}$ such that the functor*

$$- \otimes \mathrm{Sp} : \mathrm{Pr}^L_{\mathrm{ca}} \to \mathrm{Pr}^L_{\mathrm{dual}}$$

is strong symmetric monoidal and such that the fully faithful inclusion $\mathrm{Pr}^L_{\mathrm{dual}} \to \mathrm{Pr}^L_{\mathrm{ca}}$ is closed, that is preserves inner homs.

Proof. One checks immediately that $\otimes$ and $\underline{\mathrm{Hom}}^{\mathrm{ca}}$ restrict to stable compactly assembled categories. □

We will denote the inner hom in $\mathrm{Pr}^L_{\mathrm{dual}}$ also by $\underline{\mathrm{Hom}}^{\mathrm{dual}}$ to make clear that we are in the stable setting (although it agrees with $\underline{\mathrm{Hom}}^{\mathrm{ca}}$).

Definition 2.12.10. A dualizable, stable ∞-category $\mathcal{C}$ is called smooth if the functor $\mathrm{Sp} \to \mathcal{C}^\vee \otimes \mathcal{C}$ is strongly left adjoint. It is called proper if the functor $\mathcal{C}^\vee \otimes \mathcal{C} \to \mathrm{Sp}$ is strongly left adjoint.

Proposition 2.12.11. *The smooth and proper dualizable stable ∞-categories are precisely the dualizable objects of $\mathrm{Pr}^L_{\mathrm{dual}}$. For a smooth and proper dualizable ∞-category $\mathcal{C}$ we have for any dualizable stable ∞-category $\mathcal{D}$*

$$\underline{\mathrm{Hom}}^{\mathrm{dual}}(\mathcal{C},\mathcal{D}) \simeq \mathrm{Fun}^L(\mathcal{C},\mathcal{D})$$

in particular, the dual of $\mathcal{C}$ in $\mathrm{Pr}^L_{\mathrm{dual}}$ agrees with the dual in Pr^L.

Proof. The functor $\mathrm{Pr}^L_{\mathrm{dual}} \to \mathrm{Pr}^L$ is strong symmetric monoidal, thus preserves dualizable objects, duals and inner homs out of dualizable objects. □

Example 2.12.12. One interesting example is $\underline{\mathrm{Hom}}^{\mathrm{dual}}(\mathrm{Sp}_p^\wedge, \mathrm{Sp})$. We have $\mathrm{Fun}^L(\mathrm{Sp}_p^\wedge, \mathrm{Sp}) = \mathrm{Sp}_p^\wedge$, for example since $\mathrm{Sp}_p^\wedge$ is compactly generated and $(\mathrm{Sp}_p^\wedge)^\omega$ is the category of compact p-power torsion spectra, which agrees with its opposite (along Spanier-Whitehead duality, i.e. $\mathrm{map}(-,\mathbb{S})$). Indeed, the equivalence takes $X \in \mathrm{Sp}_p^\wedge$ to the functor $\mathrm{Sp}_p^\wedge \to \mathrm{Sp}$ taking $Y \mapsto \mathrm{fib}(X \otimes Y \to X \otimes Y[1/p])$.

Now by definition $\underline{\mathrm{Hom}}^{\mathrm{dual}}(\mathrm{Sp}_p^\wedge, \mathrm{Sp})$ is given by $(\mathrm{Sp}_p^\wedge, S)^{\mathrm{ca}}$ where S consists of the class of morphisms $X \to X'$ for which $X \otimes Y \to X' \otimes Y$ is compact in Sp for all compact p-power torsion spectra Y. Since those are generated as a stable subcategory by $\mathbb{S}/p$, it agrees furthermore with $(\mathrm{Sp}_p^\wedge, S')^{\mathrm{ca}}$ where we take S' to be all $X \to X'$ where $X/p \to X'/p$ is compact (yet another choice would be those where $X/p^n \to X'/p^n$ is compact for all n).

Recall that $\widetilde{\mathrm{Nuc}}(\mathbb{Z}_p)$ was similarly defined as $(\mathcal{D}(\mathbb{Z})_p^\wedge, S)^{\mathrm{ca}}$ for the class of morphisms $X \to X'$ where $X/p^n \to X'/p^n$ is compact for each n (and we could have shown that $n = 1$ suffices, actually). So we define

$$\widetilde{\mathrm{Nuc}}(\mathbb{S}_p) \coloneqq \underline{\mathrm{Hom}}^{\mathrm{dual}}(\mathrm{Sp}_p^\wedge, \mathrm{Sp}).$$

Note that we can't directly write this as limit analogous to the $\mathbb{Z}$ case, since $\mathbb{S}/p^n$ is not a ring[20]. Similarly, we can't write the $\mathbb{Z}_p$ case as $\underline{\mathrm{Hom}}^{\mathrm{dual}}$ since we would need a $\mathcal{D}(\mathbb{Z})$-linear version of $\underline{\mathrm{Hom}}^{\mathrm{dual}}$. This will be one of our next goals.

Note that the compact objects in $\widetilde{\mathrm{Nuc}}(\mathbb{S}_p)$ are given by the full subcategory of $\mathrm{Sp}_p^\wedge$ on those X where X/p is compact. These agree with the compact $\mathbb{S}_p^\wedge$-modules. However, we also have objects such as

$$\mathrm{colim}_{\alpha\in\mathbb{Q}}\, j\widehat{\bigoplus_{n\geq 1}}\mathbb{S}_p^\wedge \in \mathrm{Ind}(\mathrm{Sp}_p^\wedge)$$

where the map from $\alpha \to \alpha'$ is given on the n-th summand by multiplication with $p^{\lfloor\alpha' n\rfloor - \lfloor \alpha n\rfloor}$. This is an S-exhaustible object, hence an ω_1-compact object in $\widetilde{\mathrm{Nuc}}(\mathrm{Sp}_p^\wedge)$, but it can't be written as colimit of compact objects since none of the maps factor through a compact $\mathbb{S}_p^\wedge$-module.

Remark 2.12.13. Harr [Har23b] has recently shown that under a hypercompleteness assumption on a locally compact Hausdorff space X, if the category $\mathrm{Shv}(X;\mathrm{Sp})$ is smooth, then X is finite, c.f. Theorem 0.2 of op. cit. In other words, spectral sheaves on such spaces are almost never smooth.

[20] There is of course a free $\mathbb{E}_1$-ring $\mathbb{S}//p^n$, but this is yet another description, since $\mathbb{Z}//p^n \neq \mathbb{Z}/p^n$ even over $\mathbb{Z}$

Chapter 3

Continuous K-Theory

K-theory is by now a very classical topic. It all started with Whitehead's definition of the group $K_1(R)$ for a ring R (he had more specifically group rings in mind coming from applications to simple homotopy theory). Then later Grothendieck defined the group $K_0(R)$ for a ring or more generally for schemes. In the 70's Quillen managed to define a connective K-theory spectrum $\mathrm{k}(R) \in \mathrm{Sp}$ with $\pi_0(\mathrm{k}(R)) = K_0(R)$ and $\pi_1(\mathrm{k}(R)) = K_1(R)$. Quillen realized that K-theory really is an invariant associated with categories; nowadays usually one of small stable ∞-categories, with

$$\mathrm{k}(R) = \mathrm{k}(\mathcal{D}^{\mathrm{perf}}(R)) .$$

Several people, including Bass and Thomason, realized in various contexts that a non-connective version of K-theory is in fact needed. In the following we will denote connective K-theory by k and its non-connective variant by K. The purpose of this section is to review these definitions and their properties and extend K-theory to become an invariant of compactly assembled ∞-categories. That this is possible is an insight of Efimov.

3.1 Connective algebraic K-theory

In this section we would like to recall the definition of the (connective) algebraic K-theory of stable ∞-categories $\mathcal{C}$ or more generally ∞-categories with finite colimits. We will treat the non-connective version in the following section. The relation between these is that there is a map

$$\mathrm{k}(\mathcal{C}) \to \mathrm{K}(\mathcal{C})$$

which induces an isomorphism on π_i for $i > 0$ and an injection on π_0. The map on π_0 is an isomorphism precisely of $\mathcal{C}$ is idempotent complete, so that in this situation $\mathrm{k}(\mathcal{C})$ is the connective cover of $\mathrm{K}(\mathcal{C})$. Our main interest lies in non-connective K-theory, but we shall define it using the connective version.

Construction 3.1.1. *For any ∞-category $\mathcal{C}$ with pushouts we define another ∞-category* $\mathrm{coSpan}(\mathcal{C})$ *which is informally given as follows: Objects are objects of $\mathcal{C}$. A morphism from*

X to Y in $\mathrm{coSpan}(\mathcal{C})$ *is given by a cospan*

$$\begin{tikzcd} & W & \\ X \arrow[ur] & & Y \arrow[ul] \end{tikzcd}$$

A 2-morphism between spans is given by a diagram of the form

$$\begin{tikzcd} & W \arrow[dd, "\simeq"] & \\ X \arrow[ur] \arrow[dr] & & Y \arrow[ul] \arrow[dl] \\ & W' & \end{tikzcd}$$

Note that the vertical maps are required to be equivalences (otherwise we would get an $(\infty, 2)$-category. Higher morphisms are similarly defined. Composition of morphisms is defined by taking pushouts and identities are given by the cospan $X \to X \leftarrow X$ where both morphisms are the identity on X.

If $\mathcal{C}$ also admits an initial object and hence finite coproducts, then $\mathrm{coSpan}(\mathcal{C})$ *inherits a symmetric monoidal structure given by the coproduct of $\mathcal{C}$. Note that this is typically not a coproduct in* $\mathrm{coSpan}(\mathcal{C})$ *anymore. We will not give a rigorous definition here but instead refer the reader to [Bar17].*

Definition 3.1.2. Let $\mathcal{C}$ be a small ∞-category with finite colimits. Then we define $\mathrm{k}(\mathcal{C})$ as the connective spectrum associated with the $\mathbb{E}_\infty$-group $\Omega|\mathrm{coSpan}(\mathcal{C})|$. The K-groups are defines as $\mathrm{k}_n(\mathcal{C}) := \pi_n(\mathrm{k}(\mathcal{C}))$ for $n \geq 0$.

Since $\mathrm{coSpan}(\mathcal{C})$ is an $\mathbb{E}_\infty$-monoid in Cat_∞ (a symmetric monoidal ∞-category), the realization also inherits the structure of an $\mathbb{E}_\infty$-monoid and thus its loopspace is an $\mathbb{E}_\infty$-group. But note that $|\mathrm{coSpan}(\mathcal{C})|$ is in fact connected, since for every object X there is the cospan $X = X \leftarrow \emptyset$. Thus in fact $|\mathrm{coSpan}(\mathcal{C})|$ is already an $\mathbb{E}_\infty$-group.

Proposition 3.1.3. *We have that $\mathrm{k}_0(\mathcal{C})$ is the free abelian group generated by isomorphism classes $[X]$ of objects of $\mathcal{C}$ modulo the relation that*

$$[A] + [D] = [C] + [B]$$

for a pushout

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

in $\mathcal{C}$ and $[\emptyset] = 0$.

Proof. The proof will be given in the appendix 3.1.1 to this section. $\square$

Remark 3.1.4. The cospan perspective in this generality is due to Raptis and Steimle [RS18] and was shown by them to be equivalent to older definitions (notably the one using Waldhausen's $S_\bullet$-construction).

One can generalize the definition slightly, where instead of an ∞-category with finite colimits, $\mathcal{C}$ is an (unpointed) Waldhausen ∞-category. This is an ∞-category $\mathcal{C}$ with a chosen class of morphisms called cofibrations (satisfying certain axioms, e.g. existence of pushouts along cofibrations). Then one can similarly define a connective K-theory spectrum using a modified definition of $|\operatorname{coSpan}(\mathcal{C})|$ where we require the left leg of a cospan to be a cofibration. This applies for example to an abelian category (with cofibrations the monomorphisms) or its finitely generated projective objects (with cofibrations the split monomorphisms). We shall however not need this generality here.

Example 3.1.5. Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors preserving finite colimits. Then the functor $F \amalg G$ given by the pointwise coproduct of F and G also preserves finite coproducts, and the induced map

$$\mathrm{k}(F \amalg G) : \mathrm{k}(\mathcal{C}) \to \mathrm{k}(\mathcal{D})$$

is given by $\mathrm{k}(F) + \mathrm{k}(G)$. This follows since as functors $\operatorname{coSpan}(\mathcal{C}) \to \operatorname{coSpan}(\mathcal{D})$ it is true that $F \amalg G$ is the pointwise tensor product of F and G. One can also express this in a more fancy way by noting that the category $\mathrm{Cat}_\infty^{\mathrm{rex}}$ is semiadditive (i.e. finite coproducts and finite products agree). (Connective) Spectra also form a semiadditive category and the functor $\mathrm{k} : \mathrm{Cat}_\infty^{\mathrm{rex}} \to \mathrm{Sp}_{\geq 0}$ preserves products (as is obvious from the definition). Thus it also preserves addition on mapping spaces, which is given by $\amalg$ and $+$ respectively.

Example 3.1.6 (Eilenberg swindle)**.** Assume that a small ∞-category with finite colimits admits a colimit preserving functor $F : \mathcal{C} \to \mathcal{C}$ such that $F \amalg \mathrm{id}_{\mathcal{C}} \simeq F$. Then $\mathrm{k}(\mathcal{C}) = 0$. Indeed, the functor F induces a map of spectra $f : \mathrm{k}(\mathcal{C}) \to \mathrm{k}(\mathcal{C})$ such that $f + \mathrm{id} = f$. But we are in an additive (i.e. grouplike) setting, so this gives $\mathrm{id} = 0$ on $\mathrm{k}(\mathcal{C})$. Note that this happens whenever $\mathcal{C}$ has countable colimits; then we can take the functor

$$F : \mathcal{C} \to \mathcal{C} \quad X \mapsto \coprod_{\mathbb{N}} X\ .$$

We conclude that $\mathrm{k}(\mathcal{C}) = 0$ for ∞-categories $\mathcal{C}$ with countable colimits.

Proposition 3.1.7. *The functor*

$$\mathrm{k} : \mathrm{Cat}_\infty^{\mathrm{rex}} \to \mathrm{Sp}$$

commutes with filtered colimits.

Proof. Recall that filtered colimits in $\mathrm{Cat}_\infty^{\mathrm{rex}}$ are computed in Cat_∞. It is clear that for such a filtered colimit $\mathcal{C} = \operatorname{colim} \mathcal{C}_i$ we have an equivalence in Cat_∞ (or equivalently in symmetric monoidal categories, as their filtered colimits are also formed underlying):

$$\operatorname{colim} \operatorname{coSpan}(\mathcal{C}_i) \simeq \operatorname{colim} \operatorname{coSpan}(\mathcal{C}).$$

This follows by examining the construction. Namely, the n-th level of $\operatorname{coSpan}(\mathcal{C})$ as a complete Segal space is given by the union of path components of the mapping space

$\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathrm{Tw}[n], \mathcal{C})$ of those functors which send every square in $\mathrm{Tw}[n]$ to a pushout. Here $\mathrm{Tw}[n]$, the twisted arrow category of the poset $[n]$, comes from a finite simplicial set and is hence a compact object in Cat_∞, so that $\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathrm{Tw}[n], -)$ preserves filtered colimits, and since we chose path components of this space via a colimit condition, we can just pull the filtered colimit right through. Now the realization functor as well as the functor Ω also commute with filtered colimits, which finishes the proof. $\square$

Lemma 3.1.8. *The full subcategory $\mathrm{Cat}_\infty^{\mathrm{rex,pt}} \subseteq \mathrm{Cat}_\infty^{\mathrm{rex}}$ on pointed categories is a left Bousfield localization, with left adjoint given by*

$$(-)_+ \coloneqq \mathrm{An}_*^{\mathrm{fin}} \otimes - : \mathrm{Cat}_\infty^{\mathrm{rex}} \to \mathrm{Cat}_\infty^{\mathrm{rex,pt}}.$$

We can also describe $\mathcal{C}_+ \subseteq \mathrm{Ind}(\mathcal{C})_*$ as the full subcategory generated under finite colimits by the image of $\mathcal{C} \subseteq \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C})_*$. Moreover, if $\mathcal{C}$ already has a final object $*$, then $\mathcal{C}_+ = \mathcal{C}_{*/}$.

Proof. Recall the closed symmetric monoidal structure on $\mathrm{Cat}_\infty^{\mathrm{rex}}$ from e.g. [Lur17a, 4.8.1]; the unit is given by $\mathrm{An}^{\mathrm{fin}}$, the inner hom by $\mathrm{Fun}^{\mathrm{rex}}(-,-)$, and by definition we have an equivalence

$$\mathrm{Fun}^{\mathrm{rex}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \simeq \mathrm{Fun}^{\mathrm{rex,rex}}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$$

where the right hand side denotes the full subcategory of functors preserving finite colimits in both variables separately. Now $\mathrm{An}_*^{\mathrm{fin}}$ is the free pointed finitely cocomplete category on one generator, in the sense that $\mathrm{Fun}^{\mathrm{rex,pt}}(\mathrm{An}_*^{\mathrm{fin}}, \mathcal{D}) \simeq \mathcal{D}$ by evaluating at S^0. The remaining points are as follows:

1. $\mathrm{An}_*^{\mathrm{fin}} \otimes \mathcal{C}$ is pointed for any $\mathcal{C} \in \mathrm{Cat}_\infty^{\mathrm{rex}}$.

   To see this, we will use the following almost tautological characterization: $\mathcal{C} \in \mathrm{Cat}_\infty^{\mathrm{rex}}$ is pointed if and only if the functor $\mathcal{C} \to *$ admits both adjoints, and the adjoints agree. This is clearly satisfied for $\mathrm{An}_*^{\mathrm{fin}}$, so in particular both adjunctions are internal to $\mathrm{Cat}_\infty^{\mathrm{rex}}$, and thus preserved by $\mathcal{C} \otimes -$. In other words, $\mathcal{C} \otimes \mathrm{An}_*^{\mathrm{fin}} \to \mathcal{C} \otimes * = *$ also has both adjoints and they agree, proving that $\mathcal{C} \otimes \mathrm{An}_*^{\mathrm{fin}}$ is pointed.

2. $\mathrm{An}_*^{\mathrm{fin}} \otimes -$ is left adjoint to the inclusion.

   For $\mathcal{C} \in \mathrm{Cat}_\infty^{\mathrm{rex}}$ and $\mathcal{D} \in \mathrm{Cat}_\infty^{\mathrm{rex,pt}}$ we now have natural equivalences

   $$\mathrm{Fun}^{\mathrm{rex,pt}}(\mathcal{C} \otimes \mathrm{An}_*^{\mathrm{fin}}, \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{rex}}(\mathcal{C}, \mathrm{Fun}^{\mathrm{rex,pt}}(\mathrm{An}_*^{\mathrm{fin}}, \mathcal{D})) \simeq \mathrm{Fun}^{\mathrm{rex}}(\mathcal{C}, \mathcal{D}).$$

3. Suppose $\mathcal{C} \in \mathrm{Cat}_\infty^{\mathrm{rex}}$ admits a final object $*$. Then $\mathrm{An}_*^{\mathrm{fin}} \otimes \mathcal{C} = \mathcal{C}_{*/}$.

   Since $\mathcal{C}_{*/}$ is pointed, we obtain by a unique pointed right exact functor $\mathrm{An}_*^{\mathrm{fin}} \otimes \mathcal{C} \to \mathcal{C}_{*/}$ extending $\mathcal{C} \to \mathcal{C}_{*/},\ c \mapsto c \sqcup *$. Now we already know that in the presentable world the statement is true, i.e. that $\mathrm{Ind}(\mathcal{C}_{/*}) = \mathrm{Ind}(\mathcal{C})_* = \mathrm{An}_* \otimes \mathrm{Ind}(\mathcal{C})$, see e.g. [Lur17a, 4.8.1.21]. Moreover, we have that $\mathrm{Ind} : \mathrm{Cat}_\infty^{\mathrm{rex}} \to \mathrm{Pr}^L$ is strong symmetric monoidal by [Lur17a, 4.8.1.10], hence we obtain a commutative diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d, hook] & \mathrm{An}_*^{\mathrm{fin}} \otimes \mathcal{C} \arrow[r] \arrow[d, hook] & \mathcal{C}_{*/} \arrow[d, hook] \arrow[dr] & \\
\mathrm{Ind}(\mathcal{C}) \arrow[r] \arrow[dr] & \mathrm{Ind}(\mathrm{An}_*^{\mathrm{fin}} \otimes \mathcal{C}) \arrow[r] & \mathrm{Ind}(\mathcal{C}_{*/}) \arrow[r, "\simeq"] & \mathrm{Ind}(\mathcal{C})_* \\
& \mathrm{An}_* \otimes \mathrm{Ind}(\mathcal{C}) \arrow[u, "\simeq"'] \arrow[urr, "\simeq"'] & &
\end{tikzcd}$$

which shows that the comparison functor $\mathrm{An}_*^{\mathrm{fin}} \otimes \mathcal{C} \to \mathcal{C}_{*/}$ is fully faithful. (The bottom left triangle commutes by symmetric monoidality of Ind, and the bottom right triangle commutes since the diagonal equivalence is the unique functor that makes $\mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C})_*$ factor over $\mathrm{Ind}(\mathcal{C}) \to \mathrm{An}_* \otimes \mathrm{Ind}(\mathcal{C})$, which the other composite already satisfies). To see essential surjectivity, note that for any pointed object $* \xrightarrow{x} X$ in $\mathcal{C}_{*/}$ we have a pushout diagram in $\mathcal{C}_{*/}$:

$$\begin{tikzcd}
* \sqcup * \arrow[r] \arrow[d, "x \sqcup *"'] & * \arrow[d] \\
X \sqcup * \arrow[r] & (X, x) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Now the objects in the span are all in the image of $\mathcal{C} \to \mathcal{C}_{*/}$, hence also in the image of $\mathrm{An}_*^{\mathrm{fin}} \otimes \mathcal{C} \to \mathcal{C}_{*/}$. Moreover, since the latter functor is fully faithful, we can then lift the whole span into $\mathrm{An}_*^{\mathrm{fin}} \to \mathcal{C}$, take the pushout there, and see that it maps to (X, x) by right exactness.

4. Let $\mathcal{C}_+ \subseteq \mathrm{Ind}(\mathcal{C})_*$ denote the full subcategory generated under finite colimits by the image of $\phi : \mathcal{C} \subseteq \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C}_*)$. Then $\mathcal{C}_+ \simeq \mathrm{An}_*^{\mathrm{fin}} \otimes \mathcal{C}$.

   Clearly $\mathcal{C}_+ \in \mathrm{Cat}_\infty^{\mathrm{rex,pt}}$, and $\mathcal{C} \to \mathcal{C}_+$ is right exact. As above, we have the commutative diagram

$$\begin{tikzcd}
\mathrm{An}_*^{\mathrm{fin}} \otimes \mathcal{C} \arrow[r, hook] & \mathrm{Ind}(\mathrm{An}_*^{\mathrm{fin}} \otimes \mathcal{C}) & \mathrm{An}_* \otimes \mathrm{Ind}(\mathcal{C}) \arrow[l, "\simeq"'] \arrow[d, "\simeq"] \\
\mathcal{C} \arrow[u] \arrow[r, hook] \arrow[dr] & \mathrm{Ind}(\mathcal{C}) \arrow[u] \arrow[r] \arrow[ur] & \mathrm{Ind}(\mathcal{C})_* \\
& \mathcal{C}_+ \arrow[ur, hook] &
\end{tikzcd}$$

   Thus $\mathrm{An}_*^{\mathrm{fin}} \otimes \mathcal{C}$ is also a full subcategory of $\mathrm{Ind}(\mathcal{C})_*$ containing the image of ϕ, and we obtain a fully faithful $g : \mathcal{C}_+ \to \mathrm{An}_*^{\mathrm{fin}} \otimes \mathcal{C}$ which commutes with the maps from $\mathcal{C}$. However, by the universal property of $\mathrm{An}_*^{\mathrm{fin}} \otimes \mathcal{C}$, we see that $\mathcal{C} \to \mathcal{C}_+$ must also factor uniquely through a map $f : \mathrm{An}_*^{\mathrm{fin}} \otimes \mathcal{C} \to \mathcal{C}_+$. Now by construction $gf\eta = \eta$, hence gf is the identity on $\mathrm{An}_*^{\mathrm{fin}} \otimes \mathcal{C}$, and in particular g is also essentially surjective.

$\square$

The following result has been shown by Raptis-Steimle in the special case $\mathcal{C} = \mathrm{An}_{/X}$ for X compact.

Proposition 3.1.9. *For every ∞-category with finite colimits the functor $\mathcal{C} \to \mathcal{C}_+$ (unit of the adjunction) induces an equivalence $\mathrm{k}(\mathcal{C}) \to \mathrm{k}(\mathcal{C}_+)$.*

Proof. We first assume that $\mathcal{C}$ admits a terminal object $*$, so that $\mathcal{C}_+ = \mathcal{C}_{*/}$. To prove that $\mathrm{k}(\mathcal{C}) \to \mathrm{k}(\mathcal{C}_+)$ is an equivalence it suffices to show that the functor

$$\mathrm{coSpan}(\mathcal{C}) \to \mathrm{coSpan}(\mathcal{C}_+)$$

induces an equivalence of anima after geometric realization (since equivalence of $\mathbb{E}_\infty$-groups can be detected on underlying anima). We have a functor back $\mathcal{C}_+ \to \mathcal{C}$ that forgets the basepoint which preserves pushouts (but not the initial object or coproducts). Now we consider the composition

$$\mathcal{C}_+ \to \mathcal{C} \to \mathcal{C}_+ \qquad X \mapsto X \sqcup *$$

which comes with a canonical natural transformation $* \sqcup - \Rightarrow -$ to the identity functor induced by the basepoint of X. Similarly, the composition

$$\mathcal{C} \to \mathcal{C}_+ \to \mathcal{C} \qquad X \mapsto X \sqcup *$$

comes with a natural transformation $- \Rightarrow - \sqcup *$. Now we note that the ∞-category $\mathrm{coSpan}(\mathcal{C})$ without its $\mathbb{E}_\infty$-structure is functorial in pushout preserving functors since we only need pushouts for the definition and not arbitrary finite colimits.

Moreover, we claim that $\mathrm{coSpan}(-)$ is also 2-functorial in “cocartesian natural transformations“, where we demand every naturality square to be a pushout. Indeed, suppose that $F, G : \mathcal{C} \to \mathcal{D}$ are functors in $\mathrm{Cat}_\infty^{\mathrm{rex}}$ and $\alpha : F \Rightarrow G$ is such a natural transformation. Then we can view α as a functor $\tilde{\alpha} : \mathcal{C} \times [1] \to \mathcal{D}$ which is still in $\mathrm{Cat}_\infty^{\mathrm{rex}}$ (note that $[1]$ has all colimits). Now $\mathrm{coSpan}(-)$ preserves products, and $\mathrm{coSpan}([1])$ is the retract category hence still weakly contractible. So on realizations we obtain a commutative diagram

$$\begin{tikzcd}
{|\mathrm{coSpan}(\mathcal{C})|} \arrow[d, "\simeq"'] \arrow[drr, bend left, "{|\mathrm{coSpan}(F)|}"] & & \\
{|\mathrm{coSpan}(\mathcal{C})| \times |\mathrm{coSpan}([1])|} \arrow[rr, "{|\mathrm{coSpan}(\tilde{\alpha})|}"] & & {|\mathrm{coSpan}(\mathcal{D})|} \\
{|\mathrm{coSpan}(\mathcal{C})|} \arrow[u, "\simeq"] \arrow[urr, bend right, "{|\mathrm{coSpan}(G)|}"'] & &
\end{tikzcd}$$

In other words such natural transformations $\alpha : F \Rightarrow G$ give us homotopies $|\mathrm{coSpan}(F)| \simeq |\mathrm{coSpan}(G)|$. Applying this to our situation from above, we obtain functors

$$|\mathrm{coSpan}(\mathcal{C}_+)| \to |\mathrm{coSpan}(C)| \qquad \text{and} \qquad |\mathrm{coSpan}(C)| \to |\mathrm{coSpan}(\mathcal{C}_+)|$$

such that both compositions are homotopic to the identities, hence they are mutually inverse homotopy equivalences, as desired.

Now for an arbitrary object $\mathcal{C}$ in $\mathrm{Cat}_\infty^{\mathrm{rex}}$ we claim that it can be written as a filtered colimit

$$\mathcal{C} = \mathrm{colim}_{X \in \mathcal{C}}\, \mathcal{C}_{/X}$$

in $\mathrm{Cat}_\infty^{\mathrm{rex}}$ of slices $\mathcal{C}_{/X}$ along the finite colimit preserving postcomposition functors $\mathcal{C}_{/X} \to \mathcal{C}_{/Y}$ for $X \to Y$ in $\mathcal{C}$. Note that $\mathcal{C}$ is filtered since it has all pushouts. To see that $\mathcal{C}$ is equivalent to this colimit we note that since we can compute this colimit in Cat_∞, it can be written as the Dwyer-Kan localization of the Grothendieck construction of the functor

$$X \in \mathcal{C} \mapsto \mathcal{C}_{/X}$$

at the coCartesian edges. This Grothendieck construction is equivalent the cocartesian fibration given by the target projection $t : \mathrm{Ar}(\mathcal{C}) \to \mathcal{C}$, where coCartesian edges are given by those squares inverted by the source projection, i.e. where the top map is an equivalence. The source projection admits the fully faithful right adjoint given by the identity section $\mathcal{C} \subseteq \mathrm{Ar}(\mathcal{C}), c \mapsto \mathrm{id}_c$, and is therefore precisely the localization we are looking for. In other words, the source projection exhibits $\mathcal{C}$ as the left Bousfield (in particular Dwyer-Kan) localization of $\mathrm{Ar}(\mathcal{C})$ at the t-cocartesian edges, and thus as the colimit of $X \mapsto \mathcal{C}_{/X}$.

Now both source and target of the natural transformation $\mathrm{k}((-)_+) \to \mathrm{k}$ induced by $\mathrm{id} \to (-)_+$ preserve filtered colimits (c.f. Proposition 3.1.7), and it is an equivalence on all categories admitting a final object, hence the above discussion shows that it is always an equivalence. $\square$

Construction 3.1.10. *Let $\mathcal{C}$ be any ∞-category with finite colimits. We define the Spanier–Whitehead category of $\mathcal{C}$ as*

$$\mathrm{SW}(\mathcal{C}) \coloneqq \mathrm{colim}(\mathcal{C}_+ \xrightarrow{\Sigma} \mathcal{C}_+ \xrightarrow{\Sigma} \cdots)$$

where the colimit is taken along the functor Σ defined as $\Sigma(X) = * \amalg_X *$. Note that this is a colimit in Cat_∞ but also in $\mathrm{Cat}_\infty^{\mathrm{rex}}$.

Analogously to Lemma 3.1.8, one can show the following:

Lemma 3.1.11. *The full subcategory $\mathrm{Cat}_\infty^{\mathrm{ex}} \subseteq \mathrm{Cat}_\infty^{\mathrm{rex}}$ on stable categories is a left Bousfield localization, with left adjoint given by*

$$\mathrm{SW} = \mathrm{Sp}^{\mathrm{fin}} \otimes - : \mathrm{Cat}_\infty^{\mathrm{rex}} \to \mathrm{Cat}_\infty^{\mathrm{ex}}.$$

We can also describe $\mathrm{SW}(\mathcal{C}) \subseteq \mathrm{Sp}(\mathrm{Ind}(\mathcal{C})) = \mathrm{Ind}(\mathcal{C}) \otimes \mathrm{Sp}$ as the full stable subcategory generated by the image of $\mathcal{C} \subseteq \mathrm{Ind}(\mathcal{C}) \to \mathrm{Sp}(\mathrm{Ind}(\mathcal{C}))$.

Proposition 3.1.12. *The functor $\mathcal{C} \to \mathrm{SW}(\mathcal{C})$ induces an equivalence*

$$\mathrm{k}(\mathcal{C}) \to \mathrm{k}(\mathrm{SW}(\mathcal{C}))\ .$$

Proof. Since $\mathrm{SW}(\mathcal{C})$ is defined as a filtered colimit it suffices by Proposition 3.1.7 and Proposition 3.1.9 to check that $\Sigma : \mathcal{C}_+ \to \mathcal{C}_+$ induces an equivalence

$$\mathrm{k}(\mathcal{C}) \to \mathrm{k}(\mathcal{C}).$$

As a consequence of additivity (see [HLS23] for a modern treatment in the stable case, or use a subdivision argument to compare the cospan-Q-construction with the Waldhausen S-construction and use Waldhausen's original additivity theorem [**?**, Theorem 1.4.2]), one sees that this map is in fact given by multiplication with -1 on the level of spectra, hence an equivalence. ☐

Let $\mathcal{C} \subseteq \mathcal{D}$ be an inclusion of ∞-categories with finite colimits which is an equivalence after idempotent completion, that is $\mathcal{C} \to \mathcal{D}$ is fully faithful and every object of $\mathcal{D}$ is a retract of an object in the image. These are also called *dense* inclusions.

Proposition 3.1.13 (Cofinality)**.** *Let $\mathcal{C} \to \mathcal{D}$ be a dense inclusion. Then $\mathrm{k}_i(\mathcal{C}) \to \mathrm{k}_i(\mathcal{D})$ is an isomorphism for $i > 0$ and an injection for $i = 0$. Moreover, the class $[D] \in \mathrm{k}_0(\mathcal{D})$ of an object $D \in \mathcal{D}$ lies in the image precisely if for some $n \geq 0$ the object $\Sigma^n(D_+)$ lies in the image of $\mathcal{C}_+ \to \mathcal{D}_+$.*

Note that if $\mathcal{C} \to \mathcal{D}$ is an dense, exact inclusion of stable ∞-categories then this allows us to use k-theory to exactly determine the objects in the image. We will reduce to general case to the stable case.

Proof. We first claim that $\mathrm{SW}(\mathcal{C}) \to \mathrm{SW}(\mathcal{D})$ is also a dense inclusion. This follows since $\mathrm{Ind}(\mathrm{SW}(\mathcal{C})) = \mathrm{Sp} \otimes \mathrm{Ind}(\mathcal{C}) = \mathrm{Sp} \otimes \mathrm{Ind}(\mathcal{D}) = \mathrm{Ind}(\mathrm{SW}(\mathcal{D}))$. Thus the result follows from the 'classical' cofinality result for stable ∞-categories, see e.g. [HLS23] for a modern treatment. Note that the last assertion then follows from the stable result as well. ☐

One could also ask for a stronger statement, namely if we can use K-theory to determine whether objects $D \in \mathcal{D}$ actually lie in (the essential image of) $\mathcal{C} \subseteq \mathcal{D}$. A necessary condition is that the class $[D] \in \mathrm{k}_0(\mathcal{D})$ lies in the image. This is true if $\mathcal{C}$ and $\mathcal{D}$ are stable, but in the unstable case this is not sufficient as it only guarantees by the previous result that a suspension of D lies in the image. There are examples of finitely dominated anima that are not finite. But after suspending twice every finitely dominated anima becomes simply connected (and still finitely dominated) and it is an insight of Wall that such anima are actually finite. Abstractly his result can be formulated as follows:

Theorem 3.1.14 (Wall finiteness obstruction)**.** *Let $\mathcal{C} \subseteq \mathcal{D}$ be a dense inclusion in $\mathrm{Cat}_\infty^{\mathrm{rex}}$. Then an object $D \in \mathcal{D}$ lies in $\mathcal{C}$ precisely if the essential image of $[\mathrm{id}_D] \in \mathrm{k}_0(\mathcal{D}_{/D})$ lies in the image of $\mathrm{k}_0(\mathcal{C}_{/D}) \to \mathrm{k}_0(\mathcal{D}_{/D})$.*

Proof. [Lur14, Lecture 15]. ☐

Example 3.1.15. Consider $\mathrm{An}^{\mathrm{fin}} \subseteq \mathrm{An}^{\omega}$. Then a connected finitely dominated anima $X \in \mathrm{An}^{\omega}_{\geq 1}$ lies in the essential image precisely if

$$\mathrm{id}_X \in \mathrm{k}_0(\mathrm{An}^{\omega}_{/X}) = \mathrm{k}_0((\mathrm{Sp}^X)^{\omega})$$

lies in the image of

$$\mathrm{k}_0(\mathrm{An}^{\mathrm{fin}}_{/X}) \to \mathrm{k}_0(\mathrm{An}^{\omega}_{/X}) = \mathrm{k}_0((\mathrm{Sp}^X)^{\omega}).$$

Since X is connected, $\mathrm{An}^{\mathrm{fin}}_{/X}$ is generated under finite colimits by $* \to X$. One then easily computes that $\mathrm{k}_0(\mathrm{An}^{\mathrm{fin}}_{/X}) = \mathrm{k}_0(\mathrm{An}^{\mathrm{fin}}) = \mathbb{Z}$ using Proposition 3.1.3. Moreover, recall that $\mathrm{Sp}^X \simeq \mathrm{Mod}(\mathbb{S}[\Omega X])$ by Schwede-Shipley, hence $(\mathrm{Sp}^X)^{\omega} = \mathrm{Perf}(\mathbb{S}[\Omega X])$, and $\mathrm{k}_0((\mathrm{Sp}^X)^{\omega}) = \mathrm{k}_0(\mathbb{Z}[\pi_1 X])$. Thus the question becomes whether a certain element in

$$\tilde{\mathrm{k}}_0(\mathbb{Z}[\pi_1 X]) := \mathrm{k}_0(\mathbb{Z}[\pi_1 X])/\mathbb{Z}$$

vanishes. This element is the classical Wall finiteness obstruction.

3.1.1 Appendix: Proof of Proposition 3.1.3

Let $\mathcal{C}$ be an ∞-category with chosen object 0, and assume that for each object X we have chosen morphisms

$$a_X : X \to 0$$
$$b_X : 0 \to X$$

(In particular, $\mathcal{C}$ is weakly connected.)

Definition 3.1.16. In the above situation, we define a group $G(\mathcal{C})$ as follows:

- Generators are given by endomorphisms of 0 in $\mathcal{C}$.
- Relations are given as follows: For every pair $f : 0 \to X$ and $g : X \to 0$, we require

$$[gf] = [gb_X][a_X b_X]^{-1}[a_X f].$$

Lemma 3.1.17. *The group $G(\mathcal{C})$ agrees with $\pi_1|\mathcal{C}| = \pi_1(|\mathcal{C}|, 0)$.*

Proof. We construct morphisms in both directions and show they are mutually inverse. We clearly have a homomorphism $G = G(\mathcal{C}) \to \pi_1|\mathcal{C}|$. Indeed, we map every generator $[f]$ to the corresponding endomorphism $f : 0 \to 0$, and observe that the relations hold in $\pi_1|\mathcal{C}|$, since in $\mathcal{C}$ we have

$$(gb_X) \circ (a_X b_X)^{-1} \circ (a_X f) = gb_X b_X^{-1} a_X^{-1} a_X f = gf.$$

To get a map $\pi_1|\mathcal{C}| \to G$ it is enough to construct a functor $\mathcal{C} \to BG$, since BG is a groupoid. The map on objects is clear, on morphisms we send $f : X \to Y$ to

$$[a_Y f b_X] \cdot [a_X b_X]^{-1} .$$

We have to check that this is compatible with composition and identities. Identities are clear, for composition we have to check that for $f : X \to Y$, $g : Y \to Z$ we have

$$[a_Z g f b_X][a_X b_X]^{-1} = [a_Z g b_Y][a_Y b_Y]^{-1}[a_Y f b_X][a_X b_X]^{-1} \tag{3.1}$$

in G. Using the defining relation of G for the pair of morphisms fb_Y and $a_Z g$, we get

$$[a_Z g f b_Y] = [a_Z g b_Y][a_Y b_Y]^{-1}[a_Y f b_Y],$$

which is directly equivalent to equation 3.1.

Now note that the composite

$$G \to \pi_1|\mathcal{C}| \to G$$

is an isomorphism: if f is already an endomorphism of 0, then

$$[a_0 f b_0][a_0 b_0]^{-1} = [a_0][f][b_0][b_0]^{-1}[a_0]^{-1} = [a_0][f][a_0]^{-1},$$

so the composite is just conjugation by $[a_0]$ on G. If we show that $G \to \pi_1|\mathcal{C}|$ is surjective, it follows that both maps are isomorphisms.

Every endomorphism of 0 in $|\mathcal{C}|$ is represented by a zigzag of morphisms $f_n g_n^{-1} f_{n-1} g_{n-1}^{-1} \cdots f_0 g_0^{-1}$ in $\mathcal{C}$, ending and starting in 0. Let $f_n : X_n \to Y_{n+1}$ and $g_n : X_n \to Y_n$. Then the above zigzag represents the same morphism as the zigzag

$$a_0^{-1} f_n' (g_n')^{-1} \cdots f_0' (g_0')^{-1} a_0$$

with $f_i = a_{Y_{i+1}} f_i b_{X_i}$, $g_i = a_{Y_i} g_i b_{X_i}$. All of these morphisms are endomorphisms of 0, so they lie in the image of $G \to \pi_1|\mathcal{C}|$. □

Corollary 3.1.18. *The abelianisation of $\pi_1|\mathcal{C}|$ (in particular, $\pi_1|\mathcal{C}|$ itself if $\mathcal{C}$ admits a monoidal structure with unit 0) admits a presentation as an abelian group, with:*

- *Generators given by endomorphisms of 0.*
- *Relations given by, for each pair of morphisms $f : 0 \to X$ and $g : X \to 0$, the relation*

$$[gf] + [a_X b_X] = [g b_X] + [a_X f].$$

Remark 3.1.19. It is not really necessary to fix choices of a_X, b_X: We can more symmetrically state Corollary 3.1.18 as follows:

- Generators are given by endomorphisms of 0.
- For every object X, and every choice of maps $f, f' : 0 \to X$ and $g, g' : X \to 0$, the "cut-and-paste" relation

$$[gf] + [g'f'] = [gf'] + [g'f]$$

Indeed, this relation clearly has the relation from 3.1.18 as special case. Conversely, the relation from 3.1.18 implies

$$\begin{aligned}
[gf] + [a_X b_X] &= [g b_X] + [a_X f] \\
[gf'] + [a_X b_X] &= [g b_X] + [a_X f'] \\
[g'f'] + [a_X b_X] &= [g' b_X] + [a_X f'] \\
[g'f] + [a_X b_X] &= [g' b_X] + [a_X f]
\end{aligned}$$

and taking the alternating sum of these relations implies the cut-and-paste relation.

Proof of Proposition 3.1.3. For a small ∞-category $\mathcal{C}$ with finite colimits we want to apply Corollary 3.1.18 to coSpan($\mathcal{C}$). We chose the object $\emptyset$ as basepoint and $a_X : X \to \emptyset$ as the cospan $X \to X \leftarrow \emptyset$, and $b_X : \emptyset \to X$ as the span $\emptyset \to X \leftarrow X$. Then $a_X b_X$ is given by the object X (that is the cospan $\emptyset \to X \leftarrow \emptyset$. More generally, for any $g : X \to 0$, represented by a cospan $\emptyset \to A \leftarrow X$, the composite $a_X g$ is the endomorphism given by A and for any $f : \emptyset \to X$ represented by a span $X \to B \leftarrow \emptyset$, the composite $f b_X$ is the endomorphism given by B. The composite gf is given by the pushout of $A \leftarrow X \to B$.

Thus, we get a generators and relations description for $\pi_1|\operatorname{coSpan}(\mathcal{C})|$ of the following form:

- Generators are given by objects of $\mathcal{C}$.
- Relations are given as follows: for every pair of morphisms f, g as above, that is for every diagram $A \leftarrow X \to B$, we have

$$[A \amalg_X B] + [X] = [A] + [B].$$

This is exactly the description we have given and finishes the proof. □

3.2 Non-connective algebraic K-Theory

In this section we will introduce non-connective K-theory. The design criterion is to demand preservation of certain cofiber sequences, which we want to explain first.

We will first consider cofiber sequence of ∞-categories with finite colimits, i.e. cofiber sequences in $\mathrm{Cat}_\infty^{\mathrm{rex}}$. Note that $\mathrm{Cat}_\infty^{\mathrm{rex}}$ is pointed (even semiadditive), which allows us to talk about fibers and cofibers by taking pullbacks respectively pushouts along $* \to \mathcal{C}$ respectively $\mathcal{C} \to *$. For a given functor $f : \mathcal{C} \to \mathcal{D}$ in $\mathrm{Cat}_\infty^{\mathrm{rex}}$ we denote this cofiber by $\mathcal{D}/\mathcal{C}$. If $\mathcal{C}$ and $\mathcal{D}$ happen to be stable, this cofiber is given by the Verdier quotient of $\mathcal{D}$ by the essential image of $\mathcal{C}$.

Remark 3.2.1. In general this cofiber $\mathcal{D}/\mathcal{C}$ is best understood in terms of the associated Ind-categories: Here we obtain an adjunction

$$\begin{tikzcd}
\mathrm{Ind}(\mathcal{C}) \arrow[r, shift left, "\mathrm{Ind}(i)"] \arrow[r, phantom, "\perp"] & \mathrm{Ind}(\mathcal{D}) \arrow[l, shift left, "\mathrm{Ind}(i)^R"] \arrow[r, shift left, "\mathrm{Ind}(p)"] \arrow[r, phantom, "\perp"] & \mathrm{Ind}(\mathcal{D}/\mathcal{C}) \arrow[l, shift left, "\mathrm{Ind}(p)^R"]
\end{tikzcd}$$

Since $\mathrm{Ind} : \mathrm{Cat}_\infty^{\mathrm{rex}} \to \mathrm{Pr}^L$ preserves colimits, we see that $\mathrm{Ind}(\mathcal{D}/\mathcal{C}) = \mathrm{Ker}(\mathrm{Ind}(i)^R)$. As we will see in the next section (c.f. Lemma 3.3.11), if we assume that i (and hence $\mathrm{Ind}(i)$) is fully faithful, and that $\mathrm{Ind}(i)^R$ preserves pushouts, then we have a natural pushout square

$$\begin{tikzcd}
\mathrm{Ind}(i)\,\mathrm{Ind}(i)^R D \arrow[r] \arrow[d] & D \arrow[d] \\
\mathrm{Ind}(i)(*) \arrow[r] & \mathrm{Ind}(p)^R\,\mathrm{Ind}(p) D \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

for every $D \in \mathrm{Ind}(\mathcal{D})$. From this we will deduce for $d, d' \in \mathcal{D}$ an equivalence

$$\mathrm{Map}_{\mathcal{D}/\mathcal{C}}(pd, pd') = \mathrm{Map}_{\mathrm{Ind}(\mathcal{D})}(jd,\ jd' \amalg_{\mathrm{Ind}(i)\,\mathrm{Ind}(i)^R jd'} \mathrm{Ind}(i)(*)).$$

Note that this extra condition that $\mathrm{Ind}(i)^R$ should preserve pushouts is automatic in the stable case.

This last description of mapping spaces in $\mathcal{D}/\mathcal{C}$ will be our unstable analogue for the well-known mapping space formula in Verdier quotients (see e.g. [**?**, Theorem I.3.3]). Indeed, note that the above conditions are automatic for stable Verdier sequences. Moreover, since $\mathrm{Ind}(\mathcal{C}^{\mathrm{idem}}) = \mathrm{Ind}(\mathcal{C})$, the above statements are invariant under idempotent completion. However, we are mainly interested in cofiber sequences in $\mathrm{Cat}^{\mathrm{rex}}$ which induce cofiber sequences on connective k-theory, and since one of the main points of the latter (c.f. Proposition 3.1.13) is that it is *not* invariant under idempotent completion, we currently need to add a small technical assumption to the main kind of cofiber sequence we will be working with.

Definition 3.2.2. We say that a sequence

$$\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{E}$$

in $\mathrm{Cat}_\infty^{\mathrm{rex}}$ is a Verdier cofiber sequence if i is fully faithful, $\mathrm{Ind}(i)^R$ preserves pushouts, and i_+ has retract-closed image.

As the following Proposition shows, there are more general classes of sequences sent to cofiber sequences by connective k-theory (however for them we will generally not have a mapping space formula for the quotients).

Proposition 3.2.3. *Let $S = (\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{E})$ be a sequence in* $\mathrm{Cat}^{\mathrm{rex}}$*. Consider the following conditions:*

1. *S is a cofiber sequence, $\mathcal{C}$ and $\mathcal{D}$ admit a terminal object, i is fully faithful with retract-closed image and preserves the terminal object. (In this case also $\mathcal{E}$ admits a terminal object which is preserved by p).*

2. *S is a Verdier cofiber sequence, i.e S is a cofiber sequence, i is fully faithful, i_+ has retract-closed image, and* $\mathrm{Ind}(i)^R$ *preserves pushouts (hence weakly contractible colimits).*

2' S is a cofiber sequence, i is fully faithful, i_+ has retract-closed image, and $\mathrm{Ind}(i)$ *is strongly left adjoint.*

3. $S_+ = (\mathcal{C}_+ \xrightarrow{i_+} \mathcal{D}_+ \xrightarrow{p_+} \mathcal{E}_+)$ is a cofiber sequence and i_+ is fully faithful with retract-closed image.

4. $\mathrm{SW}(S) = (\mathrm{SW}(\mathcal{C}) \to \mathrm{SW}(\mathcal{D}) \to \mathrm{SW}(\mathcal{E}))$ is a Verdier sequence, i.e. it is a cofiber sequence and $\mathrm{SW}(i)$ is fully faithful with retract-closed image.

5. $\mathrm{k}(S) = (\mathrm{k}(\mathcal{C}) \to \mathrm{k}(\mathcal{D}) \to \mathrm{k}(\mathcal{E}))$ is a cofiber sequence.

Then the following implications hold:

$$\begin{tikzcd}
(2') \arrow[r, Rightarrow] & (2) \arrow[d, Rightarrow] & & \\
(1) \arrow[r, Rightarrow] & (3) \arrow[r, Rightarrow] & (4) \arrow[r, Rightarrow] & (5)
\end{tikzcd}$$

Moreover, we have:

(i) If $\mathcal{C}, \mathcal{D}$ and hence $\mathcal{E}$ are pointed, then (1) $\Leftrightarrow$ (3) and (2) $\Leftrightarrow$ (2'). If $\mathcal{C}, \mathcal{D}$ and hence $\mathcal{E}$ are stable, then all implications except (4) $\Rightarrow$ (5) become equivalences. The functors $(-)^{\mathrm{idem}}, (-)_+$ and $\mathrm{SW}(-)$ preserve sequences of types (1)-(4), and the latter two also type (5).

(ii) Sequences of type (3), (4) and (5) are stable under filtered colimits in $\mathrm{Cat}^{\mathrm{rex}}$.

Proof.

(1) $\Rightarrow$ (3) Since $(-)_+$ is a left adjoint, it preserves cofiber sequences. By assumption and the addendum to Lemma 3.1.8 the map i_+ is induced by i on slices $\mathcal{C}_{*/} \to \mathcal{D}_{*/}$, and it is easily seen to be fully faithful with retract-closed image.

(2) $\Rightarrow$ (3) It suffices to check that i_+ is fully faithful. By definition, we have a square with fully faithful vertical maps

$$\begin{tikzcd}
\mathrm{Ind}(\mathcal{C})_* \arrow[r, "\mathrm{Ind}(i)_*"] & \mathrm{Ind}(\mathcal{D})_* \\
\mathcal{C}_+ \arrow[u, hook] \arrow[r, "i_+"] & \mathcal{D}_+ \arrow[u, hook]
\end{tikzcd}$$

so to check that i_+ is fully faithful, we are reduced to the following general statement: Suppose $\ell : \mathcal{A} \hookrightarrow \mathcal{B}$ is a fully faithful functor in Pr^L, and ℓ^R preserves pushouts, then $\ell_* = \mathrm{An}_* \otimes \ell : \mathcal{A}_* \to \mathcal{B}_*$ is still fully faithful. Indeed, by definition we have $\ell_*(* \to a) = \ell(a)/\ell(*)$, and its right adjoint $(\ell_*)^R : \mathcal{B}_* \to \mathcal{A}_*$ is simply induced by ℓ^R

since it preserves the final object. The unit of the adjunction $\ell_* \dashv (\ell_*)^R$ is then given by the composite horizontal map

$$\begin{tikzcd}
* \arrow[r, "\simeq"'] \arrow[d] \arrow[rr, bend left, "\simeq"] & \ell^R\ell(*) \arrow[r] \arrow[d] & \ell^R(*) \arrow[d] \\
a \arrow[r, "\simeq"] & \ell^R\ell(a) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \ell^R(\ell(a)/\ell(*))
\end{tikzcd}$$

the equivalences come from the fully faithfulness of ℓ and the fact that $\ell^R(*) = *$. The right square is a pushout by definition of ℓ_* and the assumption that ℓ^R preserves pushouts. Thus also the unit for $\ell_* \dashv (\ell_*)^R$ is an equivalence, proving that ℓ_* is still fully faithful.

(3) ⇒ (4) We will show in Lemma 3.2.5 below that the class of functors which are fully faithful (resp. have retract-closed image) is closed under filtered colimits. By the definition of SW, this implies the claim.

(4) ⇒ (5) This follows immediately from 3.1.12 and the classical stable statement, for which we refer the reader to [HLS23].

(i) If S is a cofiber sequence in $\mathrm{Cat}^{\mathrm{rex}}$, then $\mathrm{Ind}(\mathcal{E}) = \mathrm{Ker}(\mathrm{Ind}(i)^R)$. From this it follows that if $\mathcal{C}, \mathcal{D}$ are pointed then so is $\mathcal{E}$, and the claimed equivalences are then clear. Similarly, if $\mathcal{C}, \mathcal{D}$ are stable, then $\mathrm{Ind}(\mathcal{E})$ is a stable subcategory of $\mathrm{Ind}(\mathcal{D})$, and hence $\mathcal{E}^{\mathrm{idem}} = \mathrm{Ind}(\mathcal{E})^\omega$ is stable. Using that $\mathcal{E}$ had finite colimits to begin with, one can check that this yields stability of $\mathcal{E}$, and the claimed equivalences are also clear.

Next, note that idempotent completion commutes with $(-)_+$ and $\mathrm{SW}(-)$ by their universal properties and all three functors preserve colimits. Moreover, we know from Proposition 3.1.9 and 3.1.12 that k inverts the maps $\mathcal{C} \to \mathcal{C}_+$ and $\mathcal{C} \to \mathrm{SW}(\mathcal{C})$). So the only non-trivial remaining claim is that $(-)_+$ preserves Verdier cofiber sequences. But this follows from $\mathrm{Ind}((-)_+) = \mathrm{An}_* \otimes \mathrm{Ind}(-) = \mathrm{Ind}(-)_*$ and the fact that $(\mathrm{Ind}(i)_*)^R$ is simply the restriction of $\mathrm{Ar}(\mathrm{Ind}(i)^R) : \mathrm{Ar}(\mathrm{Ind}(\mathcal{D})) \to \mathrm{Ar}(\mathrm{Ind}(\mathcal{C}))$ when viewing $\mathrm{Ind}(\mathcal{C})_* \subseteq \mathrm{Ar}(\mathrm{Ind}(\mathcal{C}))$ as the full subcategory on arrows with source $*$, and analogously for $\mathcal{D}$. Note this even shows that $(-)_+$ sends Verdier cofiber sequences to sequences of type (2').

(ii) This follows from the facts that $(-)_+, \mathrm{SW}(-)$ and $\mathrm{k}(-)$ (by Proposition 3.1.7) preserve filtered colimits, and that fully faithful functors with retract-closed image are closed under filtered colimits in $\mathrm{Cat}^{\mathrm{rex}}$ by Lemma 3.2.5.

□

Remark 3.2.4. We currently don't know whether in (2) and (2') the condition that i_+ has retract-closed image is already implied by i having retract closed image. This is certainly the case when $\mathcal{C}$ is idempotent-complete, since then also $\mathcal{C}_+$ will be (and i_+ is fully faithful).

Note also that the above recovers the definition of a stable Verdier sequence if $\mathcal{C}, \mathcal{D}$ and hence $\mathcal{E}$ are stable.

Lemma 3.2.5. *Denote by $\mathcal{F}, \mathcal{R} \subseteq \mathrm{Ar}(\mathrm{Cat}_\infty^{\mathrm{rex}})$ the full subcategories on those functors which are fully faithful respectively have image closed under retracts. Then $\mathcal{F}$ and $\mathcal{R}$ and in particular their intersection are closed under filtered colimits in $\mathrm{Ar}(\mathrm{Cat}_\infty^{\mathrm{rex}})$ (equivalently, in $\mathrm{Ar}(\mathrm{Cat}_\infty)$).*

Proof. Let $\alpha : \mathcal{C}_\bullet \Rightarrow \mathcal{D}_\bullet$ be a natural transformation of diagrams $I \to \mathrm{Cat}_\infty^{\mathrm{rex}}$ such that α is pointwise in $\mathcal{F}$. Write $\mathcal{C}_\infty = \mathrm{colim}_i \mathcal{C}_i$ and let $\alpha_\infty : \mathcal{C}_\infty \to \mathcal{D}_\infty$ be the induced map on colimits. Since the mapping spaces in a filtered colimit of categories are computed as a filtered colimit of the respective mapping spaces it is clear that α_∞ is again fully faithful.

Now suppose instead that α is pointwise in $\mathcal{R}$ and that $d \in \mathcal{C}_\infty$ is retract of some $\alpha_\infty(c)$. Since the minimal diagram witnessing this (a 2-simplex) is finite and hence compact in Cat_∞, this witness already exists at some finite stage of the colimit. Concretely, this means that for some i we find $d_i \in \mathcal{D}_i$ which is a retract of some $\alpha_i(c_i)$ so that $\lambda^i_{\mathcal{C}}(c_i) = c$ and $\lambda^i_{\mathcal{D}}(d_i) = d$, where λ^i denote the colimit inclusions. But then by assumption $d_i = \alpha_i(c_i')$ for some $c_i' \in \mathcal{C}_i$, and $d = \lambda^i \alpha_i c_i' = \alpha_\infty \lambda_i c_i'$, as desired. $\square$

Example 3.2.6. Given an idempotent-complete category $\mathcal{C}$, we can consider the natural cofiber sequence

$$\mathcal{C} \xrightarrow{j} \mathrm{Ind}(\mathcal{C})^{\omega_1} \to \mathrm{Ind}(\mathcal{C})^{\omega_1}/\mathcal{C}.$$

By idempotent-completeness, it is clear that this is a type (2') sequence in the language of Proposition 3.2.3. Hence we obtain a natural cofiber sequence on k-theory:

$$\mathrm{k}(\mathcal{C}) \to \mathrm{k}(\mathrm{Ind}(\mathcal{C})^{\omega_1}) \to \mathrm{k}(\mathrm{Ind}(\mathcal{C})^{\omega_1}/\mathcal{C}) \ .$$

We note that $\mathrm{k}(\mathrm{Ind}(\mathcal{C})^{\omega_1}) = 0$ by the Eilenberg swindle (Example 3.1.6) since $\mathrm{Ind}(\mathcal{C})^{\omega_1}$ has countable colimits. So if $\mathcal{C}$ is an idempotent complete ∞-category with finite colimits we have an equivalence

$$\mathrm{k}(\mathcal{C}) \simeq \Omega \mathrm{k}(\mathrm{Ind}(\mathcal{C})^{\omega_1}/\mathcal{C}) \ .$$

In particular we get $\mathrm{k}_0(\mathrm{Ind}(\mathcal{C})^{\omega_1}/\mathcal{C}) = 0$. But note that $\mathrm{Ind}(\mathcal{C})^{\omega_1}/\mathcal{C}$ doesn't need to be idempotent complete itself, so we cannot iterate this process to get further deloopings. We could however replace $\mathrm{Ind}(\mathcal{C})^{\omega_1}/\mathcal{C}$ by its idempotent completion $(\mathrm{Ind}(\mathcal{C})^{\omega_1}/\mathcal{C})^{\mathrm{idem}}$. By the cofinality theorem (Proposition 3.1.13) the induced map

$$\mathrm{k}(\mathrm{Ind}(\mathcal{C})^{\omega_1}/\mathcal{C}) \to \mathrm{k}((\mathrm{Ind}(\mathcal{C})^{\omega_1}/\mathcal{C})^{\mathrm{idem}})$$

would be an equivalence on 1-connective covers, so that for an idempotent complete ∞-category with finite colimits we have that

$$\mathrm{k}(\mathcal{C}) \simeq \tau_{\geq 0} \Omega \mathrm{k}(\mathrm{Ind}(\mathcal{C})^{\omega_1}/\mathcal{C})^{\mathrm{idem}})$$

Now $\mathrm{k}_0(\mathrm{Ind}(\mathcal{C})^{\omega_1}/\mathcal{C})^{\mathrm{idem}})$ might be non-zero. This is how negative K-theory arises, namely this group is by definition $\mathrm{K}_{-1}(\mathcal{C})$.

Definition 3.2.7. For a small idempotent-complete category with finite colimits $\mathcal{C}$ let

$$\mathrm{Calk}(\mathcal{C}) \coloneqq (\mathrm{Ind}(\mathcal{C})^{\omega_1}/\mathcal{C})^{\mathrm{idem}} \ .$$

For a small ∞-category $\mathcal{C}$ with finite colimits and $n \geq 0$ we inductively define

$$\mathrm{Calk}^0(\mathcal{C}) \coloneqq \mathcal{C}^{\mathrm{idem}} \qquad \mathrm{Calk}^{n+1}(\mathcal{C}) \coloneqq \mathrm{Calk}(\mathrm{Calk}^n(\mathcal{C})) \ .$$

Clearly this construction gives rise to a functor

$$\mathrm{Calk} : \mathrm{Cat}_\infty^{\mathrm{rex}} \to \mathrm{Cat}_\infty^{\mathrm{rex,idem}}.$$

Proposition 3.2.8. *For every $n \geq 0$ we have canonical equivalences of connective spectra*

$$\tau_{\geq 0}\Omega\mathrm{k}(\mathrm{Calk}^{n+1}(\mathcal{C})) \simeq \mathrm{k}(\mathrm{Calk}^n(\mathcal{C})) \ .$$

Proof. Apply Example 3.2.6 to $\mathrm{Calk}^n(\mathcal{C})$. □

Definition 3.2.9. We define a non-connective K-theory spectrum $\mathrm{K}(\mathcal{C})$ as the spectrum with

$$\tau_{\geq -n}\mathrm{K}(\mathcal{C}) = \Omega^n\mathrm{k}(\mathrm{Calk}^n(\mathcal{C})) \ .$$

and $\mathrm{K}(\mathcal{C}) = \operatorname{colim} \tau_{\geq -n}\mathrm{K}(\mathcal{C})$ along the canonical maps. Equivalently, this is the spectrum with n-th underlying space given by $\Omega^\infty\mathrm{Calk}^n(\mathcal{C})$ and structure maps the underlying maps of Proposition 3.2.8. We define the algebraic K-theory groups as $\mathrm{K}_n(\mathcal{C}) \coloneqq \pi_n(\mathrm{K}(\mathcal{C}))$.

Example 3.2.10. For any ∞-category $\mathcal{C}$ with finite colimits we have that $\mathrm{k}_n(\mathcal{C}) = \mathrm{K}_n(\mathcal{C})$ for $n > 0$ and $\mathrm{k}_0(\mathcal{C}) \to \mathrm{K}_0(\mathcal{C})$ is injective. It is an isomorphism if $\mathcal{C}$ is idempotent complete. We have

$$\mathrm{K}_{-1}(\mathcal{C}) = \mathrm{k}_0((\mathrm{Ind}(\mathcal{C})^{\omega_1}/\mathcal{C})^{\mathrm{idem}})$$

so that this group is an obstruction to the idempotent completeness of $\mathrm{Ind}(\mathcal{C})^{\omega_1}/\mathcal{C}$ (we have seen in Example 3.2.6 that $\mathrm{k}_0(\mathrm{Ind}(\mathcal{C})^{\omega_1}/\mathcal{C})$ vanishes for idempotent complete $\mathcal{C}$). In many situations one can show that $\mathrm{Ind}(\mathcal{C})^{\omega_1}/\mathcal{C}$ is idempotent complete, e.g. of $\mathcal{C} = \mathrm{Perf}(R)$ for a regular ring, and then $K_{-1} = 0$. In fact one can then often even show that all negative K-groups vanish so that non-connective K-theory $\mathrm{K}(\mathcal{C})$ is even equivalent to $\mathrm{k}(\mathcal{C})$.

We will require the following Proposition to bootstrap our results from connective k-theory to non-connective theory K as defined above.

Proposition 3.2.11. *Using the terminology of Proposition 3.2.3, we have:*

1. *Given a filtered diagram $\mathcal{C}_\bullet : I \to \mathrm{Cat}^{\mathrm{rex,idem}}$, the filtered colimit of the natural type (2') sequences*

$$\operatorname{colim}_i \mathcal{C}_i \to \operatorname{colim}_i \mathrm{Ind}(\mathcal{C}_i)^{\omega_1} \to \operatorname{colim}_i \mathrm{Calk}(\mathcal{C}_i)$$

   is itself a type (2') sequence.

2. *The functor* $\mathrm{Calk} : \mathrm{Cat}^{\mathrm{rex,idem}} \to \mathrm{Cat}^{\mathrm{rex,idem}}$ *preserves sequences of type (5).*

3. *We have the following natural* k*-equivalences of endofunctors of* $\mathrm{Cat}^{\mathrm{rex,idem}}$*:*

$$\mathrm{Calk}(-)_+ \Rightarrow \mathrm{Calk}((-)_+) \qquad \textit{and} \qquad \mathrm{SW}(\mathrm{Calk}(-)) \Rightarrow \mathrm{Calk}(\mathrm{SW}(-)).$$

4. *The functor* $\mathrm{Calk} : \mathrm{Cat}^{\mathrm{rex,idem}} \to \mathrm{Cat}^{\mathrm{rex,idem}}$ *preserves filtered colimits up to* k*-equivalence.*

Proof.

1. Since the first functor will again be fully faithful with retract-closed image by Lemma 3.2.5, it suffices to check the following general statement: If $\mathcal{C}_\bullet : I \to \mathrm{Pr}^L_{\mathrm{ca}}$ is a filtered diagram, then the map

   $$\phi : \mathrm{colim}_i\, \mathcal{C}_i \to \mathrm{colim}_i\, \mathrm{Ind}(\mathcal{C}_i^{\omega_1}) \simeq \mathrm{Ind}(\mathrm{colim}_i\, \mathcal{C}_i^{\omega_1})$$

   induced by all the $\widehat{\jmath}$'s is again a strong left adjoint. Indeed, here we would only need to apply it to maps of the form $\mathrm{Ind}(\mathcal{C}_i) \xrightarrow{\mathrm{Ind}(j)=\widehat{\jmath}} \mathrm{Ind}(\mathrm{Ind}(\mathcal{C}_i)^{\omega_1})$.

   To see this, note that for each $i \to j$ in I, since the induced functor $\mathcal{C}_i \to \mathcal{C}_j$ is compactly assembled, we obtain the commutative rectangle on the left, and by passing to right adjoints the commutative rectangle on the right:

   $$\begin{tikzcd} \mathcal{C}_i \arrow[r, hook, "\widehat{\jmath}"] \arrow[d] & \mathrm{Ind}(\mathcal{C}_i^{\omega_1}) \arrow[r, "k"] \arrow[d] & \mathcal{C}_i \arrow[d] & \rightsquigarrow & \mathcal{C}_i & \mathrm{Ind}(\mathcal{C}_i^{\omega_1}) \arrow[l, "k"'] & \mathcal{C}_i \arrow[l, hook', "j"'] \\ \mathcal{C}_j \arrow[r, hook, "\widehat{\jmath}"] & \mathrm{Ind}(\mathcal{C}_i^{\omega_1}) \arrow[r, "k"] & \mathcal{C}_j & \rightsquigarrow & \mathcal{C}_j \arrow[u] & \mathrm{Ind}(\mathcal{C}_j^{\omega_1}) \arrow[l, "k"] \arrow[u] & \mathcal{C}_j \arrow[l, hook', "j"] \arrow[u] \end{tikzcd}$$

   Taking the limit of the right diagram and using that mapping spaces in a limit are computed as the limit of mapping spaces, we identify the limit of the j's as right adjoint to the limit of the k's. But the latter is by definition the right adjoint we are interested in, i.e. the right adjoint of ϕ, proving that ϕ is strongly left adjoint as claimed.

2. So suppose $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ induces a cofiber sequence on k-theory, and consider the following diagram:

   $$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{D} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathrm{Ind}(\mathcal{C})^{\omega_1} \arrow[r] \arrow[d] & \mathrm{Ind}(\mathcal{D})^{\omega_1} \arrow[r] \arrow[d] & \mathrm{Ind}(\mathcal{E})^{\omega_1} \arrow[d] \\ \mathrm{Calk}(\mathcal{C}) \arrow[r] & \mathrm{Calk}(\mathcal{D}) \arrow[r] & \mathrm{Calk}(\mathcal{E}) \end{tikzcd}$$

   All vertical maps induce cofiber sequences on k-theory, the top row does by assumption, and the middle row is sent to 0 by the Eilenberg swindle, hence also to a cofiber sequence. Since cofibers commute, it follows that the bottom row is also sent to a cofiber sequence, as desired.

3. We begin with stabilization. The natural map is induced by the universal property, as one can check that $\mathrm{Calk}(\mathrm{SW}(-))$ is stable. Now let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{rex,idem}}$ and consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{SW}(\mathcal{C}) \arrow[r] \arrow[d, equal] & \mathrm{SW}(\mathrm{Ind}(\mathcal{C})^{\omega_1}) \arrow[r] \arrow[d] & \mathrm{SW}(\mathrm{Calk}(\mathcal{C})) \arrow[d] \\
\mathrm{SW}(\mathcal{C}) \arrow[r] & \mathrm{Ind}(\mathrm{SW}(\mathcal{C}))^{\omega_1} \arrow[r] & \mathrm{Calk}(\mathrm{SW}(\mathcal{C}))
\end{tikzcd}$$

The bottom sequence is clearly type (2'), and the top one is as well by Proposition 3.2.3(i). So we obtain a diagram of cofiber sequences after applying connective k-theory. Moreover, both middle terms admit countable colimits, hence by the Eilenberg swindle and the 5-Lemma we see that the right vertical map is sent to an equivalence by k-theory.

For the pointed case, one can argue analogously. Alternatively, note that $\mathrm{SW}((-)_+) = \mathrm{SW}(-)$, hence using Proposition 3.1.12 and the above we have a sequence of natural k-equivalences

$$\mathrm{Calk}(-)_+ \sim_{\mathrm{k}} \mathrm{SW}(\mathrm{Calk}(-)) \sim_{\mathrm{k}} \mathrm{Calk}(\mathrm{SW}(-)) \sim_{\mathrm{k}} \mathrm{SW}(\mathrm{Calk}((-)_+)) \sim_{\mathrm{k}} \mathrm{Calk}((-)_+).$$

4. So let $\mathcal{C}_\bullet : I \to \mathrm{Cat}_\infty^{\mathrm{rex,idem}}$ be a filtered diagram with colimit $\mathcal{C} \coloneqq \mathrm{colim}_i\, \mathcal{C}_i$. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{colim}_i\, \mathcal{C}_i \arrow[r, hook] \arrow[d, "\simeq"'] & \mathrm{colim}_i\, \mathrm{Ind}(\mathcal{C}_i)^{\omega_1} \arrow[r] \arrow[d] & \mathrm{colim}_i\, \mathrm{Calk}(\mathcal{C}_i) \arrow[d] \\
\mathcal{C} \arrow[r, hook] & \mathrm{Ind}(\mathcal{C})^{\omega_1} \arrow[r] & \mathrm{Calk}(\mathcal{C})
\end{tikzcd}$$

The bottom sequence is of type (2') by definition, and the top one is too by Proposition 3.2.11(1), so that by Propositions 3.2.3 both are sent to cofiber sequences by k. Therefore the Eilenberg-Swindle and the 5-Lemma show that the right vertical map is sent to an equivalence by k.

$\square$

Corollary 3.2.12. *The functor* $\mathrm{K} : \mathrm{Cat}_\infty^{\mathrm{rex}} \to \mathrm{Sp}$ *enjoys the following properties:*

1. *It is invariant under idempotent completion. There is a natural map* $\mathrm{k}(\mathcal{C}) \to \mathrm{K}(\mathcal{C})$ *that induces an injection on* π_0 *and an isomorphism on* π_n *for* $n \geq 1$*. If* $\mathcal{C}$ *is already idempotent-complete, then* $\mathrm{k}(\mathcal{C}) \to \mathrm{K}(\mathcal{C})$ *is a connective cover.*

2. *It commutes with filtered colimits and finite products.*

3. *It satisfies an Eilenberg swindle: assume that there exists a functor* $F : \mathcal{C} \to \mathcal{C}$ *preserving finite colimits such that* $F \amalg \mathrm{id} \simeq F$*. Then* $\mathrm{K}(\mathcal{C}) \simeq 0$*.*

4. *If $S = (\mathcal{C} \to \mathcal{D} \to \mathcal{E})$ is a sequence in $\mathrm{Cat}^{\mathrm{rex,idem}}$ which induces a cofiber sequence on connective* k*-theory, then it will also induce a cofiber sequence on non-connective* K*-theory. In particular, if S is of any of the types discussed in Proposition 3.2.3, then* $\mathrm{K}(S)$ *is a cofiber sequence.*[1]

5. *It inverts the canonical maps $\mathcal{C} \to \mathcal{C}_+$ and $\mathcal{C} \to \mathrm{SW}(\mathcal{C})$.*

Note that given k*, the properties 1 and 4 already determine the functor* K *uniquely.*

Proof.

1. This is clear from the definition and cofinality (see Proposition 3.1.13).

2. By semiadditivity of $\mathrm{Cat}^{\mathrm{rex}}$ it is clear that Calk preserves finite products, hence the fact that K preserves finite products follows from the corresponding statement for connective k-theory.

   To see that K also commutes with filtered colimits, note that we can use (1) to reduce to considering idempotent complete categories. The claim then follows from the fact that k commutes with filtered colimits by Proposition 3.1.7, and Calk preserves filtered colimits up to k-equivalence by Proposition 3.2.11(4).

3. Since K commutes with finite products, it then follows that we have a map $\mathrm{K}(F) : \mathrm{K}(\mathcal{C}) \to \mathrm{K}(\mathcal{C})$ such that $\mathrm{K}(F) + \mathrm{id}_{\mathrm{K}(\mathcal{C})} = \mathrm{id}_{\mathrm{K}(\mathcal{C})}$ in $\pi_0(\mathrm{Map}_{\mathrm{Sp}}(\mathrm{K}(\mathcal{C}), \mathrm{K}(\mathcal{C})))$. Since the latter is a *group*, it follows that $\mathrm{id}_{\mathrm{K}(\mathcal{C})} = 0$, so $\mathrm{K}(\mathcal{C}) = 0$.

4. This follows from the fact that Calk preserves such sequences, see Proposition 3.2.11(2).

5. By Proposition 3.2.11(3) the functor Calk commutes with $(-)_+$ and SW up to k-equivalence, hence the claim follows from the analogous statements for connective k-theory, see Propositions 3.1.9 and 3.1.12.

□

3.3 The continuous Calkin category

In this section we would like to prove that there is a canonical extension

$$\begin{tikzcd} \mathrm{Cat}_\infty^{\mathrm{rex}} \arrow[r, "\mathrm{Calk}"] \arrow[d, hook, "\mathrm{Ind}"'] & \mathrm{Cat}_\infty^{\mathrm{rex}} \\ \mathrm{Pr}^L_{\mathrm{ca}} \arrow[ur, dashed, "\mathrm{Calk}^{\mathrm{cont}}"'] & \end{tikzcd}$$

[1]For sequences of types (1)-(4) one does not need to assume idempotent-completeness a priori to obtain a cofiber sequence on K, however for type (5) we do. This is related to which sequences are preserved by $(-)^{\mathrm{idem}}$.

called the continuous Calkin category. Let $\mathcal{C}$ be a compactly assembled ∞-category. Then we consider the cofiber sequence

$$\mathcal{C} \xrightarrow{\widehat{\jmath}} \mathrm{Ind}(\mathcal{C}^{\omega_1}) \to \mathrm{Ind}(\mathcal{C}^{\omega_1})/\mathcal{C}$$

in $\mathrm{Pr}^L_{\mathrm{ca}}$, i.e. the pushout against the terminal map $\mathcal{C} \to \mathrm{pt}$. Recall from Theorem 2.6.5 that cofibers in $\mathrm{Pr}^L_{\mathrm{ca}}$ are computed as the colimits in Pr^L and thus by the limit of the right adjoint diagram. In other words,

$$\mathrm{Ind}(\mathcal{C}^{\omega_1})/\mathcal{C} = \ker(k : \mathrm{Ind}(\mathcal{C}^{\omega_1}) \to \mathcal{C}) = k^{-1}(*).$$

Here ker again means the pullback against the unique map $\mathrm{pt} \to \mathcal{C}$ picking out the terminal object $* \in \mathcal{C}$. This gives us the description of $\mathrm{Ind}(\mathcal{C}^{\omega_1})/\mathcal{C} \subseteq \mathrm{Ind}(\mathcal{C}^{\omega_1})$ as the full subcategory spanned by those objects in $\mathrm{Ind}(\mathcal{C}^{\omega_1})$ with vanishing colimit.

Lemma 3.3.1. *The functor* $\mathrm{Ind}(\mathcal{C}^{\omega_1}) \to \mathrm{Ind}(\mathcal{C}^{\omega_1})/\mathcal{C}$ *is a Bousfield localization and* $\mathrm{Ind}(\mathcal{C}^{\omega_1})/\mathcal{C}$ *is compactly generated.*

Proof. The right adjoint is given by the fully faithful inclusion $\mathrm{Ind}(\mathcal{C}^{\omega_1})/\mathcal{C} \subseteq \mathrm{Ind}(\mathcal{C}^{\omega_1})$, so we have a Bousfield localization. The functor $\mathrm{Ind}(\mathcal{C}^{\omega_1}) \to \mathrm{Ind}(\mathcal{C}^{\omega_1})/\mathcal{C}$ is compactly assembled, hence sends compact objects to compact objects. Moreover, it sends generators to generators (since the right adjoint is conservative), hence it follows that $\mathrm{Ind}(\mathcal{C}^{\omega_1})/\mathcal{C}$ is generated by compacts. $\square$

Definition 3.3.2. For a compactly assembled ∞-category $\mathcal{C}$ we define a small ∞-category

$$\mathrm{Calk}^{\mathrm{cont}}(\mathcal{C}) := (\mathrm{Ind}(\mathcal{C}^{\omega_1})/\mathcal{C})^{\omega} \in \mathrm{Cat}^{\mathrm{rex}}_{\infty}.$$

Note that $\mathrm{Calk}^{\mathrm{cont}}(\mathcal{C})$ is automatically idempotent-complete.

Proposition 3.3.3. *For a compactly assembled ∞-category $\mathcal{C}$ there is a natural right exact functor $p : \mathcal{C}^{\omega_1} \to \mathrm{Calk}^{\mathrm{cont}}(\mathcal{C})$ which is a homological epimorphism*[2] *and we have a natural cofiber sequence*

$$\mathcal{C} \xrightarrow{\widehat{\jmath}} \mathrm{Ind}(\mathcal{C}^{\omega_1}) \xrightarrow{\mathrm{Ind}(p)} \mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}}(\mathcal{C}))$$

in $\mathrm{Pr}^L_{\mathrm{ca}}$. *Moreover,* $\mathrm{Ind}(p)^R$ *preserves pushouts (equivalently: weakly contractible colimits).*

Proof. By definition we have that $\mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}}(\mathcal{C})) = \mathrm{Ind}(\mathcal{C}^{\omega_1})/\mathcal{C}$. The functor p is induced by the structure map $\mathrm{Ind}(\mathcal{C}^{\omega_1}) \to \mathrm{Ind}(\mathcal{C}^{\omega_1})/\mathcal{C}$ which is compactly assembled, hence sends compact objects to compact objects. The only thing that remains to be shown is that the right adjoint of $\mathrm{Ind}(p)$ preserves pushouts. This right adjoint is given by the kernel inclusion

$$\mathrm{Ker}(k) \subseteq \mathrm{Ind}(\mathcal{C}^{\omega_1}).$$

Since both k and $\mathrm{pt} \to \mathrm{Ind}(\mathcal{C}^{\omega_1})$ preserve pushouts, it follows that also the above inclusion does (in fact, this is how colimits in a limit of categories are computed, compare [Lur17b, 5.4.5.5]). $\square$

[2] Here we use the notion of homological epimorphism slightly more generally than initially introduced to mean a functor on $\mathrm{Cat}^{\mathrm{rex}}_{\infty}$ that induced a Bousfield localization after Ind.

Remark 3.3.4. If $\mathcal{C}$ is dualizable (equivalently stable and compactly assembled) then also $\mathrm{Calk}^{\mathrm{cont}}(\mathcal{C})$ is stable and the whole discussion takes place in $\mathrm{Pr}^L_{\mathrm{dual}}$.

Lemma 3.3.5. *For $\mathcal{C} \in \mathrm{Cat}^{\mathrm{rex}}$ we have $\mathrm{Calk}^{\mathrm{cont}}(\mathrm{Ind}(\mathcal{C})) = \mathrm{Calk}(\mathcal{C})$.*

Proof. Note that by definition we have the natural cofiber sequence

$$\mathrm{Ind}(\mathcal{C}) \xrightarrow{\widehat{\jmath}} \mathrm{Ind}(\mathrm{Ind}(\mathcal{C})^{\omega_1}) \xrightarrow{\mathrm{Ind}(p)} \mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}}(\mathrm{Ind}(\mathcal{C}))).$$

However in this special case we have $\widehat{\jmath} = \mathrm{Ind}(j)$ for $j : \mathcal{C} \subseteq \mathrm{Ind}(\mathcal{C})^{\omega_1}$, so the entire sequence lies in $\mathrm{Pr}^L_\omega \simeq \mathrm{Cat}^{\mathrm{rex,idem}}_\infty$. Taking compact objects, we obtain

$$\mathrm{Calk}^{\mathrm{cont}}(\mathrm{Ind}(\mathcal{C})) = (\mathrm{Ind}(\mathcal{C})^{\omega_1}/\mathcal{C}^{\mathrm{idem}})^{\mathrm{idem}} = \mathrm{Calk}(\mathcal{C}).$$

(Recall that if $\mathcal{C}$ is not yet idempotent-complete then by definition $\mathrm{Calk}(\mathcal{C}) = \mathrm{Calk}(\mathcal{C}^{\mathrm{idem}})$). □

To gain a better understanding of the continuous Calkin construction, we will investigate similar cofiber sequences in $\mathrm{Pr}^L_{\mathrm{ca}}$ in general. More specifically, we want to gain a better understanding of the cofibers $\mathcal{D}/\mathcal{C}$ in such sequences. It will turn out that this works best for the Verdier cofiber sequences we now define:

Definition 3.3.6. A Verdier cofiber sequence is a cofiber sequence in $\mathrm{Pr}^L_{\mathrm{ca}}$

$$\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{E}$$

so that i is fully faithful and i^R preserves pushouts (hence weakly contractible colimits).

Note that this is the direct analogue of the Verdier cofiber sequences in $\mathrm{Cat}^{\mathrm{rex}}$. Let us first generalize the addendum of Proposition 3.3.3.

Lemma 3.3.7. *Let $\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a Verdier cofiber sequence. Then also p^R preserves weakly contractible colimits.*

Proof. Since i^R preserves weakly contractible colimits, it is clear that $\ker(i^R) = (i^R)^{-1}(*)$ is closed under weakly contractible colimits, i.e. p^R preserves them. □

Warning 3.3.8. *The analogue of Lemma 2.9.11 for the unstable setting fails; even if i is strongly left adjoint, p need not be. Indeed, even if i^R preserves all colimits, $(i^R)^{-1}(*)$ clearly need not be closed under e.g coproducts in $\mathcal{D}$. Specifically, if i^R preserves colimits, then $i^R(\emptyset) = \emptyset$, hence p^R preserves the initial object if and only if $\mathcal{C}$ is pointed.*

Moreover, let us mention another difference to the case of Verdier sequences in Pr^L:

Warning 3.3.9. *It is generally not true that a Verdier cofiber sequence* $\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{D}/\mathcal{C}$ *is also a fiber sequence (even if* i *is strongly left adjoint). In general, one can take the kernel in* Pr^L

$$\mathcal{K} = \ker(p) = p^{-1}(\emptyset)$$

which contains $\mathcal{C}$ *as a full subcategory, but it might be larger as the example below shows. We will give a more explicit description of* $\mathcal{K}$ *in Corollary 3.3.15 below. The induced sequence*

$$\mathcal{K} \to \mathcal{D} \to \mathcal{D}/\mathcal{C}$$

is also a cofiber sequence in Pr^L *as one checks by universal properties.*

We don't know whether $\mathcal{K}$ *is generally itself compactly assembled or even whether it agrees with the kernel in* $\mathrm{Pr}^L_{\mathrm{ca}}$*. So there might even be another* ∞*-category in between* $\mathcal{C}$ *and* $\mathcal{K}$ *which is the kernel in* Pr^L_{ca}*.*

Example 3.3.10. Let $\mathcal{C} = \mathrm{Grp}$ be the ordinary category of non-abelian groups. This is compactly generated, hence compactly assembled. We consider $\mathrm{Calk}^{\mathrm{cont}}(\mathrm{Grp})$ as a full subcategory of

$$\mathrm{Ind}(\mathrm{Grp}^{\omega_1})$$

given by the kernel of k. We claim that the constant Ind object on the group Σ_∞ lies in the kernel of $\mathrm{Ind}(\mathrm{Grp}^{\omega_1}) \to \mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}}(\mathrm{Grp}))$, in fact in the kernel of $p : \mathrm{Grp}^{\omega_1} \to \mathrm{Calk}^{\mathrm{cont}}(\mathrm{Grp})$, but clearly it does not lie in the image of

$$\mathrm{Grp} \xrightarrow{\hat{\jmath}} \mathrm{Ind}(\mathrm{Grp}^{\omega_1}),$$

since this functor is given by Ind of the inclusion $\mathrm{Grp}^{\omega} \subseteq \mathrm{Grp}^{\omega_1}$. To see that Σ_∞ lies in the kernel, we note that we have a cofiber sequence

$$\Sigma_2 \to \Sigma_\infty \to \{e\}$$

in Grp^{ω_1}. So applying p gives a cofiber sequence

$$\{e\} \to p(\Sigma_\infty) \to \{e\}$$

which proves the claim. We will see in Corollary 3.3.15 below that generally the kernel of the projection p in a Verdier cofiber sequence is closed under such 'extensions'.

As announced above, we will now take a closer look at the quotient $\mathcal{D}/\mathcal{C}$ in a Verdier cofiber sequence $\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{D}/\mathcal{C}$ in $\mathrm{Pr}^L_{\mathrm{ca}}$. More specifically, we wish to have a formula for mapping spaces in the quotient, analogous to the one for stable Verdier sequences. It becomes easier to work with $p : \mathcal{D} \to \mathcal{D}/\mathcal{C}$ when viewing it as an endofunctor

$$p^R p : \mathcal{D} \to \mathcal{D}/\mathcal{C} \subseteq \mathcal{D}.$$

The unit $D \to p^R pD$ exhibits $p^R pD$ as the universal object under D which lies in the kernel of i^R.

Lemma 3.3.11. *Let $\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a Verdier cofiber sequence in $\mathrm{Pr}^L_{\mathrm{ca}}$. Then there is a natural pushout square*

$$\begin{tikzcd} ii^R D \arrow[r, "\varepsilon_i"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & D \arrow[d, "\eta_p"] \\ i(*) \arrow[r] & p^R pD \end{tikzcd}$$

where the morphisms are given by the counit $\varepsilon_i : ii^R \to \mathrm{id}$ and the unit $\eta_p : \mathrm{id} \to p^R p$.

Proof. Since i is fully faithful and i^R preserves pushouts, we get

$$i^R \left(D \amalg_{ii^R D} i(*)\right) = i^R D \amalg_{i^R D} * = *,$$

hence the pushout lies in the kernel of i^R, which is the image of p^R. In order to verify the universal property, assume that we have an object $K \in \ker(i^R)$. Then we compute

$$\begin{aligned} \mathrm{Map}_{\mathcal{D}}(D \amalg_{ii^R D} i(*), K) &= \mathrm{Map}_{\mathcal{D}}(D, K) \times_{\mathrm{Map}_{\mathcal{D}}(ii^R D, K)} \mathrm{Map}_{\mathcal{D}}(i(*), K) \\ &= \mathrm{Map}_{\mathcal{D}}(D, K) \times_{\mathrm{Map}_{\mathcal{C}}(i^R D, *)} \mathrm{Map}_{\mathcal{C}}(*, *) \\ &= \mathrm{Map}_{\mathcal{D}}(D, K), \end{aligned}$$

which finishes the proof. □

Corollary 3.3.12. *Let $\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{D}/\mathcal{C}$ be a Verdier cofiber sequence in $\mathrm{Pr}^L_{\mathrm{ca}}$. Then:*

1. *for $D, D' \in \mathcal{D}$ the functor $p : \mathcal{D} \to \mathcal{D}/\mathcal{C}$ induces an equivalence of mapping spaces*

$$p : \mathrm{Map}_{\mathcal{D}}(D, D' \amalg_{ii^R D'} i(*)) \xrightarrow{\simeq} \mathrm{Map}_{\mathcal{D}/\mathcal{C}}(pD, pD').$$

2. *We have $i(*) \underset{\simeq}{\xrightarrow{\eta}} p^R pi(*) = p^R \emptyset$, so $i(*)$ represents the initial object in $\mathcal{D}/\mathcal{C}$.*

3. *the kernel $p^{-1}(\emptyset)$ of $p : \mathcal{D} \to \mathcal{D}/\mathcal{C}$ (in Pr^L) is given by those objects $D \in \mathcal{D}$ for which the bottom map in the canonical pushout square is an equivalence:*

$$\begin{tikzcd} ii^R D \arrow[r, "\varepsilon_i"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & D \arrow[d] \\ i(*) \arrow[r, "\simeq"'] & p^R pD \end{tikzcd}$$

4. *The projection $p : \mathcal{D} \to \mathcal{D}/\mathcal{C}$ is a localization at those morphisms $f : D \to D'$ which have the property that the morphism $c : ii^R D' \amalg_{ii^R D} D \to D'$ becomes an equivalence after base-change along $ii^R D' \to i(*)$. The morphism c measures the deviation from the following square being a pushout:*

$$\begin{tikzcd} ii^R D \arrow[r] \arrow[d] & D \arrow[d] \\ ii^R D' \arrow[r] & D' \end{tikzcd}$$

In particular, if this square is a pushout, then we have a local equivalence.

Proof. The first three follow immediately from the above Lemma. For the fourth, simply consider the diagram of iterated pushouts:

$$\begin{tikzcd}
ii^R D \arrow[r] \arrow[d] & D \arrow[d] & \\
ii^R D' \arrow[r] \arrow[d] \arrow[ur, phantom, "\ulcorner" very near end] & ii^R D' \amalg_{ii^R D} D \arrow[r, "c"] \arrow[d] & D' \arrow[d] \\
i(*) \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & p^R p D \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & p^R p D
\end{tikzcd}$$

□

Let $\mathcal{C}$ be compactly assembled and consider the canonical Verdier cofiber sequence.

$$\mathcal{C} \xrightarrow{\widehat{\jmath}} \mathrm{Ind}(\mathcal{C}^{\omega_1}) \xrightarrow{\mathrm{Ind}(p)} \mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}}(\mathcal{C}))$$

with $p : \mathcal{C}^{\omega_1} \to \mathrm{Calk}^{\mathrm{cont}}(\mathcal{C})$ (recall $\widehat{\jmath}$ is even strongly left adjoint). For $X, Y \in \mathcal{C}^{\omega_1}$ we have

$$\mathrm{Map}_{\mathrm{Calk}^{\mathrm{cont}}(\mathcal{C})}(pX, pY) = \mathrm{Map}_{\mathrm{Ind}(\mathcal{C}^{\omega_1})}(jX, jY \amalg_{\widehat{\jmath}Y} \widehat{\jmath}(*)).$$

In the stable world this yields the following description of mapping spectra, which explains the relation to the Calkin algebra in functional analysis.

Corollary 3.3.13. *If $\mathcal{C}$ is dualizable, then for $X, Y \in \mathcal{C}^{\omega_1}$ we have*

$$\mathrm{map}_{\mathrm{Calk}^{\mathrm{cont}}(\mathcal{C})}(pX, pY) = \mathrm{map}_{\mathcal{C}}(X, Y)/\,\mathrm{map}^{\mathrm{ca}}_{\mathcal{C}}(X, Y).$$

Proof. Observe that in the stable case we have $\widehat{\jmath}(*) = \widehat{\jmath}0 = 0$ and that the formula for the mapping anima immediately implies the same formula for the mapping spectra since everything in sight is exact (e.g. replace X be shifts and observe the formulas are exact in X):

$$\begin{aligned}
\mathrm{map}_{\mathrm{Calk}^{\mathrm{cont}}(\mathcal{C})}(pX, pY) &= \mathrm{map}_{\mathrm{Ind}(\mathcal{C}^{\omega_1})}(jX, jY/\widehat{\jmath}Y) \\
&= \mathrm{map}_{\mathrm{Ind}(\mathcal{C}^{\omega_1})}(jX, jY)/\,\mathrm{map}_{\mathrm{Ind}(\mathcal{C}^{\omega_1})}(jX, \widehat{\jmath}Y) \\
&= \mathrm{map}_{\mathcal{C}}(X, Y)/\,\mathrm{map}^{\mathrm{ca}}_{\mathcal{C}}(X, Y).
\end{aligned}$$

□

The above also lets us get a better understanding of the kernel of p. To describe it, we will need the following definition.

Definition 3.3.14. Let $\mathcal{C} \subseteq \mathcal{D}$ be a fully faithful inclusion of ∞-categories. We say that it is closed under extensions if for every pushout

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}$$

in $\mathcal{D}$ with $A, C, D \in \mathcal{C}$ we also have $B \in \mathcal{C}$.

Given any full subcategory $\mathcal{C} \subseteq \mathcal{D}$ we can form the smallest subcategory $\overline{\mathcal{C}} \subseteq \mathcal{D}$ closed under extensions and pushouts.

Corollary 3.3.15. *Let $\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{D}/\mathcal{C}$ be a Verdier cofiber sequence in $\mathrm{Pr}^L_{\mathrm{ca}}$. Then the kernel of p is the smallest subcategory of $\mathcal{D}$ containing $i(\mathcal{C})$ which is closed under extensions and pushouts. In fact, every object of the kernel is given by a single extension.*

Proof. It is immediate from Corollary 3.3.12(3) that every D in the kernel of p is an extension of objects in the image of i. Conversely it is clear that the kernel is closed under these sort of extensions. □

3.4 Efimov K-theory

In the stable setting, all results in this section are due to Efimov.

Definition 3.4.1. Let $\mathcal{C}$ be a compactly assembled category. We define its continuous K-theory as

$$\mathrm{K}^{\mathrm{cont}}(\mathcal{C}) \coloneqq \Omega\mathrm{K}(\mathrm{Calk}^{\mathrm{cont}}(\mathcal{C})).$$

We can reduce this to the connective k-theory as before. Namely, for $\mathcal{C}$ compactly assembled and $n \geq 1$, we define

$$\mathrm{Calk}^{\mathrm{cont},n}(\mathcal{C}) \coloneqq \mathrm{Calk}^{n-1}(\mathrm{Calk}^{\mathrm{cont}}(\mathcal{C})).$$

Then $\mathrm{K}^{\mathrm{cont}}(\mathcal{C})$ is the spectrum with

$$\mathrm{K}^{\mathrm{cont}}(\mathcal{C})_n = \mathrm{k}(\mathrm{Calk}^{\mathrm{cont},n}(\mathcal{C}))$$

for $n \geq 1$. However, there is no immediate formula for $n = 0$, as we already have to use a Calkin construction once to bring us from the big to the small world.

Theorem 3.4.2. *The functor $\mathrm{K}^{\mathrm{cont}} : \mathrm{Pr}^L_{\mathrm{ca}} \to \mathrm{Sp}$ enjoys the following properties:*

1. *There is a natural map $\mathrm{K}(\mathcal{C}^\omega) \to \mathrm{K}^{\mathrm{cont}}(\mathcal{C})$ which is an equivalence if $\mathcal{C}$ is compactly generated.*
2. *It commutes with filtered colimits and finite products.*
3. *It satisfies an Eilenberg swindle: assume that there exists a compactly assembled functor $F : \mathcal{C} \to \mathcal{C}$ such that $F \amalg \mathrm{id} \simeq F$. Then $\mathrm{K}^{\mathrm{cont}}(\mathcal{C}) \simeq 0$.*
4. *If $S = (\mathcal{C} \to \mathcal{D} \to \mathcal{E})$ is a sequence in $\mathrm{Pr}^L_{\mathrm{ca}}$ which becomes a Verdier upon stabilization (e.g. a Verdier cofiber sequence) then it is sent to a cofiber sequence by $\mathrm{K}^{\mathrm{cont}}$.*
5. *It inverts the canonical maps $\mathcal{C} \to \mathrm{An}_* \otimes \mathcal{C}$ and $\mathcal{C} \to \mathrm{Sp} \otimes \mathcal{C}$.*

Note that given K, the functor $\mathrm{K}^{\mathrm{cont}}$ is uniquely determined by the first two points.

The proof of this theorem will follow immediately from the analogous results about the interaction of the Calkin constructions with various kinds of cofiber sequences. Namely, the following result is an analogue of Proposition 3.2.3 for the world of large categories, and we will also see in what generality the constructions Ind and $\mathrm{Calk}^{\mathrm{cont}}$ passing between these worlds preserve such cofiber sequences.

Proposition 3.4.3. *Let $S = (\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{E})$ be a sequence in $\mathrm{Pr}^L_{\mathrm{ca}}$ and consider the following statements:*

1. *S is a cofiber sequence and i is fully faithful and preserves the terminal object.*

2. *S is a Verdier cofiber sequence, i.e. it is a cofiber sequence with i fully faithful and i^R preserving pushouts.*

2' *S is a cofiber sequence and i is a fully faithful strong left adjoint.*

3. *$\mathrm{An}_* \otimes S = (\mathcal{C}_* \to \mathcal{D}_* \to \mathcal{E}_*)$ is a cofiber sequence and $\mathrm{An}_* \otimes i$ is fully faithful.*

4. *$\mathrm{Sp} \otimes S = (\mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{D}) \to \mathrm{Sp}(\mathcal{E}))$ is a Verdier sequence, i.e. it is a cofiber sequence and $\mathrm{Sp} \otimes i$ is fully faithful.*

5. *$\mathrm{K}^{\mathrm{cont}}(S) = (\mathrm{K}^{\mathrm{cont}}(\mathcal{C}) \to \mathrm{K}^{\mathrm{cont}}(\mathcal{D}) \to \mathrm{K}^{\mathrm{cont}}(\mathcal{E}))$ is a cofiber sequence.*

Then the following implications hold:

$$\begin{tikzcd}
 & (2') \arrow[r, Rightarrow] & (2) \arrow[d, Rightarrow] & & \\
 & (1) \arrow[r, Rightarrow] & (3) \arrow[r, Rightarrow] & (4) \arrow[r, Rightarrow] & (5)
\end{tikzcd}$$

Moreover, we have:

(i) If $\mathcal{C}, \mathcal{D}, \mathcal{E}$ are pointed, then (1) $\Leftrightarrow$ (3) and (2) $\Leftrightarrow$ (2'). If $\mathcal{C}, \mathcal{D}, \mathcal{E}$ are stable, then all implications except (4) $\Rightarrow$ (5) become equivalences. In particular, $\mathrm{An}_* \otimes -$ and $\mathrm{Sp} \otimes -$ preserve sequences of all types.

(ii) The natural cofiber sequence defining $\mathrm{Calk}^{\mathrm{cont}}$ is of type (2'):

$$\mathcal{C} \xrightarrow{\hat{\jmath}} \mathrm{Ind}(\mathcal{C}^{\omega_1}) \to \mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}}(\mathcal{C})).$$

(iii) Sequences of type (3), (4) and (5) are stable under filtered colimits. A filtered colimit of the above natural type (2') sequences is again a sequence of type (2').

(iv) $\mathrm{Ind} : \mathrm{Cat}^{\mathrm{rex,idem}} \to \mathrm{Pr}^L_{\mathrm{ca}}$ preserves sequences of all types[3] and $\mathrm{Calk}^{\mathrm{cont}} : \mathrm{Pr}^L_{\mathrm{ca}} \to \mathrm{Cat}^{\mathrm{rex,idem}}$ preserves Verdier cofiber sequences. As a consequence, also Calk preserves Verdier cofiber sequences.

[3]Here idempotent-completeness is only important for type (5) sequences.

(v) We have the following natural K*-equivalences of functors* $\mathrm{Pr}^L_{\mathrm{ca}} \to \mathrm{Cat}^{\mathrm{rex,idem}}$*:*

$$\mathrm{Calk}^{\mathrm{cont}}(-)_+ \Rightarrow \mathrm{Calk}^{\mathrm{cont}}(\mathrm{An}_* \otimes -) \qquad \textit{and} \qquad \mathrm{SW}(\mathrm{Calk}^{\mathrm{cont}}(-)) \Rightarrow \mathrm{Calk}^{\mathrm{cont}}(\mathrm{Sp} \otimes -).$$

(vi) $\mathrm{Calk}^{\mathrm{cont}}$ *preserves filtered colimits up to* K*-equivalence.*

Proof. The proof of all implications except (4) $\Rightarrow$ (5) is exactly as in the small case, see Proposition 3.2.3.

(iv) That Ind preserves these sequences is obvious. So consider a Verdier cofiber sequence $\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{E}$ in $\mathrm{Pr}^L_{\mathrm{ca}}$. Since applying $\mathrm{Ind}((-)^{\omega_1})$ gives another Verdier cofiber sequence by Proposition 3.4.4 below, the fact that cofibers commute tells us that

$$\begin{aligned}\mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}}(\mathcal{E})) &= \mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}}(\mathcal{D}))/\,\mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}}(\mathcal{C}))\\ &= \mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}}(\mathcal{D})/\mathrm{Calk}^{\mathrm{cont}}(\mathcal{C}))\end{aligned}$$

Thus $\mathrm{Calk}^{\mathrm{cont}}(\mathcal{C}) \to \mathrm{Calk}^{\mathrm{cont}}(\mathcal{D}) \to \mathrm{Calk}^{\mathrm{cont}}(\mathcal{E})$ is again a cofiber sequence.

Since $\mathrm{Calk}^{\mathrm{cont}}$ lands in idempotent-complete categories, it remains to see that $\mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}}(i))$ is fully faithful and that its right adjoint preserves pushouts. Since i is compactly assembled and we can identify $\mathrm{Ind}(p)^R : \mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}}(\mathcal{C})) \to \mathrm{Ind}(\mathcal{C}^{\omega_1})$ with the inclusion $\mathrm{Ker}(k) \subseteq \mathrm{Ind}(\mathcal{C}^{\omega_1})$, we see that $\mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}}(i))$ and its right adjoint are obtained via restriction from $\mathrm{Ind}(i) \dashv \mathrm{Ind}(i)^R$:

$$\begin{tikzcd} \mathcal{C} & \mathrm{Ind}(\mathcal{C}^{\omega_1}) \arrow[l, "k"'] & \mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}}(\mathcal{C})) \arrow[l, hook', "\mathrm{Ind}(p)^R"'] \arrow[d, dashed, "\mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}}(i))"'] \\ \mathcal{D} \arrow[from=u, hook, "i"'] & \mathrm{Ind}(\mathcal{D}^{\omega_1}) \arrow[l, "k"] \arrow[from=u, hook, "\mathrm{Ind}(i)"'] & \mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}}(\mathcal{D})) \arrow[l, hook', "\mathrm{Ind}(p)^R"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} & \mathrm{Ind}(\mathcal{C}^{\omega_1}) \arrow[l, "k"'] & \mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}}(\mathcal{C})) \arrow[l, hook', "\mathrm{Ind}(p)^R"'] \\ \mathcal{D} \arrow[u, "i^R"] & \mathrm{Ind}(\mathcal{D}^{\omega_1}) \arrow[l, "k"] \arrow[u, "\mathrm{Ind}(i)^R"'] & \mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}}(\mathcal{D})) \arrow[l, hook', "\mathrm{Ind}(p)^R"] \arrow[u, dashed, "\mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}}(i))^R"'] \end{tikzcd}$$

By the left rectangle, $\mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}}(i))$ and hence $\mathrm{Calk}^{\mathrm{cont}}(i)$ is fully faithful, and by the right one we see that its right adjoint preserves pushouts, as $\mathrm{Ind}(i)^R$ does by Proposition 3.4.4 below, and both $\mathrm{Ind}(p)^R$ do by Lemma 3.3.7.

(v) We can construct the diagram analogous to the one in the proof of Proposition 3.2.11(3) so that applying $\mathrm{Calk}^{\mathrm{cont}}$ yields via $\mathrm{Calk}^{\mathrm{cont}}(\mathrm{Ind}(-)) = \mathrm{Calk}$ (see Lemma 3.3.5) and (iv) the following diagram of horizontal Verdier cofiber sequences

$$\begin{array}{ccccc} \mathrm{Calk}^{\mathrm{cont}}(\mathrm{Sp}\otimes\mathcal{C}) & \hookrightarrow & \mathrm{Calk}(\mathrm{SW}(\mathcal{C}^{\omega_1})) & \longrightarrow & \mathrm{Calk}(\mathrm{SW}(\mathrm{Calk}^{\mathrm{cont}}(\mathcal{C}))) \\ \| & & \downarrow & & \downarrow \\ \mathrm{Calk}^{\mathrm{cont}}(\mathrm{Sp}\otimes\mathcal{C}) & \hookrightarrow & \mathrm{Calk}((\mathrm{Sp}\otimes\mathcal{C})^{\omega_1}) & \longrightarrow & \mathrm{Calk}(\mathrm{Calk}^{\mathrm{cont}}(\mathrm{Sp}\otimes\mathcal{C})) \end{array}$$

Again the middle categories admit an Eilenberg swindle, so the right vertical map gives an equivalence on K-theory by Corollary 3.2.12 and the 5-Lemma. Since $\mathrm{K}(\mathrm{Calk}(-)) = \Omega\mathrm{K}$, this yields the claim.

(4) ⇒ (5) So let $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ be a type (4) sequence. Note that by (v) and Corollary 3.2.12 we have natural equivalences

$$\Sigma \mathrm{K}^{\mathrm{cont}} \simeq \mathrm{K}(\mathrm{Calk}^{\mathrm{cont}}(-)) \simeq \mathrm{K}(\mathrm{SW}(\mathrm{Calk}^{\mathrm{cont}}(-))) \simeq \mathrm{K}(\mathrm{Calk}^{\mathrm{cont}}(\mathrm{Sp} \otimes -)).$$

Since tensoring with Sp turns the given sequence into a type Verdier cofiber sequence and $\mathrm{Calk}^{\mathrm{cont}}$ preserves these by (iv), the claim follows since K sends Verdier cofiber sequences in $\mathrm{Cat}^{\mathrm{rex,idem}}$ to cofiber sequences by Corollary 3.2.12(4).

(iii) For types (3) and (4) this follows from the fact that fully faithful functors are closed under filtered colimits in $\mathrm{Pr}^L_{\mathrm{ca}}$, see Corollary 2.5.12. For type (5), this follows from (vi) together with the fact that K preserves filtered colimits by Corollary 3.2.12(2).

For the specific type (2') natural sequences, it remains to check that if $\mathcal{C}_\bullet : I \to \mathrm{Pr}^L_{\mathrm{ca}}$ is a filtered diagram, then the map

$$\phi : \mathrm{colim}_i\, \mathcal{C}_i \to \mathrm{colim}_i\, \mathrm{Ind}(\mathcal{C}_i^{\omega_1}) \simeq \mathrm{Ind}(\mathrm{colim}_i\, \mathcal{C}_i^{\omega_1})$$

induced by all the $\widehat{\jmath}$'s is again a strong left adjoint. We have already done this in the proof of Proposition 3.2.11(1).

(i) The only non-trivial claims are that $\mathrm{An}_* \otimes -$ preserves Verdier cofiber and type (5) sequences and that $\mathrm{Sp} \otimes -$ preserves type (5) sequences. In fact, $\mathrm{An}_* \otimes -$ sends Verdier cofiber sequences to sequences of type (2'), by an analogous argument as in the case of small categories, see Proposition 3.2.3(i). That $\mathrm{An}_* \otimes -$ and $\mathrm{Sp} \otimes -$ preserve type (5) sequences follows from (v) and the invariance of K under $(-)_+$ and $\mathrm{SW}(-)$, see Corollary 3.2.12(5).

(vi) Let $\mathcal{C}_\bullet : I \to \mathrm{Pr}^L_{\mathrm{ca}}$ be a filtered diagram with colimit $\mathcal{C} \coloneqq \mathrm{colim}_i\, \mathcal{C}_i$. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{colim}_i\, \mathcal{C}_i \arrow[r] \arrow[d, equal] & \mathrm{Ind}(\mathrm{colim}_i\, \mathcal{C}_i^{\omega_1}) \arrow[r] \arrow[d] & \mathrm{Ind}(\mathrm{colim}_i\, \mathrm{Calk}^{\mathrm{cont}}(\mathcal{C}_i)) \arrow[d] \\
\mathcal{C} \arrow[r] & \mathrm{Ind}(\mathcal{C}^{\omega_1}) \arrow[r] & \mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}}(\mathcal{C}))
\end{tikzcd}$$

The bottom sequence is of type (2') by definition, and the top one is too by (iii)[4], hence $\mathrm{Calk}^{\mathrm{cont}}$ sends both to Verdier cofiber sequences by (iv). Again the middle terms then afford an Eilenberg swindle, so that by the 5-Lemma we see that $\mathrm{K}(\mathrm{Calk}^{\mathrm{cont}}(\mathrm{Ind}(-))) \simeq \mathrm{K}(\mathrm{Calk}(-)) \simeq \Omega \mathrm{K}$ inverts the map $\mathrm{colim}_i\, \mathrm{Calk}^{\mathrm{cont}}(\mathcal{C}_i) \to \mathrm{Calk}^{\mathrm{cont}}(\mathcal{C})$, as desired.

□

The following Proposition was needed in the above proof to deduce that $\mathrm{Calk}^{\mathrm{cont}}$ sends Verdier cofiber sequences in $\mathrm{Pr}^L_{\mathrm{ca}}$ to type Verdier cofiber sequences in $\mathrm{Cat}^{\mathrm{rex,idem}}$, which was a fact crucial to prove all the remaining results on $\mathrm{Calk}^{\mathrm{cont}}$ above. A stable version of this following result is due to Ramzi, see [Ram, Proposition A.26].

[4]Despite the mutual reference, this is not circular.

Proposition 3.4.4. *For uncountable regular κ, the functor $(-)^\kappa : \mathrm{Pr}^L_{\mathrm{ca}} \to \mathrm{Cat}^{\mathrm{rex}}_\infty$ preserves Verdier cofiber sequences.*

Proof. Consider a Verdier cofiber sequence $\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{E}$ in $\mathrm{Pr}^L_{\mathrm{ca}}$. It is clear that $i : \mathcal{C}^\kappa \to \mathcal{D}^\kappa$ is again fully faithful with image closed under retracts. Moreover, the right adjoint of $\mathrm{Ind}(i)$ factors as

$$\mathrm{Ind}(\mathcal{D}^\kappa) \xrightarrow{\mathrm{Ind}(i^R)} \mathrm{Ind}(\mathcal{C}) \xrightarrow{\ell^*} \mathrm{Ind}(\mathcal{C}^\kappa)$$

where $\ell : \mathcal{C}^\kappa \subseteq \mathcal{C}$ is the inclusion. Since i^R and hence $\mathrm{Ind}(i^R)$ preserve pushouts, and also ℓ^* preserves all colimits[5], it follows that $\mathrm{Ind}(i)^R$ preserves pushouts.

It remains to see that the canonical comparison functor $\phi : \mathcal{D}^\kappa/\mathcal{C}^\kappa \to \mathcal{E}^\kappa$ is an equivalence. Let $q : \mathcal{D}^\kappa \to \mathcal{D}^\kappa/\mathcal{C}^\kappa$ be the quotient map, so that $\phi q = p : \mathcal{D}^\kappa \to \mathcal{E}^\kappa$. Given $d, d' \in \mathcal{D}^\kappa$, the mapping space formula for Verdier quotients in $\mathrm{Pr}^L_{\mathrm{ca}}$ (Corollary 3.3.12) gives us a commutative diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{D}^\kappa/\mathcal{C}^\kappa}(qd, qd') \arrow[r, "j", "\simeq"'] \arrow[dd, "\phi"'] & \mathrm{Map}_{\mathrm{Ind}(\mathcal{D}^\kappa/\mathcal{C}^\kappa)}(\mathrm{Ind}(q)jd, \mathrm{Ind}(q)jd') \arrow[d, "\mathrm{Ind}(\phi)"'] & \mathrm{Map}_{\mathrm{Ind}(\mathcal{D}^\kappa)}(jd, jd' \amalg_{jii^Rd'} \mathrm{Ind}(i)(*)) \arrow[l, "\mathrm{Ind}(q)"', "\simeq"] \arrow[dl, "\mathrm{Ind}(p)"] \arrow[dd, "k"] \\
 & \mathrm{Map}_{\mathrm{Ind}(\mathcal{E}^\kappa)}(jpd, jpd') \arrow[d, "\simeq"', "k"] & \\
\mathrm{Map}_{\mathcal{E}^\kappa}(pd, pd') \arrow[r, "\simeq"'] & \mathrm{Map}_{\mathcal{E}}(pd, pd') & \mathrm{Map}_{\mathcal{D}}(d, d' \amalg_{ii^Rd'} i(*)) \arrow[l, "\simeq", "p"']
\end{tikzcd}$$

Since $\mathcal{C}$ is κ-compactly generated, we see that $j : \mathcal{C} \subseteq \mathrm{Ind}(\mathcal{C}^\kappa)$ preserves finite and κ-filtered colimits (c.f. Corollary 2.1.27). Write $* = \mathrm{colim}_\ell\, c_\ell \in \mathcal{C}$ as κ-filtered colimit of κ-compact objects c_ℓ. Then

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Ind}(\mathcal{D}^\kappa)}(jd, jd' \amalg_{jii^Rd'} \mathrm{Ind}(i)(*)) &\simeq \mathrm{colim}_\ell\, \mathrm{Map}_{\mathrm{Ind}(\mathcal{D}^\kappa)}(jd, j(d' \amalg_{ii^Rd'} ic_\ell)) \\
&\underset{\simeq}{\xrightarrow{k}} \mathrm{colim}_\ell\, \mathrm{Map}_{\mathcal{D}}(d, d' \amalg_{ii^Rd'} ic_\ell) \\
&\simeq \mathrm{Map}_{\mathcal{D}}(d, d' \amalg_{ii^Rd'} i(*)),
\end{aligned}$$

where in the last step we use that d is κ-compact. So the right vertical, and hence also the left vertical map in the above diagram is an equivalence, proving ϕ is fully faithful.

To see that ϕ is essentially surjective, note that we have a factorization

$$\begin{tikzcd}
\mathcal{D} \arrow[rr, two heads, "p"] \arrow[d, equal] & & \mathcal{E} \arrow[d, equal] \\
\mathrm{Ind}_\kappa(\mathcal{D}^\kappa) \arrow[r, "\mathrm{Ind}_\kappa(q)"] & \mathrm{Ind}_\kappa(\mathcal{D}^\kappa/\mathcal{C}^\kappa) \arrow[r, hook, "\mathrm{Ind}_\kappa(\phi)"] & \mathrm{Ind}_\kappa(\mathcal{E}^\kappa)
\end{tikzcd}$$

Since p is essentially surjective, also $\mathrm{Ind}_\kappa(\phi)$ is, i.e. it is an equivalence. Taking κ-compacts, we see that also ϕ is an equivalence (since everything is idempotent-complete). □

[5] by Lemma 2.1.35, ℓ^* (there called i^*) is the Ind-extension of the finite colimit preserving $R : \mathcal{C} \subseteq \mathrm{Ind}(\mathcal{C}^\kappa)$ from Corollary 2.1.27.

We can in fact be more general than K-theory. To this extent we would like to formalise the needed structure.

Definition 3.4.5. Let $\mathcal{D}$ be a stable ∞-category. A localizing invariant with values in $\mathcal{D}$ is a functor $F : \mathrm{Cat}_\infty^{\mathrm{rex}} \to \mathcal{D}$ satisfying the following properties:

1. $F(*) = 0$.
2. F sends Verdier cofiber sequences to cofiber sequences in $\mathcal{D}$.
3. F inverts the natural map $\mathcal{C} \to \mathrm{SW}(\mathcal{C})$.

A localizing invariant is called finitary if it moreover preserves filtered colimits. We denote the respective ∞-categories by

$$\mathrm{Loc}_\omega(\mathcal{D}) \subseteq \mathrm{Loc}(\mathcal{D}) \subseteq \mathrm{Fun}(\mathrm{Cat}_\infty^{\mathrm{rex}}, \mathcal{D}).$$

Analogously, a continuous localizing invariant is a functor $F : \mathrm{Pr}^L_{\mathrm{ca}} \to \mathcal{D}$ satisfying the analogous properties: It sends the point to 0, sends Verdier cofiber sequences to cofiber sequences in $\mathcal{D}$, and inverts the natural maps $\mathcal{C} \to \mathrm{Sp}(\mathcal{C})$. It is called finitary if it preserves filtered colimits. We denote the respective ∞-categories by

$$\mathrm{Loc}^{\mathrm{cont}}_\omega(\mathcal{D}) \subseteq \mathrm{Loc}^{\mathrm{cont}}(\mathcal{D}) \subseteq \mathrm{Fun}(\mathrm{Pr}^L_{\mathrm{ca}}, \mathcal{D}).$$

Remark 3.4.6. Note that for finitary invariants the condition that $\mathcal{C} \to \mathrm{SW}(\mathcal{C})$ gets sent to an equivalence is equivalent to the assertion that $\mathcal{C} \to \mathcal{C}_+$ get sent to an equivalence. This follows since $\mathrm{SW}(\mathcal{C}) = \mathrm{colim}(\mathcal{C}_+ \xrightarrow{\Sigma} \mathcal{C}_+ \xrightarrow{\Sigma} \cdots)$ and that Σ acts by -1 as a result of the localization property. A similar statement is true in the continuous world.

Also note that by virtue of their idempotence-invariance and the condition that $\mathcal{C} \to \mathrm{SW}(\mathcal{C})$ gets mapped to an equivalence, we easily see that localizing invariants are in fact determined on idempotent-complete stable ∞-categories, i.e. by their values on $\mathrm{Cat}_\infty^{\mathrm{perf}}$ respectively $\mathrm{Pr}^L_{\mathrm{dual}}$. Typically in the literature this is the generality in which they are defined and we will also sometimes adopt this perspective.

Remark 3.4.7. The condition (3) that localizing invariants are invariant under stabilization is so that we recover the classical notion in the literature. It might be interesting to consider the class of examples where this condition is left out, however we currently do not know of an interesting example.

Now conditions (2) and (3) together with the implications from Proposition 3.4.3 and the analogous results in the small world show that if $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ is a type (4) sequence, i.e. one whose stabilization is a Verdier sequence, then it will also be sent to a cofiber sequence by any localizing invariant.

Lemma 3.4.8. *Every localizing invariant and every continuous localizing invariant preserves finite products and satisfies the Eilenberg swindle.*

Proof. Consider the type (2') cofiber sequence

$$\mathcal{C} \hookrightarrow \mathcal{C} \times \mathcal{D} \to \mathcal{D}$$

This admits a splitting. It follows that every localizing and continuous localizing invariant preserves finite products. Moreover, the Eilenberg swindle is a formal consequence once products are preserved (since the target is additive). □

Theorem 3.4.9 (Efimov)**.** *The restriction along* $\mathrm{Ind} : \mathrm{Cat}_\infty^{\mathrm{rex}} \to \mathrm{Pr}^L_{\mathrm{ca}}$ *induces an equivalence*

$$\mathrm{Loc}^{\mathrm{cont}}(\mathcal{D}) \xrightarrow{\simeq} \mathrm{Loc}(\mathcal{D})$$

which also restricts to an equivalence

$$\mathrm{Loc}^{\mathrm{cont}}_\omega(\mathcal{D}) \xrightarrow{\simeq} \mathrm{Loc}_\omega(\mathcal{D})$$

We denote the inverse applied to a localizing invariant F by F^{cont}.

Proof. Given $F \in \mathrm{Loc}(\mathcal{D})$, the canonical cofiber sequence $\mathcal{C} \to \mathrm{Ind}(\mathcal{C}^{\omega_1}) \to \mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}}(\mathcal{C}))$ together with the Eilenberg swindle force us to define F^{cont} as

$$F^{\mathrm{cont}}(\mathcal{C}) \coloneqq \Omega F(\mathrm{Calk}^{\mathrm{cont}}(\mathcal{C}))$$

analogously to what we did for continuous K-theory. Clearly $F^{\mathrm{cont}}(\mathrm{Ind}(-)) = F$, and F^{cont} is completely determined by this. Using that $\mathrm{Calk}^{\mathrm{cont}}$ preserves Verdier cofiber sequences by Proposition 3.4.3(iv), the fact that F is localizing also yields that F^{cont} is localizing. Moreover, the analogous proof as for $\mathrm{K}^{\mathrm{cont}}$ in Theorem 3.4.2(4) shows that if F is finitary, then so is F^{cont}. □

In the next section, we will see an important example of a localizing invariant that is naturally defined on $\mathrm{Pr}^L_{\mathrm{dual}}$.

3.5 Topological Hochschild homology

One important localizing invariant is topological Hochschild homology and we will see how this extends to the dualizable world and to H-unital ring spectra. Let us first review the definition here.

For a ring spectrum R classical topological Hochschild homology is defined as

$$\mathrm{THH}(R) \simeq R \otimes_{R \otimes_{\mathbb{S}} R^{\mathrm{op}}} R \in \mathrm{Sp}$$

where R is considered as a left module over the ring $R \otimes_{\mathbb{S}} R^{\mathrm{op}}$ by the 'obvious' multiplication from both sides and as a right module over $R \otimes_{\mathbb{S}} R^{\mathrm{op}}$ by the flipped multiplication. For an R-bimodule M we also have the variant with coefficients in M defined as:

$$\mathrm{THH}(R; M) \simeq R \otimes_{R \otimes_{\mathbb{S}} R^{\mathrm{op}}} M \ .$$

so that we have $\mathrm{THH}(R) = \mathrm{THH}(R; R)$.

Topological Hochschild homology is an important invariant which is one of the key tools used nowadays to compute algebraic K-theory. The way this connects to K-theory is that there is a 'trace' map

$$K(R) \to \mathrm{THH}(R) \ .$$

We will not get into the computational aspect here but will discuss a definition of the trace in the next section.

Remark 3.5.1. When R is an algebra over some commutative base ring spectrum k then we can also form the variant relative to k defined as

$$\mathrm{THH}(R/k) \coloneqq R \otimes_{R \otimes_k R^{\mathrm{op}}} R$$

and similar with coefficients. These relative variants are k-modules. For example when k is a field and R a discrete k-algebra then this is the definition of classical Hochschild homology. We will not need this here.

We have the following lemma:

Lemma 3.5.2. *The spectrum* $\mathrm{THH}(R)$ *is equivalent to the geometric realization of a simplicial object* $\Delta^{\mathrm{op}} \to \mathrm{Sp}$ *given as*

$$\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} R \otimes R \otimes R \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} R \otimes R \substack{\longrightarrow\\\longrightarrow} R$$

with face maps $\partial_i : R^{\otimes n+1} \to R^{\otimes n}$ *for* $0 \leq i < n$ *given as*

$$\partial_i(r_0 \otimes \ldots \otimes r_n) = r_0 \otimes \ldots \otimes r_i r_{i+1} \otimes \ldots \otimes r_n \quad \textit{and} \qquad \partial_n(r_0 \otimes \ldots \otimes r_n) = r_n r_0 \otimes r_1 \otimes \ldots$$

Similarly we have that $\mathrm{THH}(R; M)$ *is given by the geometric realization of*

$$\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} R \otimes R \otimes M \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} R \otimes M \substack{\longrightarrow\\\longrightarrow} M$$

with similarly defined face maps.

Proof. Consider the resolution of R as a right $R \otimes_{\mathbb{S}} R^{\mathrm{op}}$-module:

$$\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} R^{\otimes 4} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} R^{\otimes 3} \substack{\longrightarrow\\\longrightarrow} R^{\otimes 2} \longrightarrow R$$

where the face maps multiply adjacent maps together and the $R \otimes_{\mathbb{S}} R^{\mathrm{op}}$ module structure is by multiplication from the outside (after flipping to make it a right mpdue). To see that this is a colimit either construct a extra degeneracy or note that the realization of this simplicial object obtained by forgetting the augmentation is simply $R \otimes_R R$. Now tensor this resolution with R (resp. M) over $R \otimes_{\mathbb{S}} R^{\mathrm{op}}$ to obtain the result. $\square$

The simplicial object from the last lemma is called the cyclic Bar construction. In the logic of the previous lemma the cyclic Bar construction arose as the base change of a simplicial diagram in R-bimodules. One can also give a direct construction of the cyclic Bar construction as follows: there is a universal symmetric monoidal ∞-category $\mathcal{A}$ containing an associative algebra object $A \in \mathcal{A}$. Universality means the following: for any symmetric monoidal ∞-category $\mathcal{C}$ and a symmetric monoidal functor $F : \mathcal{A} \to \mathcal{C}$ we get an induced algebra object $F(A) \in \mathrm{Alg}(\mathcal{C})$. This assignment defines a functor

$$\mathrm{Fun}^{\otimes}(\mathcal{A}, \mathcal{C}) \to \mathrm{Alg}(\mathcal{C})$$

which is an equivalence. Concretely $\mathcal{A}$ is the symmetric monoidal envelope of the associative operad, that is $\mathcal{A} = \mathrm{Ass}^{\otimes}_{\mathrm{act}}$, see [Lur17a]. Now the explicit formulas show that $\mathcal{A}$ is in fact a 1-category[6] and therefore a simplicial object in $\mathcal{A}$ can be defined by defining fact and degeneracy objects. Using this one can show that the cyclic Bar construction indeed defines a functor

$$\Delta^{\mathrm{op}} \to \mathcal{A}$$

and thus for any algebra corresponding to a functor $F : \mathcal{A} \to \mathcal{C}$ we get an induced simplicial object in $\mathcal{C}$.

Remark 3.5.3. One benefit of the cyclic Bar construction description of $\mathrm{THH}(R)$ over the tensor product description is that it comes with extra structure. Namely the cyclic Bar construction $\Delta^{\mathrm{op}} \to \mathcal{C}$ extends to cyclic objects, that is extends over a functor $\Delta^{\mathrm{op}} \to \Lambda^{\mathrm{op}}$ where Λ is Connes' cyclic category. We will not get into the details here, but one result is that $\mathrm{THH}(R)$ carries an action of S^1.

One immediate observation is that for the definition involving the cyclic Bar construction we do not need the unit of R. The unit only contributes to the degeneracy maps of the diagram. Indeed, one can form a non-unital version of it using a non-unital universal category in place of $\mathcal{A}$ as above.

Definition 3.5.4. Let R be an H-unital ring spectrum. Define $\mathrm{THH}(R)$ to be the realization of the (semi-simplicial) cyclic Bar construction

$$\mathrm{THH}(R) = \mathrm{colim}_{\Delta^{\mathrm{op}}_{\mathrm{inj}}} \left(\cdots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} R^{\otimes 3} \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} R^{\otimes 2} \substack{\longrightarrow\\ \longrightarrow} R \right) .$$

The reason for restricting to H-unital ring spectra here is that only for H-unital ring spectra $\mathrm{THH}(R)$ has good formal properties and can be modified in a reasonable way. For the example we have the next statement:

Lemma 3.5.5. *For an H-unital ring spectrum R we have*

$$\mathrm{THH}(R) \simeq R \otimes_{R^+ \otimes_{\mathbb{S}} (R^+)^{\mathrm{op}}} R \simeq \mathrm{THH}(R^+; R).$$

[6] Concretely it is the 1-category whose objects are finite sets and whose morphisms are maps of finite sets together with a linear order on preimages on points.

Proof. We have that

$$R = R \otimes_{R^+} R = R \otimes_R R \tag{3.2}$$

where the latter is a non-unital relative tensor product given as

$$\dots \Rrightarrow R^{\otimes 4} \Rrightarrow R^{\otimes 3} \rightrightarrows R^{\otimes 2}$$

The first equivalence in (3.2) is the definition of H-unitality and the second follows by left Kan extension along $\Delta^{\mathrm{op}}_{\mathrm{inj}} \to \Delta^{\mathrm{op}}$. More precisely, we claim that for a left R-module M and a right R-module N the Bar construction diagram

$$\dots \Rrightarrow M \otimes R^+ \otimes R^+ \otimes N \Rrightarrow M \otimes R^+ \otimes N \rightrightarrows M \otimes N$$

is the left Kan extension of the non-unital Bar construction

$$\dots \Rrightarrow M \otimes R \otimes R \otimes N \Rrightarrow M \otimes R \otimes N \rightrightarrows M \otimes N$$

which is a semi-simplicial object. This readily follows from the formula for the left Kan extension along $\Delta^{\mathrm{op}}_{\mathrm{inj}} \to \Delta^{\mathrm{op}}$ which in level $[n] \in \Delta^{\mathrm{op}}$ is given by the coproduct over the set of surjections $[n] \twoheadrightarrow [k]$ in Δ^{op}, by a cofinality argument.

This gives us two different resolutions of R as an $R^+ - R^+$-bimodule, namely

$$\dots \Rrightarrow R^{\otimes 4} \Rrightarrow R^{\otimes 3} \rightrightarrows R^{\otimes 2} \longrightarrow R$$

and

$$\dots \Rrightarrow R \otimes R^+ \otimes R^+ \otimes R \Rrightarrow R \otimes R^+ \otimes R \rightrightarrows R \otimes R \longrightarrow R$$

Using the fact that for any spectrum M we have

$$(R \otimes M \otimes R)_{R^+ \otimes (R^+)^{\mathrm{op}}} R = M \otimes R,$$

base-changing the first resolution allows us to interpret $R \otimes_{R^+ \otimes (R^+)^{\mathrm{op}}} R$ as the definition of $\mathrm{THH}(R)$, whereas the second then gives the definition of $\mathrm{THH}(R^+; R)$. Clearly it suffices to show the mentioned fact for $M = \mathbb{S}$, where it is now a consequence of H-unitality and cofinality of the diagonal for Δ^{op}:

$$\begin{aligned} |R_1 \otimes R_2 \otimes (R^+ \otimes (R^+)^{\mathrm{op}})^{\otimes \bullet} \otimes R_3| &\simeq |R_1 \otimes (R^+)^{\otimes \bullet} \otimes R_3 \otimes (R^+)^{\otimes \bullet} \otimes R_2| \\ &\simeq R_1 \otimes_{R^+} R_3 \otimes_{R^+} R_2 \\ &\simeq R \end{aligned}$$

where we labeled the different copies of R for clarity on how to permute them. $\square$

Recall that for a dualizable object $\mathcal{C}$ in an ambient symmetric monoidal ∞-category $\mathcal{X}$ the dimension is defined as the endomorphism

$$\dim_{\mathcal{X}}(\mathcal{C}) : \mathbb{1} \to \mathbb{1}$$

which is given as $\mathrm{ev} \circ \mathrm{coev} : \mathbb{1} \to \mathcal{C} \otimes \mathcal{C}^\vee \to \mathbb{1}$.[7] More generally for a morphisms $F : \mathcal{C} \to \mathcal{C}$ in $\mathcal{X}$ the trace is given by

$$\mathrm{tr}_{\mathcal{X}}(F) : \mathbb{1} \to \mathbb{1}$$

which is given as

$$\mathrm{ev} \circ (F \otimes \mathrm{id}) \circ \mathrm{coev} : \mathbb{1} \to \mathcal{C} \otimes \mathcal{C}^\vee \to \mathcal{C} \otimes \mathcal{C}^\vee \to \mathbb{1}$$

By definition we have $\dim_{\mathcal{X}}(\mathcal{C}) = \mathrm{tr}_{\mathcal{X}}(\mathrm{id}_{\mathcal{C}})$. In the case that we work in the ambient category $\mathrm{Pr}^L_s t$ we have that $\mathbb{1} = \mathrm{Sp}$ and a map $\mathrm{Sp} \to \mathrm{Sp}$ is given by a spectrum itself.

Definition 3.5.6. For a dualizable ∞-category $\mathcal{C}$ we define continuous THH as

$$\mathrm{THH}^{\mathrm{cont}}(\mathcal{C}) = \dim_{\mathrm{Pr}^L_{\mathrm{st}}}(\mathrm{Sp}(\mathcal{C})) \in \mathrm{Sp}$$

where dim is meant as the dimension in the dualizable sense. For a functor $F : \mathcal{C} \to \mathcal{C}$ in $\mathrm{Pr}^L_{\mathrm{st}}$ we define

$$\mathrm{THH}^{\mathrm{cont}}(\mathcal{C}; F) = \mathrm{tr}_{\mathrm{Pr}^L_{\mathrm{st}}}(F) \ .$$

Theorem 3.5.7. *For R an H-unital ring spectrum we have a canonical equivalence*

$$\mathrm{THH}^{\mathrm{cont}}(\mathrm{Mod}_H(R)) \simeq \mathrm{THH}(R) \ .$$

Proof. We claim that the dual of $\mathrm{Mod}_H(R)$ is given by $\mathrm{Mod}_H(R^{\mathrm{op}})$. The tensor product of $\mathrm{Mod}_H(R)$ and $\mathrm{Mod}_H(S)$ is given by

$$\mathrm{Mod}_{H-H}(R, S) \subseteq \mathrm{Mod}(R^+ \otimes S^+)$$

which is the full subcategory given by those $R^+ \otimes S^+$-modules that are H-unital separately as an R-module and as an S-module separately. This follows from the description of functors

$$\mathrm{Mod}_H(R) \to \mathcal{C}$$

for any $\mathcal{C} \in \mathrm{Pr}^L_{\mathrm{st}}$ as H-unital R-modules in $\mathcal{C}$. Under this equivalence the evaluation and coevaluation of the duality are given by the functors

$$\mathrm{coev} : \mathrm{Sp} \to \mathrm{Mod}_{H-H}(R, R^{\mathrm{op}})$$

sending $\mathbb{S}$ to R and conversely the functor $\mathrm{Mod}_{H-H}(R, R^{\mathrm{op}}) \to \mathrm{Sp}$ which is determined by the $R^{\mathrm{op}} - R$-module R, so by Morita-Theory given by $R \otimes_{(R^+)^{\mathrm{op}} \otimes R^+} -$. To see that these are indeed evaluation and coevaluation we simply verify the snake identities: for the first note that the functors

$$\mathrm{Mod}_H(R) \xrightarrow{\mathrm{coev} \otimes \mathrm{id}} \mathrm{Mod}_{H-H-H}(R, R^{\mathrm{op}}, R) \xrightarrow{\mathrm{id} \otimes \mathrm{ev}} \mathrm{Mod}_H(R)$$

[7] Technically one has to twist $\mathcal{C} \otimes \mathcal{C}^\vee \simeq \mathcal{C}^\vee \otimes \mathcal{C}$ to match the domain of $\mathrm{ev} : \mathcal{C}^\vee \otimes \mathcal{C} \to \mathbb{1}$. However, in the *symmetric* monoidal case, this is no issue and just clutters the notation.

in the composition are given as follows. The first is given by sending $M \in \mathrm{Mod}_H(R)$ to $R \otimes M \in \mathrm{Mod}_{H-H-H}(R, R^{\mathrm{op}}, R)$ where the first two R's act on the first tensor factor and the last on the second. The second functor is given by sending $N \in \mathrm{Mod}_{H-H-H}(R, R^{\mathrm{op}}, R)$ to $R \otimes_{(R^+)^{\mathrm{op}} \otimes R^+} N \in \mathrm{Mod}_H(R)$ where for the tensor product the second and third actions are used. Thus the composite of these two functors is given by

$$M \mapsto R \otimes_{(R^+)^{\mathrm{op}} \otimes R^+} (R \otimes M) = M$$

The other ZigZag identity works similar.

Now finally in order to compute the dimension we simply have to compute the composition

$$\mathrm{Sp} \to \mathrm{Mod}_{H-H}(R, R^{\mathrm{op}}) \to \mathrm{Sp}$$

which then directly yields the claim. □

We have not said anything about the functoriality of $\mathrm{THH}^{\mathrm{cont}}(\mathcal{C}; F)$ but this is our next goal. Note that for fixed $\mathcal{C}$ it is obviously functorial in F.

Lemma 3.5.8. *For a fixed dualizable $\mathcal{C}$ the assignment $F \mapsto \mathrm{THH}(\mathcal{C}; F) \in \mathrm{Sp}$ is colimit preserving in F.*

Proof. This functor is the composition

$$\mathrm{Fun}^L(\mathcal{C}, \mathcal{C}) \xrightarrow{-\otimes \mathcal{C}^\vee} \mathrm{Fun}^L(\mathcal{C} \otimes \mathcal{C}^\vee, \mathcal{C} \otimes \mathcal{C}^\vee) \xrightarrow{\mathrm{ev} \circ - \circ \mathrm{coev}} \mathrm{Fun}^L(\mathrm{Sp}, \mathrm{Sp}) = \mathrm{Sp}$$

Using the universal property of the tensor product of Pr^L one checks that the first functor preserves colimits, and the second does since ev preserves colimits. □

Proposition 3.5.9. *For dualizable categories $\mathcal{C}$ and $\mathcal{D}$ and functors $F : \mathcal{D} \to \mathcal{C}, G : \mathcal{C} \to \mathcal{D}$ we have a canonical equivalence*

$$\mathrm{THH}^{\mathrm{cont}}(\mathcal{C}; F \circ G) \simeq \mathrm{THH}^{\mathrm{cont}}(\mathcal{D}; G \circ F).$$

We note that this statement holds more generally for traces in ambient symmetric monoidal categories and is called the 'trace property'.

Proof. We work more generally in a general symmetric monoidal ∞-category $\mathcal{X}$. We can compute the trace as the upper composition in the diagram

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "\mathrm{ev}"] \arrow[dr, "\mathrm{ev}"'] & \mathcal{C} \otimes \mathcal{C}^\vee \arrow[r, "G\otimes\mathrm{id}"] & \mathcal{D} \otimes \mathcal{C}^\vee \arrow[r, "F\otimes\mathrm{id}"] \arrow[dr, "\mathrm{id}\otimes F^\vee"'] & \mathcal{C} \otimes \mathcal{C}^\vee \arrow[r, "\mathrm{coev}"] & \mathbb{1} \\
& \mathcal{D} \otimes \mathcal{D}^\vee \arrow[ur, "\mathrm{id}\otimes G^\vee"'] & & \mathcal{D} \otimes \mathcal{D}^\vee \arrow[ur, "\mathrm{coev}"'] &
\end{tikzcd}$$

which commutes by definition of the dual maps. Thus we get that

$$\dim_{\mathcal{X}}(FG) = \dim_{\mathcal{X}}(F^\vee G^\vee)$$

and the claim now follows from the assertion that for any endomorphism $H : \mathcal{C} \to \mathcal{C}$ we have

$$\dim_{\mathcal{X}}(H) = \dim_{\mathcal{X}}(H^\vee)$$

which in turn follows from the commutativity of

$$\begin{tikzcd} \mathbb{1} \arrow[r, "\mathrm{ev}"] \arrow[dr, "\mathrm{ev}"'] & \mathcal{C} \otimes \mathcal{C}^\vee \arrow[rr, "H\otimes\mathrm{id}"] \arrow[d, "\simeq"] & & \mathcal{C} \otimes \mathcal{C}^\vee \arrow[r] \arrow[d, "\simeq"] & \mathbb{1} \\ & \mathcal{C}^\vee \otimes \mathcal{C} \arrow[rr, "H^\vee\otimes\mathrm{id}"] & & \mathcal{C}^\vee \otimes \mathcal{C} \arrow[ur] & \end{tikzcd}$$

which the finishes the proof. ∎

Using this trace property we can construct the functoriality. In fact, we shall claim that the assignment $\mathcal{C} \mapsto \mathrm{THH}^{\mathrm{cont}}(\mathcal{C})$ is functorial in strongly left adjoint functors. To see this let

$$F : \mathcal{C} \to \mathcal{D}$$

be a strongly left adjoint functor with right adjoint R (which is also left adjoint). We have unit $\mathrm{id}_{\mathcal{C}} \to RF$ and counit $FR \to \mathrm{id}_{\mathcal{D}}$ of the adjunction and thus obtain a map

$$\mathrm{THH}(\mathcal{C}) = \mathrm{THH}(\mathcal{C}; \mathrm{id}_{\mathcal{C}}) \to \mathrm{THH}(\mathcal{C}; RF) \simeq \mathrm{THH}(\mathcal{D} : FR) \to \mathrm{THH}(\mathcal{D}; \mathrm{id}_{\mathcal{D}}) = \mathrm{THH}(\mathcal{D}) \ .$$

For a composite $F' \circ F : \mathcal{C} \to \mathcal{D} \to \mathcal{E}$ it is easy to verify that the induced map

$$\mathrm{THH}(\mathcal{C}) \to \mathrm{THH}(\mathcal{D}) \to \mathrm{THH}(\mathcal{E})$$

is indeed the composition of maps, indeed:

Proposition 3.5.10. *This extends to a functor*

$$\mathrm{THH}^{\mathrm{cont}} : \mathrm{Pr}^L_{\mathrm{dual}} \to \mathrm{Sp} \ .$$

In fact, it even carries a natural S^1-action, i.e. lifts through $\mathrm{Sp}^{BS^1} \to \mathrm{Sp}$.

Proof. Omitted, this requires a more throughout treatment of trace theories, see e.g. [Nik18]. ∎

Proposition 3.5.11. *The assignment*

$$\mathrm{Pr}^L_{\mathrm{dual}} \to \mathrm{Sp} \qquad \mathcal{C} \mapsto \mathrm{THH}^{\mathrm{cont}}(\mathcal{C})$$

is a finitary localizing invariant.

Proof. First consider a Verdier sequence

$$\mathcal{C} \xrightarrow{i} \mathcal{E} \xrightarrow{p} \mathcal{D}$$

in $\mathrm{Pr}^L_{\mathrm{dual}}$. Then we have the right adjoint functors R_i and R_p and get a fiber sequence

$$i \circ R_i \to \mathrm{id}_{\mathcal{E}} \to R_p \circ p$$

of endofunctors of $\mathcal{E}$. Thus applying THH yields a fiber sequence

$$\mathrm{THH}(\mathcal{E}; i \circ R_i) \to \mathrm{THH}(\mathcal{E}; \mathrm{id}) \to \mathrm{THH}(\mathcal{E}; R_p \circ p)$$

Since $R_i \circ i = \mathrm{id}_{\mathcal{C}}$ and $p \circ R_p = \mathrm{id}_{\mathcal{D}}$ the desired fiber sequence

$$\mathrm{THH}(\mathcal{C}) \to \mathrm{THH}(\mathcal{D}) \to \mathrm{THH}(\mathcal{E}) \ .$$

For a filtered colimit, we first recall from Construction 2.9.19 that $(-)^\vee : \mathrm{Pr}^L_{\mathrm{dual}} \simeq \mathrm{Pr}^L_{\mathrm{dual}}$ is a covariant equivalence, and hence $\mathcal{C} \otimes \mathcal{C}^\vee \simeq \mathrm{colim}_i\, \mathcal{C}_i \otimes \mathcal{C}_i^\vee$. It is then also easy to check that the $\mathrm{coev}_i, \mathrm{ev}_i$ induce the duality for $\mathcal{C}$, e.g. we can write $\mathrm{coev} = \mathrm{colim}_i\, \mathrm{coev}_i$ in $\mathrm{Ar}(\mathrm{Pr}^L_{\mathrm{st}})$ (using that since the diagram is filtered $\mathrm{Sp} = \mathrm{colim}_i\, \mathrm{Sp}$), and analogously for ev. Overall, we can then write

$$\dim_{\mathrm{Pr}^L_{\mathrm{st}}}(\mathcal{C}) = (\mathrm{colim}_i\, \mathrm{Sp} \to \mathrm{colim}_i\, \mathcal{C}_i \otimes \mathcal{C}_i^\vee \to \mathrm{colim}_i\, \mathrm{Sp}) = \mathrm{colim}_i \dim_{\mathrm{Pr}^L_{\mathrm{st}}}(\mathcal{C}_i)$$

as arrows in $\mathrm{Pr}^L_{\mathrm{st}}$, and hence as objects in $\mathrm{Fun}^L(\mathrm{Sp}, \mathrm{Sp}) = \mathrm{Sp}$. □

Corollary 3.5.12. *For a homological epimorphism of unital rings $R \to S$ with fiber I we have that*

$$\mathrm{THH}(\mathrm{Mod}(R, I)) \simeq \mathrm{THH}(R; I) \ .$$

If I is H-unital then this is equivalent to $\mathrm{THH}(I) = \mathrm{THH}(I^+; I)$.

Proof. We have the Verdier sequence

$$\mathrm{Mod}(R, I) \to \mathrm{Mod}(R) \to \mathrm{Mod}(S)$$

and so we get a fiber sequence

$$\mathrm{THH}(\mathrm{Mod}(R, I)) \to \mathrm{THH}(R) \to \mathrm{THH}(S) \ .$$

Now

$$\mathrm{THH}(S) = \mathrm{THH}(S; S) = \mathrm{THH}(S; S \otimes_R S) = \mathrm{THH}(R; S \otimes_S S) = \mathrm{THH}(R; S),$$

where we again employ the trace-property of THH. Thus the first claim follows from the fiber sequence of R-bimodules $I \to R \to S$ with the induced sequence

$$\mathrm{THH}(R; I) \to \mathrm{THH}(R) \to \mathrm{THH}(S) \ .$$

For the second claim we note that in the H-unital case we have a pullback square

$$\begin{tikzcd} \mathrm{Mod}(I^+) \arrow[r] \arrow[d] & \mathrm{Mod}(\mathbb{S}) \arrow[d] \\ \mathrm{Mod}(R) \arrow[r] & \mathrm{Mod}(R/I) \end{tikzcd}$$

with the horizontal functors being homological epimorphisms. Thus the claim follows by taking THH. □

Note that one could prove the previous statement by analysing carefully the duality datum associated with $\mathrm{Mod}(R, I)$, whose dual is $\mathrm{Mod}(R^{\mathrm{op}}, I^{\mathrm{op}})$.

3.6 The K-theory of Sheaves

Let X be a locally compact Hausdorff space and $\mathcal{C}$ be a compactly assembled ∞-category. According to Corollary 2.12.3 we find that $\mathrm{Shv}(X;\mathcal{C})$ is also compactly assembled, so that we can take its continuous K-theory. In this section we want to identify this K-theory.

Theorem 3.6.1 (Efimov). *We have that*

$$\mathrm{K}^{\mathrm{cont}}(\mathrm{Shv}(X;\mathcal{C})) \simeq \Gamma_c\left(X, \underline{\mathrm{K}^{\mathrm{cont}}(\mathcal{C})}\right) .$$

Corollary 3.6.2. *For any dualizable ∞-category $\mathcal{C}$, we have*

$$\mathrm{K}^{\mathrm{cont}}(\mathrm{Shv}(\mathbb{R}^n;\mathcal{C})) \simeq \Omega^n \mathrm{K}^{\mathrm{cont}}(\mathcal{C}) .$$

In order the prove this theorem we would like to use Verdier duality as indicated in the introduction. In order to do that we recall that for any map $X \to Y$ of locally compact Hausdorff spaces we have the induced adjunction

$$f^* : \mathrm{Shv}(Y;\mathcal{C}) \rightleftarrows \mathrm{Shv}(X;\mathcal{C}) : f_*$$

where f_* is given by precomposing along $(f^{-1})^{\mathrm{op}} : \mathrm{Open}(Y)^{\mathrm{op}} \to \mathrm{Open}(X)^{\mathrm{op}}$, and then f^* exists by the adjoint functor theorem. To efficiently reason about sheaf categories, the special case of when f is an open immersion, i.e. the inclusion of some open subset, will be crucial.

Remark 3.6.3. Let $i : X \to Y$ be an open immersion. It induces an inclusion $i : \mathrm{Open}(X) \subseteq \mathrm{Open}(Y)$ which is left adjoint to the usual map $i^{-1} : \mathrm{Open}(Y) \to \mathrm{Open}(X)$. This yields an adjoint triple

$$\begin{tikzcd} \mathrm{PShv}(X;\mathcal{C}) \arrow[r, hook, shift left=4ex, "\mathrm{Lan}_{i^{\mathrm{op}}}"] \arrow[r, phantom, shift left=2ex, "\perp"] \arrow[r, phantom, shift right=2ex, "\perp"] \arrow[r, hook, shift right=4ex, "((i^{-1})^{\mathrm{op}})^*"'] & \mathrm{PShv}(Y;\mathcal{C}) \arrow[l, "(i^{\mathrm{op}})^*"'] \end{tikzcd}$$

The formula for left Kan extensions shows that $\mathrm{Lan}_{i^{\mathrm{op}}}$ is given by "extending by zero":

$$(\mathrm{Lan}_{i^{\mathrm{op}}} \mathcal{F})(U) = \begin{cases} \mathcal{F}(V), & U = i(V) \\ \emptyset, & \text{else} \end{cases} .$$

The restriction functors clearly restrict to sheaves, and the left kan extension does if we postcompose with sheafificaton, ultimately giving rise to the adjoint triple on sheaf categories

$$\begin{tikzcd} \mathrm{Shv}(X;\mathcal{C}) \arrow[r, hook, shift left=4ex, "i_!"] \arrow[r, phantom, shift left=2ex, "\perp"] \arrow[r, phantom, shift right=2ex, "\perp"] \arrow[r, hook, shift right=4ex, "i_*"'] & \mathrm{Shv}(Y;\mathcal{C}) \arrow[l, "i^*"'] \end{tikzcd}$$

In particular, we see that $i_!$ is strongly left adjoint, hence compactly assembled.

Considering the contravariant pullback functoriality

$$\mathrm{LCHaus}^{\mathrm{op}} \to \mathrm{Pr}^L, \qquad X \mapsto \mathrm{Shv}(X;\mathcal{C}), \tag{3.3}$$

we have the following assertion.

Proposition 3.6.4. *The assignment* (3.3) *defines a sheaf of (presentable) ∞-categories on* LCHaus*, that is:*

1. $\mathrm{Shv}(\emptyset;\mathcal{C}) = \mathrm{pt}$ *and* $\mathrm{Shv}(X \amalg Y;\mathcal{C}) \xrightarrow{\simeq} \mathrm{Shv}(X;\mathcal{C}) \times \mathrm{Shv}(Y;\mathcal{C})$.
2. *For a space X with open subsets $U, V \subseteq X$ the following square is a pullback:*

$$\begin{tikzcd} \mathrm{Shv}(U \cup V;\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Shv}(U;\mathcal{C}) \arrow[d] \\ \mathrm{Shv}(V;\mathcal{C}) \arrow[r] & \mathrm{Shv}(U \cap V;\mathcal{C}) \end{tikzcd}$$

Moreover, all the functors are Bousfield localizations.

3. *For an increasing filtered union $U = \bigcup U_i$ of opens in X we have*

$$\mathrm{Shv}(U;\mathcal{C}) \xrightarrow{\simeq} \lim \mathrm{Shv}(U_i;\mathcal{C}).$$

Proof. The first assertion is immediate and left as an exercise. For the second, we have seen above that for an inclusion $i : U \subseteq X$ of an open set we have that $i^*i_* \simeq \mathrm{id}$ which shows that i^* is a Bousfield localization. Denote the inclusion $W \subseteq X$ of an open subset by i_W, and let $j : U \cap V \subseteq U$. We now obtain an adjunction

$$L : \mathrm{Shv}(U \cup V;\mathcal{C}) \rightleftarrows \mathrm{Shv}(U;\mathcal{C}) \times_{\mathrm{Shv}(U\cap V;\mathcal{C})} \mathrm{Shv}(V;\mathcal{C}) : R.$$

with

$$L(\mathcal{F}) = (i_U^*\mathcal{F}, i_V^*\mathcal{F}) \quad \text{and} \quad R(\mathcal{F},\mathcal{G}) = (i_U)_*\mathcal{F} \times_{(i_{U\cap V})_* j^*\mathcal{F}} (i_V)_*\mathcal{G}$$

where the unit is induced by the units of $i_U, i_V, i_{U\cap V}$

$$\eta_{\mathcal{F}} : \mathcal{F} \to (i_U)_* i_U^*\mathcal{F} \times_{(i_{U\cap V})_* i_{U\cap V}^*\mathcal{F}} (i_V)_* i_V^*\mathcal{F},$$

and similarly the counit (projected to $\mathrm{Shv}(V;\mathcal{C})$) is given by

$$\mathrm{pr}_{\mathrm{Shv}(V;\mathcal{C})}\, \varepsilon_{(\mathcal{F},\mathcal{G})} : i_V^*((i_U)_*\mathcal{F} \times_{(i_{U\cap V})_* j^*\mathcal{F}} (i_V)_*\mathcal{G}) \to \mathcal{G}.$$

Using the descriptions from Remark 3.6.3 and the sheaf condition one readily checks that both unit and counit are equivalences, so L and R are mutual inverses.

Analogously, given a filtered union $\bigcup_i U_i = X$, we can again construct an adjunction

$$L : \mathrm{Shv}(X;\mathcal{C}) \rightleftarrows \lim \mathrm{Shv}(U_i;\mathcal{C}) : R$$

with L induced by restrictions and $R(\mathcal{F}_i) = \lim_i (i_{U_i})_*\mathcal{F}_i$, and check that unit and counit are equivalences by the sheaf conditions. ∎

We can also consider the category $\mathrm{LCHaus}_{\mathrm{open}}$ of locally compact Hausdorff spaces and open embeddings. For each map f in there, we have an adjoint triple $f_! \dashv f^* \dashv f_*$ by Remark 3.6.3. Then the assignment $X \mapsto \mathrm{Shv}(X;\mathcal{C})$ becomes a covariant functor on this category by means of the extension functoriality. As we have just seen, it in fact lands in $\mathrm{Pr}^L_{\mathrm{ca}}$, giving a cosheaf.

Corollary 3.6.5. *The functor* $\mathrm{Shv}(-;\mathcal{C}) : \mathrm{LCHaus}_{\mathrm{open}} \to \mathrm{Pr}^L_{\mathrm{ca}}$ *is a cosheaf, that is*

1. $\mathrm{Shv}(\emptyset;\mathcal{C}) = \mathrm{pt}$ *and* $\mathrm{Shv}(X \amalg Y;\mathcal{C}) \xrightarrow{\simeq} \mathrm{Shv}(X;\mathcal{C}) \times \mathrm{Shv}(Y;\mathcal{C})$. *(recall from Proposition 2.7.12 that* $\mathrm{Pr}^L_{\mathrm{ca}}$ *is semiadditive).*
2. *For a space* X *with open subsets* $U, V \subseteq X$ *the following square is a pushout in* $\mathrm{Pr}^L_{\mathrm{ca}}$*:*

$$\begin{tikzcd}
\mathrm{Shv}(U \cap V;\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Shv}(U;\mathcal{C}) \arrow[d] \\
\mathrm{Shv}(V;\mathcal{C}) \arrow[r] & \mathrm{Shv}(U \cup V;\mathcal{C})
\end{tikzcd}$$

3. *For an increasing filtered union* $U = \bigcup U_i$ *of opens in* X *we have*

$$\operatorname{colim} \mathrm{Shv}(U_i;\mathcal{C}) \xrightarrow{\simeq} \mathrm{Shv}(U;\mathcal{C})$$

   in $\mathrm{Pr}^L_{\mathrm{ca}}$.

Proof. Immediate from the last result by passing to left adjoints. □

We note that we can restrict the cosheaf $\mathrm{Shv}(-;\mathcal{C})$ to the category $\mathrm{Open}(X)$ for a fixed locally compact Hausdorff space X and then get a cosheaf in the classical sense.

Corollary 3.6.6. *For compactly assembled* $\mathcal{C}$*, the functor*

$$\mathrm{LCHaus}_{\mathrm{open}} \to \mathrm{Sp}, \quad X \mapsto \mathrm{K}^{\mathrm{cont}}(\mathrm{Shv}(X;\mathcal{C}))$$

is a cosheaf.

Proof. By Theorem 3.4.2 K^{cont} commutes with filtered colimits and finite products. Given open $U, V \subseteq X$ we obtain a commutative diagram of horizontal Verdier cofiber sequences

$$\begin{tikzcd}
\mathrm{Shv}(U \cap V;\mathcal{C}) \arrow[r, hook] \arrow[d] & \mathrm{Shv}(U;\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Shv}(U;\mathcal{C})/\mathrm{Shv}(U \cap V;\mathcal{C}) \arrow[d, "\simeq"] \\
\mathrm{Shv}(V;\mathcal{C}) \arrow[r, hook] \arrow[ur, phantom, "\ulcorner"] & \mathrm{Shv}(U \cup V;\mathcal{C}) \arrow[r] & \mathrm{Shv}(U \cup V;\mathcal{C})/\mathrm{Shv}(V;\mathcal{C})
\end{tikzcd}$$

where the right vertical map is an equivalence since the left square is a pushout by the cosheaf property for $\mathrm{Shv}(-;\mathcal{C})$. Applying $\mathcal{K}$, we get another diagram of horizontal cofiber sequences, but since we are in Sp it then *follows* that the left square is a pushout in Sp. Thus overall $\mathcal{K}(\mathrm{Shv}(-;\mathcal{C}))$ is also a cosheaf. □

Now we want to look at the evaluation on compact Hausdorff spaces X. Any map between compact Hausdorff spaces is proper, and more generall we have the following Lemma:

Lemma 3.6.7. *Let $\mathcal{C}$ be compactly assembled, and $f : X \to Y$ a proper map of locally compact Hausdorff spaces. Then $f^* : \mathrm{Shv}(X;\mathcal{C}) \to \mathrm{Shv}(Y;\mathcal{C})$ is compactly assembled. Equivalently, f_* preserves filtered colimits.*

Proof. Recall from Proposition 2.8.7 that we can identify $f^* = f^*_{\mathcal{C}}$ with $f^*_{\mathrm{An}} \otimes \mathcal{C}$. So by Proposition 2.12.2 we can reduce to $\mathcal{C} = \mathrm{An}$. The claim then follows from Proposition 2.7.18 since left Kan extension preserves representables, so that f^* sends $\underline{U} \to \underline{V}$ for $U \subseteq K \subseteq V$ to the compact map $\underline{f^{-1}(U)} \to \underline{f^{-1}(V)}$ (since by properness $f^{-1}(K)$ is still compact). $\square$

Warning 3.6.8. *In the stable setting the functor $f_* : \mathrm{Shv}(X;\mathcal{C}) \to \mathrm{Shv}(Y;\mathcal{C})$ has a further right adjoint $f^!$ for proper maps. Unstably this is not true.*

Proposition 3.6.9 (Profinite descent)**.** *Assume that we have a cofiltered limit $X = \lim X_i$ of locally compact Hausdorff spaces and proper maps. Then the pushforward functors induce an equivalence*

$$\mathrm{Shv}(X;\mathcal{C}) \xrightarrow{\simeq} \lim_i^{\mathrm{Pr}^R} \mathrm{Shv}(X_i;\mathcal{C})$$

where the limit is taken along the functors $(f_{ij})_* : \mathrm{Shv}(X_i;\mathcal{C}) \to \mathrm{Shv}(X_j;\mathcal{C})$. Equivalently, the pullback functors induce an equivalence

$$\mathrm{colim}^{\mathrm{Pr}^L_{ca}} \mathrm{Shv}(X_i;\mathcal{C}) \xrightarrow{\simeq} \mathrm{Shv}(X;\mathcal{C})$$

where the colimit is taken along the functors $(f_{ij})^* : \mathrm{Shv}(X_j;\mathcal{C}) \to \mathrm{Shv}(X_i;\mathcal{C})$.

Proof. Since $\mathcal{C} \otimes - : \mathrm{Pr}^L_{\mathrm{ca}} \to \mathrm{Pr}^L_{\mathrm{ca}}$ preserves colimits, we can reduce to the case $\mathcal{C} = \mathrm{An}$ since $f^*_{\mathcal{C}} = f^*_{\mathrm{An}} \otimes \mathcal{C}$ by Proposition 2.8.7.

Now for $\mathcal{C} = \mathrm{An}$, we will show the first formulation, so consider the covariant pushforward functoriality $\mathrm{Shv}(-) : \mathrm{LCHaus} \to \mathrm{Pr}^R$. Recall that one has adjunctions

$$\begin{tikzcd}
\mathrm{Top} \arrow[rr, shift left=2, "\mathrm{Open}(-)"] \arrow[rr, phantom, "\perp"] & & \mathrm{Loc} \arrow[ll, shift left=2, "\mathrm{pt}"] \arrow[rr, hook, shift right=2, "\mathrm{Shv}(-)"'] \arrow[rr, phantom, "\perp"] & & \mathrm{RTop} \arrow[ll, shift right=2, "\tau_{\leq -1}"'] \\
\mathrm{LCHaus} \arrow[u, hook] \arrow[rr, "\simeq"] & & \mathrm{Loc}_{\mathrm{lcr}} \arrow[u, hook] & &
\end{tikzcd}$$

Here Loc is the category of locales, and RTop denotes Lurie's category of ∞-topoi and geometric morphisms from [Lur17b, Definition 6.3.1.5]. It is well-known that the adjunction between topological spaces identifies sober topological spaces with spatial locales. This further restricts to an identification of locally compact Hausdorff spaces with so-called locally compact regular locales, whose full subcategory we denote by $\mathrm{Loc}_{\mathrm{lcr}}$. A locale is locally compact precisely when its underlying poset is compactly assembled; and the locale $\mathrm{Open}(X)$ is regular precisely when X is regular, i.e. T_3, i.e. admits neighborhood bases of closed sets. We refer the reader to [Joh82, Sections III.1, VII.4] for details.

Now it is shown in [Lur17b, Theorem 6.3.3.1] that the forgetful functor $\mathrm{RTop} \to \mathrm{Pr}^R$ preserves cofiltered limits, so it remains to check that $\mathrm{LCHaus} \simeq \mathrm{Loc}_{\mathrm{lcr}} \subseteq \mathrm{Loc}$ is closed under inverse limits along proper maps. Regular locales are even closed under arbitrary limits, see [Joh82, Lemma III.1.2(i), Proposition III.1.6, Remark III.1.8]. That locally compact locales are closed under inverse limits along proper maps was shown in [HP02]; the idea is essentially that an inverse limit in Loc is computed sufficiently similar to a filtered colimit of the posets of opens, so that $\{\mathrm{pr}_i^{-1}(U) \mid i \in I,\ U \in \mathrm{Open}(X_i)\}$ still forms a basis of the limit, and then the proof becomes analogous to the proof that the limit $X = \lim_i X_i$ in spaces is again locally compact. □

Corollary 3.6.10. *For a cofiltered limit $X = \lim X_i$ of locally compact Hausdorff spaces and proper maps, and a compactly assembled ∞-category $\mathcal{C}$, we have*

$$\mathrm{K}^{\mathrm{cont}}(\mathrm{Shv}(\lim X_i; \mathcal{C})) = \operatorname{colim} \mathrm{K}^{\mathrm{cont}}(\mathrm{Shv}(X_i; \mathcal{C})).$$

If all the X_i (and hence X) are furthermore compact, then we also have an equivalence natural in $E \in \mathrm{Sp}$:

$$\Gamma(X, \underline{E}) = \operatorname{colim}_i \Gamma(X_i, \underline{E})$$

Proof. The first claim is immediate from the previous Proposition 3.6.9 and the fact that K^{cont} is finitary. For the second, note that we can adjoin a final object $-\infty$ to I and extend the diagram so that $X_{-\infty} = *$ without changing the limit. However, we need to assume that all spaces X_i are compact to guarantee that the transition maps in this new diagram are still proper. We can then apply Lemma 2.5.10 to obtain

$$\Gamma(X; \underline{(-)}) = (\mathrm{pr}_{-\infty})_*(\mathrm{pr}_{-\infty})^* = \operatorname{colim}_i (X_i \to *)_*(X_i \to *)^* = \operatorname{colim}_i \Gamma(X_i; \underline{(-)}).$$

□

Proof of Theorem 3.6.1. Fix the space X and consider the assignment

$$U \mapsto \mathrm{K}^{\mathrm{cont}}(\mathrm{Shv}(U); \mathcal{C})$$

which is a cosheaf of spectra on X. We now use Verdier duality

$$\mathbb{D} : \mathrm{coShv}(X; \mathrm{Sp}) \xrightarrow{\simeq} \mathrm{Shv}(X; \mathrm{Sp})$$

and that for a cosheaf $\mathcal{F}$ on a locally compact Hausdorff space X we have that

$$\mathbb{D}(\mathcal{F})(K) = \mathcal{F}(X)/\mathcal{F}(X \setminus K)$$

for every compact subset $K \subseteq X$. Here the evaluation of a sheaf on a subset are the global sections of the pullback to that subset (in this case the compact K). In order to identify this for the concrete cosheaf $\mathcal{F}(U) = \mathrm{K}^{\mathrm{cont}}(\mathrm{Shv}(U); \mathcal{C})$, note that for every open U we have a Verdier cofiber sequence

$$\mathrm{Shv}(U; \mathcal{C}) \to \mathrm{Shv}(X; \mathcal{C}) \to \mathrm{Shv}(X \setminus U; \mathcal{C}).$$

Applying $\mathrm{K}^{\mathrm{cont}}$ yields a cofiber sequence of spectra

$$\mathrm{K}^{\mathrm{cont}}(\mathrm{Shv}(U;\mathcal{C})) \to \mathrm{K}^{\mathrm{cont}}(\mathrm{Shv}(X;\mathcal{C})) \to \mathrm{K}^{\mathrm{cont}}(\mathrm{Shv}(X \setminus U;\mathcal{C}))$$

which for $U = X \setminus K$ shows that

$$\mathbb{D}(\mathcal{F})(K) = \mathrm{K}^{\mathrm{cont}}(\mathrm{Shv}(K;\mathcal{C})).$$

For a compact K, the map $f : K \to *$ is proper and thus induces a compactly assembled $f^* : \mathcal{C} \to \mathrm{Shv}(K;\mathcal{C})$, which gives a functor $\mathrm{K}^{\mathrm{cont}}(\mathcal{C}) \to \mathrm{K}^{\mathrm{cont}}(\mathrm{Shv}(K;\mathcal{C})) = \mathbb{D}(\mathcal{F})(K)$, which adjoins over to a map from the constant sheaf $K^*\mathrm{K}^{\mathrm{cont}}(\mathcal{C}) \to \mathbb{D}(\mathcal{F})|_K$. Since X is locally compact, we can glue these together into a map

$$\phi : X^*\mathrm{K}^{\mathrm{cont}}(\mathcal{C}) = \underline{\mathrm{K}^{\mathrm{cont}}(\mathcal{C})} \to \mathbb{D}(\mathcal{F}).$$

To prove the theorem, it suffices to show that this map is an equivalence. Indeed, the inverse Verdier equivalence

$$\mathbb{D} : \mathrm{Shv}(X;\mathrm{Sp}) \xrightarrow{\simeq} \mathrm{coShv}(X;\mathrm{Sp})$$

sends a sheaf $\mathcal{F}$ to the cosheaf $\mathbb{D}\mathcal{F}$ whose evaluation at an open U is given by compactly supported sections of $\mathcal{F}$ on U, which would then give the desired description of $\mathcal{F}$.

To see that ϕ is an equivalence, we can again reduce to the compact case since sheaves on locally compact spaces are determined by the evaluation on compact subsets. Moreover, if we know that ϕ is an equivalence for some compact space X, we also know it for each closed subspace of X. Now every compact Hausdorff space X is a subspace of the Hilbert cube $[0,1]^I$ for some set I. To see this, simply take $I = C^0(X,[0,1])$ and embedd X by the canonical 'double dual' map which is an embedding by Urysohn's Lemma.

Thus it suffice to show the claim for the Hilbert cube $[0,1]^I$. By transfinite induction, the Hilbert cube is an iterated inverse limit of the spaces $X_n = [0,1]^n$, so by Corollary 3.6.10 it suffices to show it for those. But these spaces are hypercomplete, so that an equivalence of sheaves can by checked on stalks. The stalk of both sheaves are just $\mathrm{K}^{\mathrm{cont}}(\mathcal{C})$, hence we are done. $\square$

As we have learned from Dustin Clausen, the steps taken in the proof can actually be formalised quite nicely. To this end, we consider the category CHaus of compact Hausdorff spaces.

Theorem 3.6.11 (Clausen)**.** *Every functor $F : \mathrm{CHaus}^{\mathrm{op}} \to \mathrm{Sp}$ satisfying closed and profinite descent is determined by its value on the point. Here the descent conditions mean the following:*

1. *(Closed descent) For every pair of closed subsets $K, L \subseteq X$ we have that the square*

$$\begin{tikzcd} F(K \cup L) \arrow[r] \arrow[d] & F(K) \arrow[d] \\ F(L) \arrow[r] & F(K \cap L) \end{tikzcd}$$

is a pullback.

2. (Profinite descent) For every cofiltered diagram $X = \lim X_i$ we have that

$$F(X) = \mathrm{colim}_i \, F(X_i) \ .$$

The determination on the point means that we have an equivalence

$$\mathrm{ev}_{\mathrm{pt}} : \mathrm{Fun}^{\mathrm{desc}}(\mathrm{CHaus}^{\mathrm{op}}, \mathrm{Sp}) \xrightarrow{\simeq} \mathrm{Sp}$$

given by evaluation on the point. The inverse takes $E \in \mathrm{Sp}$ to the functor $X \mapsto \Gamma(X, \underline{E})$.

This result can be seen as a sort of characterisation of 'cohomology theories' on CHaus. In contrast to the usual Eilenberg-Steenrod axioms we do not need to require 'homotopy invariance' for such a cohomology theory, but rather get some version of homotopy invariance for free, which we find very surprising. More precisely, we get that the cohomology theory F doesn't depend on the topological space X but only its shape. For CW complexes this implies that it only depends on the homotopy type/underlying anima of X.

Proof. We fix some compact Hausdorff space X and consider the assignment

$$F_X : \mathcal{K}(X)^{\mathrm{op}} \to \mathrm{Sp} \qquad K \mapsto F(K),$$

where $\mathcal{K}(X)$ is the poset of compact subsets. This satisfies finite descent and also

$$F_X(K) = \mathrm{colim}_{K \ll L} \, F_X(L)$$

where $K \ll L$ means that there is an open in between. To see this, we use that F satisfies profinite descent and that

$$K = \bigcap_{K \ll L} L = \lim_{K \ll L} L.$$

This equality can be seen by noting that every point $x \in K$ lies in some compact neighborhood and by compactness we can then even find a compact neighborhood of K, giving $K \subseteq \bigcap_{K \ll L} L$. For the converse inclusion, let $x \in X \setminus K$. Pick for each $k \in K$ a compact neighborhood L_k of k which does not contain x. By compactness of K, finitely many L_k's suffice to cover K, giving some $K \ll L \subseteq X \setminus \{x\}$.

Recall that on (locally) compact Hausdorff spaces X, there is an equivalence $\mathrm{Shv}(X; \mathrm{Sp}) \simeq \mathrm{Shv}_{\mathcal{K}}(X; \mathrm{Sp})$ between sheaves and $\mathcal{K}$-sheaves, see e.g. [Lur17b, Theorem 7.3.4.9] or [Vol23, Section 5.1] for a quick overview. What we have proved above is precisely proves that F_X is a $\mathcal{K}$-sheaf on X, so conclude that there exists a sheaf $\mathcal{F}_X$ on X with $\mathcal{F}_X|_K(K) = F_X(K)$ for all compacts $K \subseteq X$. Moreover, the map $F(\mathrm{pt}) \to F(X) = \mathcal{F}_X(X)$ induced by $X \to \mathrm{pt}$ yields by adjunction a map of sheaves

$$\phi : X^* F(\mathrm{pt}) = \underline{F(\mathrm{pt})} \to \mathcal{F}_X$$

which is a stalkwise isomorphism. We conclude that if X is hypercomplete, the sheaf $\mathcal{F}_X$ is constant with value $F(\mathrm{pt})$. For a general compact Hausdorff space X, we again choose an embedding into a Hilbert cube and argue as before to conclude that ϕ is an equivalence. We thereby see that the functor F is naturally equivalent to

$$X \mapsto \Gamma(X, \underline{F(\mathrm{pt})}) \ .$$

$\square$

Remark 3.6.12. Note that the version of Efimov's theorem for compact Hausdorff spaces X follows more or less directly from Clausen's result. Then the version of Efimov's theorem for a locally compact Hausdorff space X can be deduced by investigating the Verdier sequence associated with a compactification $X \to \overline{X} \leftarrow S$. However, one can also formulate a version of Clausen's theorem that directly implies Efimov's result. For this one has to consider functors from locally compact Hausdorff spaces to spectra that are contravariant in proper maps and covariant in open immersions (or more generally local homeomorphisms). Formally these are functors

$$F : \mathrm{Span}(\mathrm{LCHaus}, \mathrm{proper}, \mathrm{open}) \to \mathrm{Sp}$$

whose morphisms are $X \overset{p}{\leftarrow} Y \overset{i}{\to} Z$ with p proper and i an open immersion (or more generally a local homeomorphism). These functors have to satisfy: closed (finite) descent, closed profinite descent, open codescent and finally send the sequence

$$U \to X \leftarrow Z \tag{3.4}$$

to a fiber sequence $F(U) \to F(X) \to F(Z)$.[8] The result is that such functors are also determined on the point. The proof proceeds exactly as the previous one and will be left as an exercise.

Now we would like to examine the generality in which the results of Clausen and Efimov hold. To this end we will need the following assertion.

Lemma 3.6.13. *Assume that $\mathcal{D}$ is compactly assembled. Then equivalences between hypersheaves*[9] *on some topological space X with values in $\mathcal{D}$ can be detected on stalks.*

Proof. We first note that the assertion is true for $\mathcal{D} = \mathrm{An}$, since by definition of hypercompleteness we can detect equivalences of hypersheaves on homotopy sheaves, and then the claim follows from the usual statement about sheaves of sets (see also [Lur17a, Lemma A.3.9]). Now we assume that $\mathcal{D}$ is compactly generated and that $\mathcal{F} \to \mathcal{G}$ is a morphism of hypersheaves with values in $\mathcal{D}$, which is an equivalence on all stalks.

Then for each compact $d \in \mathcal{D}^\omega$ the assignments

$$U \subseteq X \mapsto \mathrm{Map}_{\mathcal{D}}(d, \mathcal{F}(U)) \qquad \text{and} \qquad U \subseteq X \mapsto \mathrm{Map}_{\mathcal{D}}(d, \mathcal{G}(U))$$

[8] Note that either open descent or finite closed descent and profinite descent suffice since they imply each other using (3.4), at least the versions for inclusions which are the only relevant things for the proof.

[9] These are sheaves satisfying hyperdescent, i.e. descent with respect to all hypercovers, see e.g. [Lur17b, Section 6.5.3].

are hypersheaves of anima on X and the map $\mathcal{F} \to \mathcal{G}$ induces a map of such. Since d is compact, the induced map on stalks at x is given by $\mathrm{Map}_{\mathcal{D}}(d, \mathcal{F}_x) \to \mathrm{Map}_{\mathcal{D}}(d, \mathcal{G}_x)$ and is thus an equivalence by assumption. From the previous case of coefficients in An, we deduce that the map of hypersheaves of anima

$$\mathrm{Map}_{\mathcal{D}}(d, \mathcal{F}) \to \mathrm{Map}_{\mathcal{D}}(d, \mathcal{G})$$

is an equivalence. Since $d \in \mathcal{D}^\omega$ was arbitrary and $\mathcal{D}$ is compactly generated, we conclude that $\mathcal{F} \to \mathcal{G}$ is an equivalence, as desired.

Finally, if $\mathcal{D}$ is compactly assembled, we can write it as a retract $\mathcal{D} \xrightarrow{i} \mathcal{C} \xrightarrow{p} \mathcal{D}$ in Pr^L, where $\mathcal{C}$ is compactly generated. Then given a map $\phi : \mathcal{F} \to \mathcal{G}$ of hypersheaves in $\mathrm{Shv}(X; \mathcal{D})$ which is an equivalence on all stalks, we can postcompose and hypersheafify to get map of hypersheaves $(i_*\phi)^{\mathrm{hyshv}} : (i\mathcal{F})^{\mathrm{hyshv}} \to (i\mathcal{G})^{\mathrm{hyshv}}$ in $\mathrm{Shv}(X; \mathcal{C})$. Since i preserves colimits and hypersheafification doesn't change stalks, we see that this is still an equivalence on all stalks, so an equivalence of hypersheaves in $\mathrm{Shv}(X; \mathcal{C})$ by the previous case. Finally, note that since p is both a left and right adjoint, postcomposing with it commutes with hypersheafification, so that $p_*(i_*\phi)^{\mathrm{hyshv}} = (p_*i_*\phi)^{\mathrm{hyshv}} = \phi^{\mathrm{hyshv}} = \phi$ is an equivalence, as desired. □

We note that in Theorems 3.6.1 and 3.6.11 the use of stalks it the key factor. We get the following versions:

Theorem 3.6.14. *Let $F : \mathrm{Pr}^L_{\mathrm{ca}} \to \mathcal{D}$ be a finitary localizing invariant with compactly assembled target $\mathcal{C}$. Then for any locally compact Hausdorff space X and compactly assembled ∞-category $\mathcal{C}$, we have*

$$F\left(\mathrm{Shv}(X; \mathcal{C})\right) \simeq \Gamma_c\left(X, \underline{F(\mathcal{C})}\right) .$$

Proof. Use exactly the same proof as for Theorem 3.6.1, just for sheaves with values in $\mathcal{D}$. The key for the descent properties is to use that F is finitary and localizing. Then for the stalkwise check use the previous Lemma 3.6.13. □

Theorem 3.6.15 (Clausen)**.** *Let $\mathcal{D}$ be a compactly assembled category. Then every functor $F : \mathrm{CHaus}^{\mathrm{op}} \to \mathcal{D}$ with the property that it has closed descent and profinite descent is determined by its value on the point. More precisely the functor given by evaluation on the point*

$$\mathrm{ev}_{\mathrm{pt}} : \mathrm{Fun}^{\mathrm{desc}}(\mathrm{CHaus}^{\mathrm{op}}, \mathcal{D}) \to \mathcal{D}$$

is an equivalence. The inverse takes $E \in \mathrm{Sp}$ to the functor $X \mapsto \Gamma(X, \underline{E})$.

Proof. In view of Lemma 3.6.13, we can use the same steps as in the proof of Theorem 3.6.11. □

Corollary 3.6.16. *The presentable categories $\mathrm{Pr}^L_{\mathrm{ca}}$ and $\mathrm{Pr}^L_{\mathrm{dbl}}$ are not compactly assembled.*

Proof. Everything we do below works exactly the same whether we use $\mathrm{Pr}^L_{\mathrm{ca}}$ or $\mathrm{Pr}^L_{\mathrm{dbl}}$, hence we stick to the former. We will prove the following observations, from which the statement follows immediately from Theorem 3.6.15:

1. The functor $\mathrm{Shv}(-;\mathrm{Sp}) : \mathrm{CHaus}^{\mathrm{op}} \to \mathrm{Pr}^L_{\mathrm{ca}}$ satisfies profinite and closed descent.

2. If $\mathcal{C}$ is presentable, then for any object $E \in \mathcal{C}$, we have $\Gamma(\mathbb{R}; \underline{E}) = \mathbb{R}_* \mathbb{R}^* E = E$.

3. $\Gamma(\mathbb{R}, \underline{\mathrm{Sp}}) = \mathrm{Sp} \in \mathrm{Pr}^L_{\mathrm{ca}}$ is not equivalent to $\mathrm{Shv}(\mathbb{R}; \mathrm{Sp})$.

The last point is clear; $\mathrm{Shv}(\mathbb{R})$ is not even compactly generated. For the second point, note that $\mathbb{R}$ admits a basis $i : \mathcal{B} \subseteq \mathrm{Open}(\mathbb{R})$ consisting of the open intervals, and that these are closed under finite intersections. It is then not hard to verify that right Kan extension along i restricts to a fully faithful functor $\mathrm{Shv}(\mathcal{B}; \mathcal{C}) \subseteq \mathrm{Shv}(\mathbb{R}; \mathcal{C})$ with left adjoint given by restricting along i. Moreover, it is straightforward to see that a constant presheaf on $\mathcal{B}$ is already a sheaf, and that the constant sheaf on $\mathbb{R}$ is the right Kan extension of the constant sheaf on $\mathcal{B}$. In particular, we see that for $E \in \mathcal{C}$ we have

$$\Gamma(\mathbb{R}; \underline{E}) = (\mathbb{R}^* E)(\mathbb{R}) = (\mathbb{R}^* E)(i(\mathbb{R})) = (\mathrm{const}_{\mathcal{B}^{\mathrm{op}}} E)(\mathbb{R}) = \mathcal{E}.$$

Finally, for the first claim, we have already seen that $\mathrm{Shv}(-)$ satisfies profinite descent in Proposition 3.6.9, so it remains to see closed descent. For readability, we will from now on write $\mathrm{Shv}(X)$ for $\mathrm{Shv}(X; \mathrm{Sp})$. So let $X \in \mathrm{CHaus}$ with two closed subsets $K, L \subseteq X$, and consider the square of restriction functors:

$$\begin{tikzcd}
\mathrm{Shv}(K \cup L) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Shv}(K) \arrow[d] \\
\mathrm{Shv}(L) \arrow[r] & \mathrm{Shv}(K \cap L)
\end{tikzcd}$$

Since the forgetful functor to Pr^L reflects pullbacks by 2.7.12, it suffices to check that the corresponding diagram is a pullback in Pr^L, i.e. in Cat_∞. The restriction functors induce a map $L : \mathrm{Shv}(K \cup L) \to \mathcal{C} \coloneqq \mathrm{Shv}(K) \times_{\mathrm{Shv}(K \cap L)} \mathrm{Shv}(L)$ which admits a right adjoint

$$R : \mathcal{C} \to \mathrm{Shv}(K \cup L),\ (\mathcal{F}, \mathcal{G}) \mapsto (i_K)_* \mathcal{F} \times_{(i_{K \cap L})_* j^* \mathcal{F}} (i_L)_* \mathcal{G}.$$

The counit, when projected to $\mathrm{Shv}(K)$, is given at $(\mathcal{F}, \mathcal{G}) \in \mathcal{C}$ by $\mathrm{pr}_{\mathrm{Shv}(K)}\, \varepsilon_{\mathcal{F},\mathcal{G}} : i_K^* R(\mathcal{F}, \mathcal{G}) \to \mathcal{F}$. Since we are working stable, i_K^* preserves pullbacks, and by proper basechange one then concludes that this map is an equivalence. Analogously, $\mathrm{pr}_{\mathrm{Shv}(L)}\, \varepsilon_{\mathcal{F},\mathcal{G}}$ and hence the counit itself is an equivalence.

Hence it remains to see that L is conservative. So suppose that $\mathcal{F} \in \mathrm{Shv}(K \cup L)$ with $i_K^* \mathcal{F} = 0$ and $i_L^* \mathcal{F}|_L$. Recall that on compact Hausdorff spaces X there is an equivalence $\mathrm{Shv}(X) \simeq \mathrm{Shv}_{\mathcal{K}}(X)$ between sheaves and $\mathcal{K}$-sheaves, see e.g. [Lur17b, Theorem 7.3.4.9] or [Vol23, Section 5.1] for a good overview without proof. By inspecting the functor inducing said equivalence one sees $\mathcal{F}_{\mathcal{K}}(C) = (i_K^* \mathcal{F})(C) = 0$ for any compact $C \subseteq K$, and analogously $\mathcal{F}_{\mathcal{K}}(C) = 0$ for any compact $C \subseteq L$. By closed descent, one then obtains $\mathcal{F}_{\mathcal{K}} = 0$, hence $\mathcal{F} = 0$, as desired. $\square$

Remark 3.6.17. We will see in the next chapter that Theorem 3.6.14 is true without the assertion that $\mathcal{D}$ is compactly assembled. The idea is to show that there is a universal finitary localizing invariant and that its target is dualizable.

However the above Corollary also shows that this fails for Theorem 3.6.15, and that compactly assembled is generally a "tight bound", c.f. Theorem 2.6.8. Nevertheless, it is still an interesting question (to which we currently don't know the answer) whether an analogue may hold for stable bicomplete categories such as $\mathrm{Sp}^{\mathrm{op}}$.

3.7 Non-commutative motives

Generalizing from K-theory, we have previously defined the notion of a localizing invariant. In this section we would like to give a formal version of it. To this end, recall that a functor

$$\mathrm{Pr}^L_{\mathrm{ca}} \to \mathcal{D}$$

to a stable ∞-category $\mathcal{D}$ is called a localizing invariant of it sends 0 to 0 and Verdier cofiber sequence to cofiber sequences. Moreover, we call it finitary if it also preserves filtered colimits.

Definition 3.7.1 (Blumberg–Gepner–Tabuada)**.** We define the ∞-category of non-commutative motives

$$\mathrm{NcMot} := \mathrm{Loc}_\omega(\mathrm{Sp}^{\mathrm{op}})^{\mathrm{op}} \subseteq \mathrm{Fun}((\mathrm{Pr}^L_{\mathrm{ca}})^{\mathrm{op}}, \mathrm{Sp}).$$

Proposition 3.7.2. *The ∞-category* NcMot *is a stable, presentable ∞-category.*

Proof. The fact that it is stable and has all colimits is clear by definition. Moreover, it is a general fact that the category of ω-accessible functors between two presentable categories is itself accessible. $\square$

The inclusion

$$\mathrm{NcMot} \to \mathrm{Fun}_{\omega_1-\lim}\left((\mathrm{Pr}^L_{\mathrm{ca}})^{\mathrm{op}}, \mathrm{Sp}\right)$$

has a left adjoint L by the adjoint functor theorem. This way we obtain a functor

$$M : \mathrm{Pr}^L_{\mathrm{ca}} \to \mathrm{NcMot} \qquad \mathcal{C} \mapsto L(\Sigma^\infty \underline{C})$$

and we have:

Proposition 3.7.3 (Blumberg–Gepner–Tabuada)**.** *The functor M exhibits* NcMot *as the universal finitary localizing invariant, that is any finitary localizing invariant*

$$L : \mathrm{Pr}^L_{\mathrm{ca}} \to \mathcal{D}$$

extends uniquely to a colimit preserving functor $\tilde{L} : \mathrm{NcMot} \to \mathcal{D}$. Moreover, we have

$$\mathrm{K}^{\mathrm{cont}}(\mathcal{C}) = \mathrm{map}_{\mathrm{NcMot}}(\mathrm{Sp}, M\mathcal{C}).$$

Definition 3.7.4. For a pair of compactly assembled ∞-categories $\mathcal{C}$ and $\mathcal{D}$ we write

$$\mathrm{KK}^{\mathrm{cont}}(\mathcal{C}, \mathcal{D}) \coloneqq \mathrm{map}_{\mathrm{NcMot}}(M\mathcal{C}, M\mathcal{D})$$

and refer to it as bivariant continuous K-theory.

One should think of this as an analogue of KK-theory as defined for C^*-algebras by Kasparov. As an example, we have that $\mathrm{KK}^{\mathrm{cont}}(\mathrm{Sp}, \mathcal{C}) = \mathrm{K}^{\mathrm{cont}}(\mathcal{C})$. Sometimes we shall also write

$$\mathrm{KK}^{\mathrm{cont}}(R, S) \coloneqq \mathrm{KK}^{\mathrm{cont}}(\mathrm{Mod}_H(R), \mathrm{Mod}_H(S))$$

for H-unital ring spectra R and S. With this notation we have $\mathrm{KK}^{\mathrm{cont}}(\mathbb{S}, R) = \mathrm{K}^{\mathrm{cont}}(R)$.

Definition 3.7.5. We say that a compactly assembled functor

$$F : \mathcal{C} \to \mathcal{D}$$

is a motivic equivalence if $M\mathcal{C} \to M\mathcal{D}$ is an equivalence in NcMot. Equivalently, if for each finitary localizing invariant L we have that $L(\mathcal{C}) \to L(\mathcal{D})$ is an equivalence.

We say that a functor $L : \mathrm{Pr}^L_{\mathrm{ca}} \to \mathcal{E}$ with $\mathcal{E}$ stable is a motivic invariant if it sends motivic equivalences to equivalences in $\mathcal{E}$. A motivic localizing invariant is a motivic invariant that is also localizing.

Example 3.7.6. Every motivic equivalence induces an equivalence in K-theory, but the converse does not hold. For example the inclusion/transfer functor

$$\mathcal{D}(\mathbb{F}_p) \to \mathcal{D}(\mathbb{Z} \text{ on } p)$$

where $\mathcal{D}(\mathbb{Z} \text{ on } p) \coloneqq \mathrm{Ind}(\mathcal{D}^{p-\mathrm{nil}}(\mathbb{Z})) = \ker(\mathcal{D}(\mathbb{Z}) \to \mathcal{D}(\mathbb{Z}[1/p]))$ induces an equivalence on K-theory by dévissage, but not on THH where $\mathrm{THH}(\mathcal{D}(\mathbb{Z} \text{ on } p)) \to \mathrm{THH}(\mathbb{Z})$ is a p-adic equivalence.

Example 3.7.7. Every finitary localizing invariant is motivic. This is true by the fact that it then factors over non-commutative motives. Postcomposition of a motivic invariant with any functor is still a motivic invariant. As an example we deduce that constructions like TC, TP and TC^- are motivic localizing invariants.

Theorem 3.7.8 (Ramzi–Sosnilo–Winges)**.** *The functor*

$$M : \mathrm{Pr}^L_{\mathrm{ca}} \to \mathrm{NcMot}$$

is a Dwyer-Kan localization at the motivic equivalences. Moreover, we have

$$\Omega^\infty(\mathrm{KK}^{\mathrm{cont}}(\mathcal{C}, \mathcal{D})) = \mathrm{colim}_{\mathcal{D} \to \hat{\mathcal{D}}} \mathrm{Map}_{\mathrm{Pr}^L_{\mathrm{ca}}}(\mathcal{C}, \hat{\mathcal{D}}),$$

where the colimit ranges over all functors $\mathcal{D} \to \hat{\mathcal{D}}$ that are fully faithful motivic equivalences.

The above theorem is shown by establishing the structure of a cofibration category on $\mathrm{Pr}^L_{\mathrm{ca}}$, where the weak equivalences are the motivic equivalences, and the cofibrations are the fully faithful functors.

In view of Proposition 2.10.14 and Theorem 2.10.16 one can also deduce the following Corollary:

Corollary 3.7.9. *The functors*

$$\mathrm{Alg}_2^{\mathrm{lu}} \to \mathrm{NcMot} \quad \textit{and} \qquad \mathrm{Alg}_2^{\mathrm{H}} \to \mathrm{NcMot}$$

which send R to $M(\mathrm{Mod}_H(R))$ are Dwyer-Kan localizations.

Corollary 3.7.10. *Every motivic localizing invariant $\mathrm{Cat}^{\mathrm{ex}}_\infty \to \mathcal{E}$ factors through a functor $\mathrm{NcMot} \to \mathcal{E}$.*

Remark 3.7.11. Ramzi–Sosnilo–Winges also show that every ω_1-finitary localizing invariant is motivic, so factors over NcMot. For κ-finitary localizing invariant and $\kappa > \omega_1$ regular this is an open question, which might actually be undecidable (according to Efimov).

One of the goals for the next chapter is to show that NcMot is in fact dualizable. This is a result of Efimov. We will in fact see that more is true, namely that it is rigid. This is one of the main reasons to investigate the notion of (local) rigidity, which roughly formalizes being self-dual in $\mathrm{Pr}^L_{\mathrm{st}}$ in a nice way.

Chapter 4

Six functors

We will now see how compactness interacts with monoidal structures, and see how this leads to the six functors.

Recall that one can view symmetric-monoidal ∞-categories as commutative algebra objects in Cat_∞. On the "big" side, a commutative algebra object in Pr^L is a so-called a closed symmetric monoidal (presentable) category, since the tensor product functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ needs to preserve colimits in both arguments due to how the tensor product and morphisms in Pr^L work. This can also be characterized by asking that $(-) \otimes x : \mathcal{C} \to \mathcal{C}$ has a right adjoint for every x, which gives the internal Hom $\underline{Hom}(x, -) : \mathcal{C} \to \mathcal{C}$.

A classical notion for small symmetric-monoidal categories is to call them *rigid* if every object is dualizable. This includes examples such as $\mathrm{Perf}(R)$, but also things such as k-finitely generated $k[G]$-modules.

For big compactly generated categories, the corresponding notion is that the compact objects coincide with the dualizable objects. Examples include the derived category $\mathcal{D}(R)$.

We will now see how to generalize this, and a weaker notion called *locally rigid*, to compactly assembled categories.

4.1 Frobenius Algebras

Locally rigid categories will in particular give Frobenius algebra objects in Pr^L. Below, we will give a short overview over the notion of Frobenius algebras. Classically, a Frobenius algebra over a field k is a finite dimensional k-algebra A with a nondegenerate bilinear pairing $b : A \otimes A \to k$ satisfying $b(xy, z) = b(x, yz)$. This gives rise to a self-duality $A^\vee \cong A$, and dualizing unit η and multiplication map μ, we obtain a counit $\varepsilon : A \to \mathbb{1}$ and a comultiplication $\Delta : A \to A \otimes A$. This superficially looks similar to a Hopf algebra, but is very different: Here, $\Delta(xy) = \Delta(x)y = x\Delta(y)$, so Δ is a bimodule homomorphism (not an algebra map as in the case of Hopf algebras). The nondegenerate pairing can be described in terms of the structure maps $\eta, \mu, \varepsilon, \Delta$ by $b = \varepsilon \circ \mu$, since $b(x, y) = b(xy, 1) = (b \circ (\mathrm{id} \otimes \eta))(xy) = \varepsilon(xy)$. So the entire structure is determined by the algebra structure (η, μ) and the counit ε.

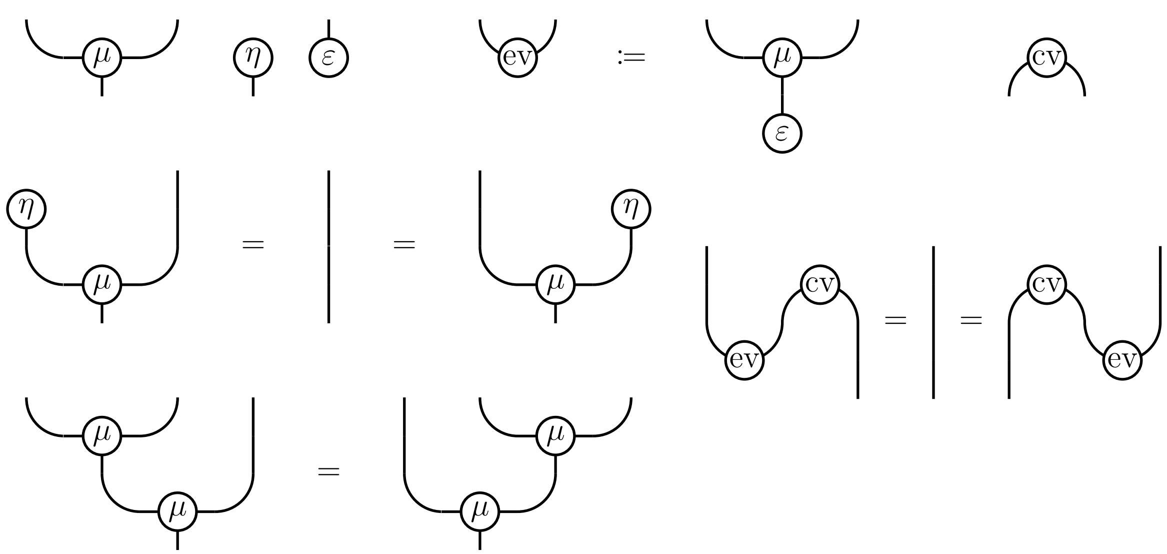
This perspective on Frobenius algebras generalizes to higher algebra: Let $\mathcal{C}$ be a symmetric-monoidal ∞-category admitting geometric realizations and so that the tensor product preserves geometric realizations separately in each variable.

Definition 4.1.1 ([Lur17a, 4.6.5.1])**.** Let $A \in \mathrm{Alg}(\mathcal{C})$ be an algebra with multiplication μ and unit η. A morphism $\varepsilon : A \to \mathbb{1}$ is nondegenerate / exhibits A as a Frobenius algebra if the composite

$$\mathrm{ev} : A \otimes A \xrightarrow{\mu} A \xrightarrow{\varepsilon} \mathbb{1}$$

exhibits A as self-dual in $\mathcal{C}$.

Remark 4.1.2. In checking identities in monoidal categories, especially those involving monoidal duality, it is often helpful to employ the visual aid of string diagrams[1]. Most identities in this section are easy enough and make a good exercise for getting comfortable with the definitions, so we will only give a visual proof of the up-to-homotopy versions using string diagrams. We read morphisms from top to bottom, and the monoidal tensorproduct from left to right (and we don't draw the unit). A straight vertical line corresponds to the identity on A. The data and basic axioms of a Frobenius algebra as above is then represented by the following diagrams:



In the first line, we give the representations of $\mu : A \otimes A \to A$, $\eta : \mathbb{1} \to A$, $\varepsilon : A \to \mathbb{1}$, and define the evaluation $\mathrm{ev} \coloneqq \varepsilon \circ \mu : A \otimes A \to \mathbb{1}$ as above. By definition of a Frobenius algebra, we also have a corresponding coevaluation $\mathrm{cv} : \mathbb{1} \to A \otimes A$. The conditions that these evaluation and coeavluation maps exhibit A as selfdual in $\mathcal{C}$ are then presented as the equalities of string diagrams on the right; they read

$$(\mathrm{ev} \otimes A) \circ (A \otimes \mathrm{cv}) \simeq \mathrm{id}_A \simeq (A \otimes \mathrm{ev}) \circ (\mathrm{cv} \otimes A).$$

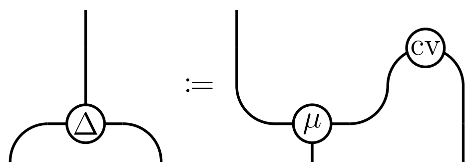
The left string diagram equalities show unitality and associativity of the multiplication.[2]

[1] See e.g. the Wikipedia article or the nlab article.

[2] Of course, the string diagrams describe e.g. associativity of μ only up to homotopy instead of coherently, however for our purposes the manipulations are easily upgraded to fully coherent statements.

Remark 4.1.3. In the case of commutative Frobenius algebras, it is more established to use a certain 3-dimension visual calculus based on bordisms. Namely, it is a surprising fact that *commutative* Frobenius algebras in a category $\mathcal{C}$ are the same as symmetric-monoidal functors $\mathrm{Bord}_1 \to \mathcal{C}$ where Bord_1 is a category with objects compact oriented 1-manifolds, and morphisms given by (anima) of 2-dimensional oriented cobordisms between them. The symmetric-monoidal structure is disjoint union. The multiplication and comultiplication come from the “pair of pants” between $S^1 \amalg S^1$ and S^1, unit and counit from a disk interpreted as morphism between and S^1, and the self-duality pairing from the cylinder on S^1 viewed as morphism between $S^1 \amalg S^1$ and $\emptyset$. In the 1-categorical world, a nice exposition can be found in [Koc]. In the noncommutative case, we can (and will) instead use string diagrams as a visual calculus.

Definition 4.1.4. Let $(A, \mu, \eta, \varepsilon)$ be a Frobenius algebra in $\mathcal{C}$ as above. Then we can define a comultiplication $\Delta : A \to A \otimes A$ by dualizing the multiplication μ, which boils down to $\Delta := (\mu \otimes A) \circ (A \otimes \mathrm{cv})$

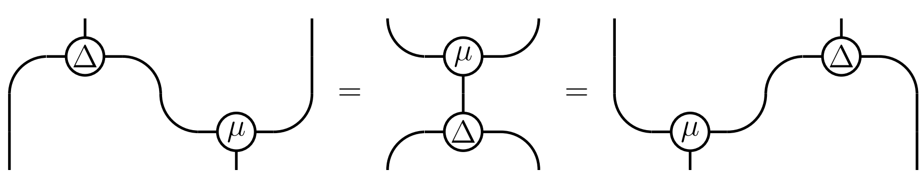


Proposition 4.1.5. *Let $(A, \mu, \Delta, \eta, \varepsilon)$ be a Frobenius algebra as above. Then*

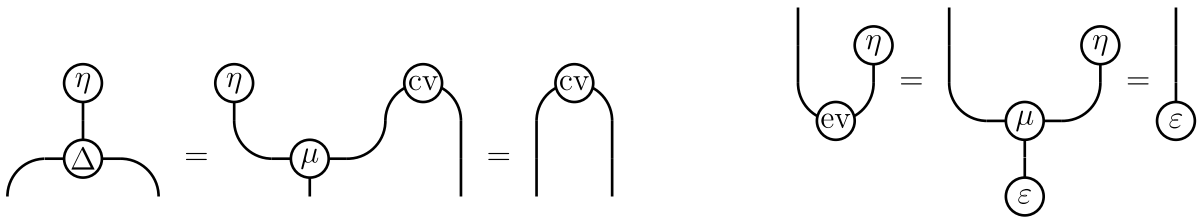
1. *We recover* $\mathrm{cv} \simeq \Delta \circ \varepsilon$.
2. *Under the self-duality of A and $\mathbb{1}$, the dual $\eta^\vee : A^\vee \to \mathbb{1}^\vee$ of η identifies with ε, and vice-versa.*
3. *(A, Δ, ε) is a coalgebra in $\mathcal{C}$. In particular, Δ is coassociative and ε is a 2-sided counit for Δ.*
4. *The comultiplication Δ admits the structure of an (A, A)-bimodule map. In particular, the Frobenius identities hold:*

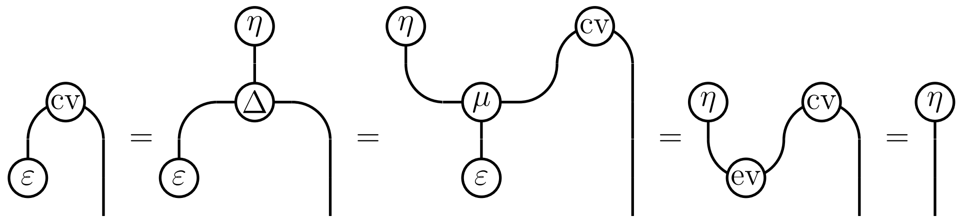
$$(A \otimes \mu) \circ (\Delta \otimes A) \simeq \Delta \circ \mu \simeq (\mu \otimes A) \circ (A \otimes \Delta).$$

   Pictorially, we write these as



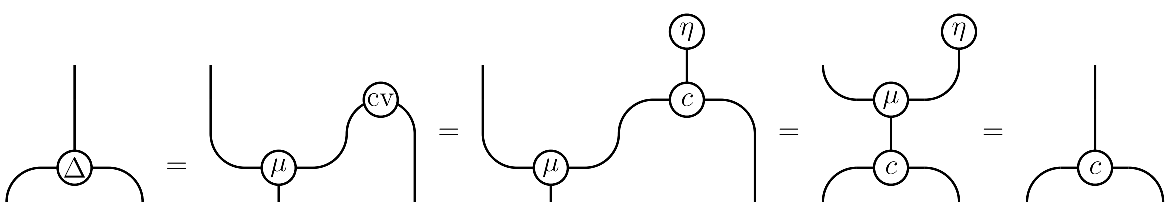
Proof. For (1.) and (2.), we have the following manipulations:





For the fully coherent version of (3), one considers the monoidal equivalence $(-)^\vee : \mathcal{C}^{\mathrm{dbl}} \simeq (\mathcal{C}^{\mathrm{dbl}})^{\mathrm{op}}$ given by taking duals on the full subcategory of dualizable objects in $\mathcal{C}$. This is well-known in the 1-categorical setting, and also exists in the ∞-categorical setting by work of Heine–Lopez-Avila–Spitzweck, c.f. [HLAS24, Theorem 5.8]. As we saw above, this sends $\mu \mapsto \Delta$ and $\eta \mapsto \varepsilon$, and hence sends the algebra (A, μ, η) to a coalgebra (A, Δ, ε).

For (4), note that the Frobenius identities are precisely the up-to-homotopy version of the statement that Δ is an (A, A)-bimodule map. For the fully coherent version, we note that by [Lur17a, 4.6.2.14][3], since $\mathrm{ev} = \varepsilon \circ \mu$ exhibits A as self-dual in $\mathcal{C}$, the morphism $\varepsilon : A \simeq A \otimes_A A \to \mathbb{1}$ also exhibits $A \in \mathrm{RMod}(A)$ as right dual to $A \in \mathrm{LMod}(A)$, which means there exists a coevaluation $c : A \to A \otimes A$ in $\mathrm{BiMod}(A, A)$ satisfying the usual triangle identities (c.f. [Lur17a, 4.6.2.1]) and so that we recover the coevaluation $\mathrm{cv} \simeq c\eta : \mathbb{1} \to A \otimes A$. We now show that forgetting the bimodule-map structure on c just yields Δ. Indeed, since the Frobenius identities hold for c, we have

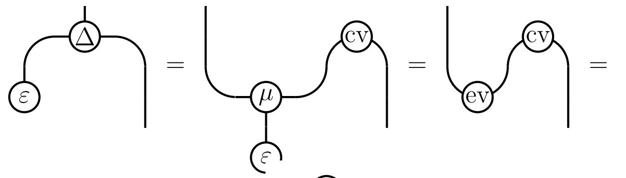


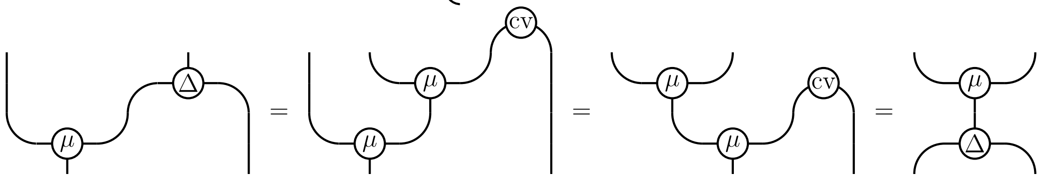
$\square$

Remark 4.1.6. In the above proof, one can also manually show the non-coherent versions of (3) and (4). This is interesting because in some cases (see next section) Δ is already a lax (A, A)-bimodule map and then the only thing left to show are these Frobenius identities.

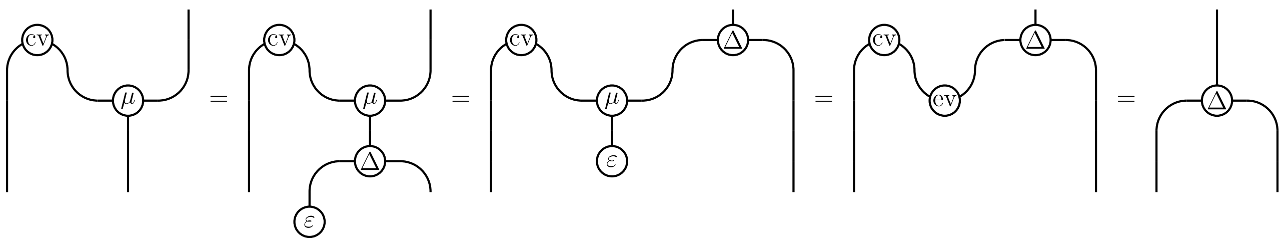
[3]their (λ, A, B, B') is specialised to our $(\varepsilon, 1, A, 1)$, with $X = A$ as right A-module and $Y = A$ as left A-module, and $e = \varepsilon : A \otimes_A A \to 1$ and $e' = \mathrm{ev} = \varepsilon \circ \mu : A \otimes A \to 1$.

To do this, we first get left-counitality of Δ and one part of the Frobenius identity as follows:

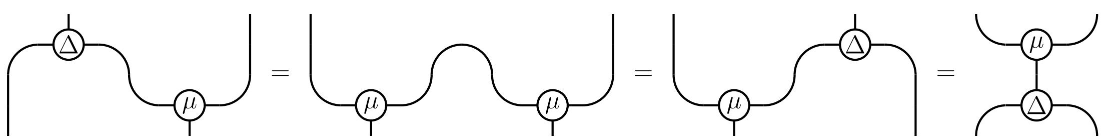




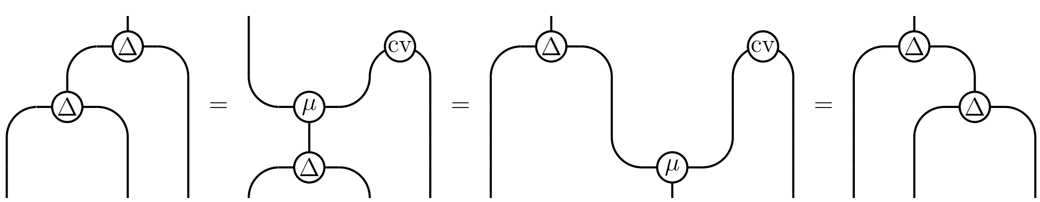
Using this, we get that we can equivalently define Δ as $(A \otimes \mu) \circ (\mathrm{cv} \otimes A)$:



Using this definition of Δ the same proof for left counitality also works for right counitality now. Moreover, we obtain the remaining Frobenius identity:



Finally, we can see coassociativity (up to homotopy) of Δ as follows:



Proposition 4.1.7 ([Lur17a, 4.6.5.14])**.** *Suppose that $\mathcal{C}$ is symmetric monoidal, $A \in \mathrm{CAlg}(\mathcal{C})$ and $\varepsilon : A \to \mathbb{1}$ nondegenerate, i.e. $(A, \mu, \eta, \varepsilon)$ is a Frobenius algebra in $\mathcal{C}$. Let $e : M \otimes_A N \to A$ be a duality datum with coevaluation $c : A \to N \otimes_A M$ in the symmetric monoidal ∞-category $\mathrm{Mod}_A(\mathcal{C})$. Then the composite map*

$$e' : M \otimes N \to M \otimes_A N \xrightarrow{e} A \xrightarrow{\varepsilon} \mathbb{1}$$

is a duality datum in $\mathcal{C}$ with coevaluation

$$c' : \mathbb{1} \xrightarrow{\eta} A \xrightarrow{c} N \otimes_A M \simeq N \otimes_A A \otimes_A M \xrightarrow{N \otimes_A \Delta \otimes_A M} N \otimes_A A \otimes A \otimes_A M \simeq N \otimes M.$$

In particular, the forgetful functor $U : \mathrm{Mod}_A(\mathcal{C}) \to \mathcal{C}$ preserves dualizable objects.

Proof. We do this in two steps. First, we claim that $e'' : M \otimes N \to M \otimes_A N \to A$ in $\mathrm{BiMod}(A, A)$ exhibits $M \in \mathrm{LMod}(A)$ as dual to $N \in \mathrm{RMod}(A)$ in the sense of [Lur17a, 4.6.2.1], with coevaluation $c'' : \mathbb{1} \xrightarrow{\eta} A \xrightarrow{c} N \otimes_A M$ in $\mathcal{C}$. is the coevaluation corresponding to e. Indeed, using the triangle identities for e and c, one easily checks that they also hold for e'' and c'', for example:

$$\begin{tikzcd}
M \arrow[rr, "M\otimes c''"] \arrow[d, "M\otimes\eta"'] & & M \otimes N \otimes_A M \arrow[rr, "e''\otimes_A M"] \arrow[d] & & M \\
M \otimes A \arrow[urr, "M\otimes c"] \arrow[d] & & M \otimes_A N \otimes_A M \arrow[urr, "e\otimes_A M"] & & \\
M \otimes_A A \arrow[urr, "M\otimes_A c"] & & & &
\end{tikzcd}$$

Now we have $e' = \varepsilon \circ e''$, and we obtain a morphism in $\mathrm{BiMod}(A, A)$

$$\Delta_{M,N} : M \otimes_A N \simeq M \otimes_A A \otimes_A N \xrightarrow{M \otimes_A \Delta \otimes_A N} M \otimes_A A \otimes A \otimes_A N \simeq M \otimes N$$

With this we can write $c' = \Delta_{N,M} \circ c''$, and we claim that c' and e' satisfy the triangle identities exhibiting M as dual to N in $\mathcal{C}$:

$$\begin{tikzcd}
M \arrow[r, "M\otimes c''"] \arrow[dr, "\simeq"'] & M \otimes N \otimes_A M \arrow[r, "\Delta_{M\otimes N,M}"] \arrow[d, "e''\otimes_A M"] & M \otimes N \otimes M \arrow[d, "e''\otimes M"] \\
 & A \otimes_A M \arrow[r, "\Delta_{A,M}"] \arrow[dr, "\simeq"'] & A \otimes M \arrow[d, "\varepsilon\otimes M"] \\
 & & M
\end{tikzcd}
\qquad
\begin{tikzcd}
N \arrow[r, "c''\otimes N"] \arrow[dr, "\simeq"'] & N \otimes_A M \otimes N \arrow[r, "\Delta_{N,M\otimes N}"] \arrow[d, "N\otimes_A e''"] & N \otimes M \otimes N \arrow[d, "N\otimes e''"] \\
 & N \otimes_A A \arrow[r, "\Delta_{N,A}"] \arrow[dr, "\simeq"'] & N \otimes A \arrow[d, "N\otimes\varepsilon"] \\
 & & N
\end{tikzcd}$$

The remainder of the proof is on arguing that these diagrams commute. The top left triangles have the obvious composite by the triangle identities for c'' and e''. Generally, note that for any two maps $f : M \to M'$ and $g : N \to N'$ in $\mathrm{Mod}_A(\mathcal{C})$ we have the following commutative square and triangles

$$\begin{tikzcd}
M \otimes_A N \arrow[r, "f\otimes_A g"] \arrow[d, "\Delta_{M,N}"'] & M' \otimes_A N' \arrow[d, "\Delta_{M',N'}"] \\
M \otimes N \arrow[r, "f\otimes g"'] & M' \otimes N'
\end{tikzcd}
\qquad
\begin{tikzcd}
M \otimes_A A \arrow[r, "\Delta_{M,A}"] \arrow[dr, "\simeq", "\mathrm{act}"'] & M \otimes A \arrow[d, "M\otimes\varepsilon"] \\
 & M
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes_A M \arrow[r, "\Delta_{A,M}"] \arrow[dr, "\simeq", "\mathrm{act}"'] & A \otimes M \arrow[d, "\varepsilon\otimes M"] \\
 & M
\end{tikzcd}$$

This is immediate from the definition for the left square. For the middle square, (and dually for the right one) this follows from the following commutative diagram

$$\begin{tikzcd}
M \otimes_A A \otimes_A A \arrow[r, "M\otimes_A \Delta \otimes_A A"] \arrow[dd, "M\otimes_A \mu"', "\simeq"] & M \otimes_A A \otimes A \otimes_A A \arrow[r, "\mathrm{act}\otimes\mu", "\simeq"'] \arrow[dd, "M\otimes_A A\otimes\mu"', "\simeq"] & M \otimes A \arrow[d, "M\otimes\varepsilon"] \\
 & & M \\
M \otimes_A A \arrow[r, "M\otimes_A \Delta"] \arrow[rr, bend right, equal] & M \otimes_A A \otimes A \arrow[r, "M\otimes_A A\otimes\varepsilon"] \arrow[uur, "\mathrm{act}\otimes A", "\simeq"'] & M \otimes_A A \arrow[u, "\simeq", "\mathrm{act}"']
\end{tikzcd}$$

Here the left square commutes by the Frobenius identity, the bottom horizontal composite is the identity by counitality of Δ, and the top-right composite from the bottom left corner to the top right one is the definition of $\Delta_{M,A}$. $\square$

Corollary 4.1.8. *With notation as above, suppose we are given another duality datum with evaluation $t : X \otimes_A Y \to A$ in Mod_A, as well as an A-linear map $f : Y \to Y'$. Then the forgetful functor $U : \mathrm{Mod}_A(\mathcal{C}) \to \mathcal{C}$ sends the dual $f^\vee : X' \to X$ to the dual $(Uf)^\vee : X' \to X$ of Uf.*

Proof. This follows from the commutativity of the diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[rrr, bend left, "X\otimes c'"] \arrow[drr, "X\otimes_A c"'] & X \otimes A \arrow[r, "X\otimes c"] & X \otimes N \otimes_A M \arrow[r, "\Delta"'] \arrow[d] & X \otimes N \otimes M \arrow[r, "f"] \arrow[d] & X \otimes Y \otimes M \arrow[d] \arrow[ddd, bend left, "t'\otimes M"] \\
 & & X \otimes_A N \otimes_A M \arrow[r, "\Delta"] \arrow[dr, "f"'] & X \otimes_A N \otimes M \arrow[r, "f"] & X \otimes_A Y \otimes M \arrow[d, "t\otimes M"] \\
 & & & X \otimes_A Y \otimes_A M \arrow[ur, "\Delta"] \arrow[dr, "t\otimes_A M"'] & A \otimes M \arrow[d] \\
 & & & & M
\end{tikzcd}$$

The diagonal composite is the definition of $U(f^\vee)$, whereas the right-down composite is the definition of $(Uf)^\vee$, where we use that the above Proposition explicitly tells us that c' is a coevaluation witnessing that UM is dual to UN and t' is an evaluation witnessing that UX is dual to UY. $\square$

4.2 Locally rigid categories

Let $\mathcal{X}$ be a symmetric monoidal $(\infty, 2)$-category (think $(\mathrm{Pr}^L_{\mathrm{st}}, \otimes, \mathrm{Sp})$, or $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^L_{\mathrm{st}})$ for a relative notion).

Definition 4.2.1. A commutative algebra object $(A, \mu, \eta : \mathbb{1} \to A)$ in $\mathcal{X}$ is called *locally rigid* if the following properties hold:

1. The multiplication $\mu : A \otimes A \to A$ admits a right adjoint $\Delta : A \to A \otimes A$ in $\mathcal{X}$.

2. The Beck-Chevalley transformations

$$(\mu \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta) \to \Delta \circ \mu$$
$$(\mathrm{id} \otimes \mu) \circ (\Delta \otimes \mathrm{id}) \to \Delta \circ \mu$$

obtained[4] from the commutative diagram of left adjoints

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mu\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mu"] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"] & A, \end{tikzcd}$$

are equivalences. (This is exactly the Frobenius identity)

3. The composite $\mathbb{1} \xrightarrow{\eta} A \xrightarrow{\Delta} A \otimes A$ is the coevaluation of a duality.

In particular, a locally rigid algebra A is dualizable, and comes with a canonical identification $A \simeq A^\vee$.

Remark 4.2.2. The multiplication $A \otimes A \to A$ is an A-A-bimodule map. This gives its right adjoint the structure of a *lax* A-A-bimodule morphism, i.e. roughly speaking one where the diagrams that need to commute for a bimodule morphism only commute up to a (not necessarily invertible) 2-morphism. These 2-morphisms include the Beck-Chevalley maps $\Delta(a) \otimes b \to \Delta(a \otimes b)$ appearing in the definition above. Thus the following can be seen to be an equivalent condition to (2.) above:

2'. The lax A-A-bimodule structure on the right adjoint $\Delta : A \to A \otimes A$ is strong.

Slightly more imprecisely, axiom (2.) thus asks for Δ to be a bimodule homomorphism, but it is important that this is just a property, not additional structure, due to Δ being the right adjoint to μ.

Proposition 4.2.3. *If A is a locally rigid algebra object, we have:*

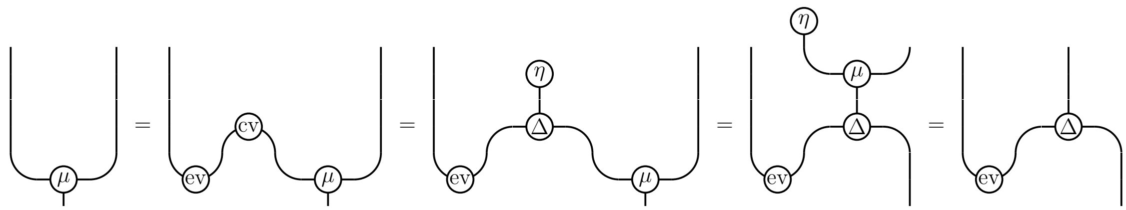
1. *The dual $\varepsilon : A \to \mathbb{1}$ of the unit $\eta : \mathbb{1} \to A$ is nondegenerate, i.e. exhibits A as Frobenius algebra in $\mathcal{X}$ in the sense of Definition 4.1.1. In other words, the evaluation* ev *corresponding to the coevaluation* $\mathrm{cv} = \Delta \circ \eta$ *is given by* $\mathrm{ev} \simeq \varepsilon \circ \mu$.

2. *The comultiplication Δ is dual to μ, and hence agrees with the comultiplication defined via the Frobenius structure as in Definition 4.1.4. In particular, all results from Proposition 4.1.5 apply.*

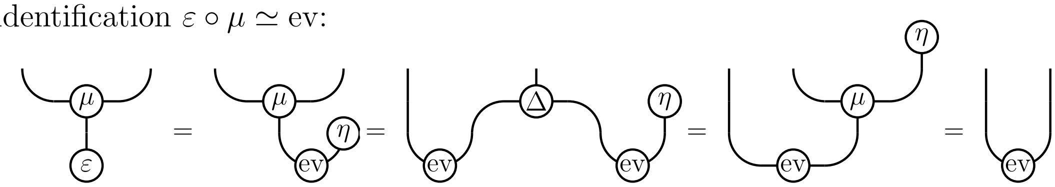
3. *The forgetful functor $\mathrm{Mod}_A(\mathcal{X}) \to \mathcal{X}$ preserves dualizable objects.*

[4] The first one is obtained by taking vertical right adjoints, the second one by taking horizontal right adjoints.

Proof. Since we have the Frobenius identities and the duality data ev and $\mathrm{cv} = \Delta \circ \eta$, one readily checks that $\mu \simeq (\mathrm{ev} \otimes A) \circ (A \otimes \Delta)$ is dual to Δ:



Completely symmetric to this is $\mu \simeq (A \otimes \mathrm{ev}) \circ (\Delta \otimes A)$. Using both of these identites, one gets the identification $\varepsilon \circ \mu \simeq \mathrm{ev}$:



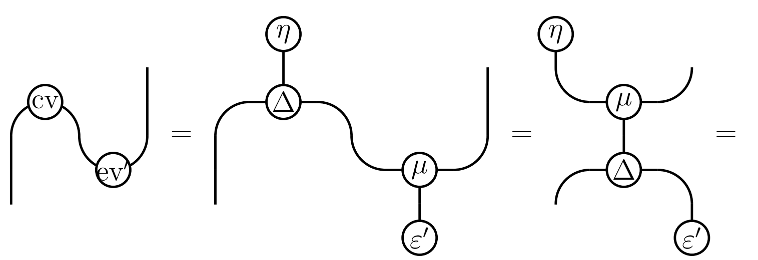
For (2.), we have just seen that Δ is dual to μ, which is the definition of the comultiplication in Definition 4.1.4. Finally (3.) now follows immediately from Proposition 4.1.7. $\square$

The following Lemma is very useful for recognizing locally rigid algebras, as it is often a lot easier to provide a counit as opposed to checking that $\mathbb{1} \to A \otimes A$ is a coevaluation.

Lemma 4.2.4. *An algebra A in $\mathcal{X}$ is locally rigid if and only if:*

1. *$\mu : A \otimes A \to A$ admits a right adjoint Δ,*
2. *The lax A-A-bimodule structure on Δ is strong,*
3. *There exists $\varepsilon' : A \to \mathbb{1}$ which is a counit for the non-unital coalgebra structure defined by Δ.*

Proof. A posteriori, we know that we must set $\mathrm{ev}' := \varepsilon' \circ \mu$. The triangle identities witnessing that ev' is an evaluation for the coevaluation $\mathrm{cv} = \Delta \circ \eta$ then follow immediately from the Frobenius identities which hold by (2):



Note also that it suffices to ask for *existence* of ε', and not choosing it, since the space of counits to Δ is contractible. For example, if ε and ε' are two counits, $\varepsilon \simeq (\varepsilon \circ \varepsilon') \circ \Delta \simeq \varepsilon'$ provides a canonical homotopy between them. $\square$

Definition 4.2.5. We call a locally rigid A *rigid* if $\eta : \mathbb{1} \to A$ admits a right adjoint. By the observation in the proof of Lemma 4.2.4, its right adjoint then automatically agrees with $\varepsilon : A \to \mathbb{1}$.

Let us consider some examples in $\mathcal{X} = (\mathrm{Pr}^L_{\mathrm{st}}, \otimes, \mathrm{Sp})$ to see this in action. In fact, we shall note already that we will later see that in this example one can give a slightly more efficient definition of being locally rigid, see Corollary 4.3.5 in the next section.

Example 4.2.6. For $A = \mathcal{D}(R)$, we may identify $A \otimes A$ with a category of $R \otimes_{\mathbb{S}} R$-modules (so R-R bimodule spectra). Under this identification, the multiplication is

$$(-) \otimes_{R \otimes_{\mathbb{S}} R} R,$$

and its right adjoint is restriction along the multiplication. The Frobenius identity is clear. Here, the unit $\mathrm{Sp} \to A$ does indeed have a right adjoint, the forgetful functor $\mathrm{Mod}(R) \to \mathrm{Sp}$. This is indeed a counit for the comultiplication. So $\mathcal{D}(R)$ is rigid.

Example 4.2.7. For a homological epi $R \to R/I$ and $A = \mathcal{D}(R, I)$, we have similar multiplication and comultiplication. But the unit does not admit a right adjoint anymore, since I is not necessarily compact, so $\mathrm{map}_{\mathcal{D}(R,I)}(I, -)$ does not preserve colimits. But the coevaluation $\mathrm{Sp} \to A \otimes A$ of the duality pairing is still given by I viewed as $R \otimes_{\mathbb{S}} R$-module, and the evaluation by $(-) \otimes_{R \otimes_{\mathbb{S}} R} R$, since the snake identity with this coevaluation candidate compares $(-) \otimes_R I$ with the identity, but they agree on $\mathcal{D}(R, I)$. From the duality we check that the counit is the forgetful functor again. So we are still locally rigid, but not rigid.

Example 4.2.8. For the p-complete category $\mathcal{D}(\mathbb{Z})^\wedge_p$, with monoidal structure given by the p-completed tensor product, the unit does again not have a right adjoint in Pr^L: the right adjoint is the forgetful functor, but this does not preserve colimits, so it is not a morphism in Pr^L. For example,

$$\mathbb{Z}/p[-1] \to \mathbb{Z}/p^2[-1] \to \ldots \to (\mathbb{Q}_p/\mathbb{Z}_p[-1])^\wedge_p \simeq \mathbb{Z}_p$$

is a colimit diagram in $\mathcal{D}(\mathbb{Z})^\wedge_p$ which is not preserved by the forgetful functor.

However, one can check that it is locally rigid, with counit $\mathcal{D}(\mathbb{Z})^\wedge_p \to \mathrm{Sp}$ given by $X \mapsto \mathrm{fib}(X \to X[p^{-1}])$. This is in fact the assembly of the forgetful functor, which for example takes $\mathbb{Z}_p$ to $\mathbb{Q}_p/\mathbb{Z}_p[-1]$ by the above colimit diagram.

Example 4.2.9. Let X be a locally compact Hausdorff space. For $A = \mathrm{Shv}(X; \mathrm{Sp})$, we may identify $A \otimes A$ with $\mathrm{Shv}(X \times X; \mathrm{Sp})$. Then:

1. The multiplication is Δ^* for the diagonal map $\Delta : X \to X \times X$.
2. Its right adjoint is Δ_*, which by properness of Δ also agrees with $\Delta_!$.
3. The Frobenius identity holds by proper base-change applied to the diagram

$$X \times X \xrightarrow{\Delta \times \mathrm{id}} X \times X \times X \xleftarrow{\mathrm{id} \times \Delta} X \times X.$$

4. The self-duality coevaluation is $\Delta_*(1)$, the pushforward of the constant sheaf on the diagonal.

5. The claim made in the introduction is that the self-duality of $\mathrm{Shv}(X;\mathrm{Sp})$ interchanges f_* with $f^!$ and f^* with $f_!$. Since the unit map $\mathrm{Shv}(\mathrm{pt};\mathrm{Sp}) \to \mathrm{Shv}(X;\mathrm{Sp})$ was t^* for $t : X \to \mathrm{pt}$ the constant map, the counit must (a posteriori) be $t_! : \mathrm{Shv}(X;\mathrm{Sp}) \to \mathrm{Shv}(\mathrm{pt};\mathrm{Sp})$. Note that t is not necessarily proper, so this is not necessarily the right adjoint of t^*. We therefore learn that the coevaluation must (a posteriori) be $t_! \circ \Delta^* : \mathrm{Shv}(X \times X;\mathrm{Sp}) \to \mathrm{Shv}(\mathrm{pt};\mathrm{Sp})$, i.e. compactly supported cohomology along the diagonal.

6. One can directly check that $t_!$ is a counit for Δ_*, since $\Delta_* = \Delta_!$ and $(-)_!$ is compatible with composition.

Note that $\mathrm{Shv}(X)$ is rigid if X is compact. Generally we want to think of rigidity as an abstract version of compactness.

In the above example, we have used (part of) the $f_!, f^!$ maps to prove that $\mathrm{Shv}(X;\mathrm{Sp})$ is locally rigid. In the following, we want to reverse the logic, and find other ways to characterize locally rigid categories. This will then produce $f_! \dashv f^!$ by dualizing $f^* \dashv f_*$ along the implied self-duality.

Finally, if we have a morphism $\mathcal{A} \to \mathcal{B}$ in $\mathrm{CAlg}(\mathrm{Pr}^L)$ then we say that $\mathcal{B}$ is locally rigid over $\mathcal{A}$ if $\mathcal{B}$ is locally rigid in $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Pr}^L)$ and similar for rigidity. Again we note that in this case one can give a slightly more efficient definition of being locally rigid, see Corollary 4.3.5 in the next section. We have already seen a number of example where $\mathcal{A} = \mathrm{Sp}$. Let us now give two more examples.

Proposition 4.2.10. *Let $f^* : \mathcal{A} \to \mathcal{B}$ be a map in $\mathrm{CAlg}(\mathrm{Pr}^L)$ and suppose that it admits a further left adjoint $f_!$ which is fully faithful and whose oplax $\mathcal{A}$-linear structure is strong. Then $\mathcal{B}$ is locally rigid over $\mathcal{A}$, and the counit is given by $f_!$.*

Proof. By assumption $f_! f^* : \mathcal{A} \to \mathcal{A}$ is an $\mathcal{A}$-linear colimit-preserving idempotent functor, and hence given by $E \otimes -$ for some idempotent algebra E in $\mathcal{A}$. Thus $f^* : \mathcal{A} \to \mathcal{B}$ is the smashing localization at E, and we can identify $\mathcal{B} \simeq \mathrm{Mod}_E(\mathcal{A})$. Under this identification, the multiplication of $\mathcal{B}$ is given by

$$\mathrm{Mod}_E(\mathcal{A}) \otimes_{\mathcal{A}} \mathrm{Mod}_E(\mathcal{A}) \simeq \mathrm{Mod}_{E\otimes E}(\mathcal{A}) \xrightarrow[\simeq]{E \otimes_{E\otimes E} -} \mathrm{Mod}_E(\mathcal{A}).$$

In particular, $\mu_{\mathcal{B}/\mathcal{A}} : \mathcal{B} \otimes_{\mathcal{A}} \mathcal{B} \to \mathcal{B}$ is an equivalence, and thus $\Delta_{\mathcal{B}/\mathcal{A}} = \mu_{\mathcal{B}/\mathcal{A}}^{-1}$ is automatically $\mathcal{B}$-$\mathcal{B}$-linear and colimit-preserving. Finally, consider the adjunctions

$$\begin{tikzcd} \mathcal{B} \arrow[r, shift right, "\mathcal{B}\otimes_{\mathcal{A}} f^*"'] \arrow[r, phantom, "\perp"] & \mathcal{B} \otimes_{\mathcal{A}} \mathcal{B} \arrow[l, shift right, "\mathcal{B}\otimes_{\mathcal{A}} f_!"'] \arrow[r, shift right, "\mu_{\mathcal{B}/\mathcal{A}}"'] \arrow[r, phantom, "\simeq"] & \mathcal{B} \arrow[l, shift right, "\Delta_{\mathcal{B}/\mathcal{A}}"'] \end{tikzcd}$$

where the left one is obtained by base-change. Since f^* is the unit for $\mu_{\mathcal{B}/\mathcal{A}}$, the lower and hence upper composite is the identity. Thus $f_!$ is the counit of $\Delta_{\mathcal{B}/\mathcal{A}}$, and $\mathcal{B}$ is locally rigid over $\mathcal{A}$ by 4.2.4. ∎

Proposition 4.2.11. *Assume that $\mathcal{B}$ is a (locally) rigid algebra over $\mathcal{A}$. Then for every map $\mathcal{A} \to \mathcal{A}'$ in $\mathrm{CAlg}(\mathrm{Pr}^L)$ the pushout*

$$\mathcal{A}' \to \mathcal{A}' \otimes_{\mathcal{A}} \mathcal{B}$$

is (locally) rigid.

Proof. This follows from the fact that the base-change $\mathcal{A}' \otimes_{\mathcal{A}} - : \mathrm{Mod}_{\mathcal{A}}(\mathrm{Pr}^L) \to \mathrm{Mod}_{\mathcal{A}'}(\mathrm{Pr}^L)$ is a symmetric monoidal 2-functor and thus preserves all terms used in the definition of (locally) rigid algebras. □

Example 4.2.12. Let X be a locally compact Hausdorff space and Y be an arbitrary space. Then pullback induces an algebra map

$$\mathrm{Shv}(Y) \to \mathrm{Shv}(X \times Y)$$

in Pr^L and this exhibits $\mathrm{Shv}(X \times Y)$ as a locally rigid $\mathrm{Shv}(Y)$-algebra. This follows by using local rigidity of $\mathrm{Shv}(X)$ and applying the above Corollary to the algebra map $\mathrm{Sp} \to \mathrm{Shv}(Y)$.

4.3 Properties of locally rigid categories

In this section, we prove the following important result, which says that for $\mathcal{B}$ locally rigid over $\mathcal{A}$, various notions agree whether we consider them relative over $\mathcal{B}$ or over $\mathcal{A}$. In the special case $\mathcal{A} = \mathrm{Sp}$, we obtain important equivalences between working $\mathcal{B}$-linearly or underlying.

Theorem 4.3.1 (Arinkin–Gaitsgory–Kazhdan–Raskin–Rozenblyum–Varshavsky)**.** *If $\mathcal{A} \to \mathcal{B}$ is a map of commutative algebras in Pr^L and $\mathcal{B}$ is locally rigid in $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Pr}^L)$, the following hold:*

1. *A $\mathcal{B}$-module $\mathcal{M}$ is dualizable in $\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^L)$ if and only if it is dualizable in $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Pr}^L)$.*
2. *A $\mathcal{B}$-algebra $\mathcal{C}$ is locally rigid in $\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^L)$ if and only if it is locally rigid in $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Pr}^L)$.*

We prove this in two statements (Propositions 4.3.3 and 4.3.10) and start with dualizability.

Lemma 4.3.2. *Suppose that $\mathcal{B}$ satisfies the first two conditions of being locally rigid in $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Pr}^L)$, i.e. its multiplication $\mu : \mathcal{B} \otimes_{\mathcal{A}} \mathcal{B} \to \mathcal{B}$ admits a strong $\mathcal{B}$-$\mathcal{B}$-linear right adjoint Δ. If $\mathcal{M}$ is a $\mathcal{B}$-module which is dualizable over $\mathcal{A}$ with $\mathcal{A}$-linear dual $\mathcal{M}^\vee$, and $\mathcal{N}$ is an arbitrary $\mathcal{B}$-module, we have a natural commutative diagram*

$$\begin{tikzcd}
\mathrm{Fun}^L_{\mathcal{B}}(\mathcal{M},\mathcal{N}) \arrow[r, "\simeq"] \arrow[d, shift left] & \mathcal{M}^\vee \otimes_{\mathcal{B}} \mathcal{N} \arrow[d, shift left] \\
\mathrm{Fun}^L_{\mathcal{A}}(\mathcal{M},\mathcal{N}) \arrow[r, "\simeq"] \arrow[u, shift left] & \mathcal{M}^\vee \otimes_{\mathcal{A}} \mathcal{N}, \arrow[u, shift left]
\end{tikzcd}$$

Proof. We have $\mathcal{M}^\vee \simeq \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{A})$. This inherits a $\mathcal{B}$-action through the $\mathcal{M}$ argument, for example by observing that $\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^L) \to \mathrm{Mod}_{\mathcal{A}}(\mathrm{Pr}^L)$, $\mathcal{N} \mapsto \mathcal{N} \otimes_{\mathcal{B}} \mathcal{M}$ has a right adjoint, which takes $\mathcal{A}$ to a $\mathcal{B}$-module with underlying $\mathcal{A}$-module $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{A})$. More generally, $\mathrm{Fun}^L_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) \simeq \mathcal{M}^\vee \otimes_{\mathcal{A}} \mathcal{N}$ by dualizability, and both sides inherit a $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{B}$ action.

Now we apply the 2-functors $\mathrm{Fun}_{\mathcal{B}\otimes_{\mathcal{A}}\mathcal{B}}(-, \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{N}))$ and $(\mathcal{M}^\vee \otimes_{\mathcal{A}} \mathcal{N}) \otimes_{\mathcal{B}\otimes_{\mathcal{A}}\mathcal{B}} (-)$ to the adjunction

$$\begin{tikzcd} \mathcal{B} \otimes_{\mathcal{A}} \mathcal{B} \arrow[r, shift left, "\mu"] & \mathcal{B}. \arrow[l, shift left, "\Delta"] \end{tikzcd}$$

and obtain adjunctions

$$\begin{tikzcd} \mathrm{Fun}^L_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) \arrow[r, shift left] & \mathrm{Fun}^L_{\mathcal{B}}(\mathcal{M}, \mathcal{N}) \arrow[l, shift left] \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{M}^\vee \otimes_{\mathcal{A}} \mathcal{N} \arrow[r, shift left] & \mathcal{M}^\vee \otimes_{\mathcal{B}} \mathcal{N}, \arrow[l, shift left] \end{tikzcd}$$

where the left hand terms agree by $\mathcal{A}$-linear dualizability of $\mathcal{M}$. Now both of these adjunctions are monadic: Preservation of colimits is automatic, and in the top one the right adjoint is conservative since it is the forgetful functor, whereas in the bottom the right adjoint is conservative since the left adjoint generates. As the composite of the adjoints on the original adjunction is just multiplication with the object $\Delta(1) \in \mathcal{B} \otimes_{\mathcal{A}} \mathcal{B}$, and everything was $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{B}$-linear, the monads on $\mathrm{Fun}^L_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) \simeq \mathcal{M}^\vee \otimes_{\mathcal{A}} \mathcal{N}$ agree, and the claim follows. □

Proposition 4.3.3 (Arinkin–Gaitsgory–Kazhdan–Raskin–Rozenblyum–Varshavsky)**.** *Let $\mathcal{A} \to \mathcal{B}$ be a map of commutative algebras in Pr^L, and assume that $\mathcal{B}$ is locally rigid as algebra in $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Pr}^L)$. Then $\mathcal{M} \in \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^L)$ is dualizable if and only if it is dualizable in $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Pr}^L)$.*

Proof. If $\mathcal{M}$ is $\mathcal{B}$-linearly dualizable, it is underlying dualizable by applying Lemma 4.1.7 to $\mathcal{B}$ viewed as Frobenius algebra in $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Pr}^L)$. We now prove the converse. For $\mathcal{N} = \mathcal{B}$, the equivalence from Lemma 4.3.2 shows that $\mathcal{M}^\vee = \mathrm{Fun}^L_{\mathcal{A}}(\mathcal{M}, \mathcal{A})$ is also the $\mathcal{B}$-linear dual $\mathrm{Fun}^L_{\mathcal{B}}(\mathcal{M}, \mathcal{B})$. The diagram

$$\begin{tikzcd} \mathrm{Fun}^L_{\mathcal{B}}(\mathcal{M}, \mathcal{B}) \arrow[r, "\simeq"] \arrow[d] & \mathcal{M}^\vee \otimes_{\mathcal{B}} \mathcal{B} \arrow[d] \\ \mathrm{Fun}^L_{\mathcal{A}}(\mathcal{M}, \mathcal{B}) \arrow[r, "\simeq"] \arrow[d, "\varepsilon"] & \mathcal{M}^\vee \otimes_{\mathcal{A}} \mathcal{B} \arrow[d, "\varepsilon"] \\ \mathrm{Fun}^L_{\mathcal{A}}(\mathcal{M}, \mathcal{A}) \arrow[r, "\simeq"] & \mathcal{M}^\vee \otimes_{\mathcal{A}} \mathcal{A}. \end{tikzcd}$$

shows that the corresponding equivalence is given by the composite $\mathrm{Fun}_{\mathcal{B}}(\mathcal{M}, \mathcal{B}) \to \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{B}) \xrightarrow{\varepsilon} \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{A})$.

Applying the lemma to $\mathcal{N} = \mathcal{M}$, we also have a coevaluation $\mathcal{B} \to \mathcal{M} \otimes_{\mathcal{B}} \mathcal{M}^\vee \simeq \mathrm{Fun}_{\mathcal{B}}(\mathcal{M}, \mathcal{M})$ given by the identity object, lifting the $\mathcal{A}$-linear coevaluation under the forgetful map. So the diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r] \arrow[d, "\eta"] & \mathcal{M} \otimes_{\mathcal{A}} \mathcal{M}^\vee \arrow[r, "\simeq"] & \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{M}) \\
\mathcal{B} \arrow[r] & \mathcal{M} \otimes_{\mathcal{B}} \mathcal{M}^\vee \arrow[r, "\simeq"] \arrow[u] & \mathrm{Fun}_{\mathcal{B}}(\mathcal{M}, \mathcal{M}) \arrow[u]
\end{tikzcd}$$

commutes. Similarly, the diagram

$$\begin{tikzcd}
\mathcal{M}^\vee \otimes_{\mathcal{A}} \mathcal{M} \arrow[r] \arrow[d] & \mathcal{A} \\
\mathcal{M}^\vee \otimes_{\mathcal{B}} \mathcal{M} \arrow[r] & \mathcal{B} \arrow[u, "\varepsilon"']
\end{tikzcd}$$

commutes, using that

$$\begin{tikzcd}
 & \mathrm{Fun}^L_{\mathcal{A}}(\mathcal{M}, \mathcal{A}) \otimes_{\mathcal{A}} \mathcal{M} \arrow[r] & \mathcal{A} \\
\mathrm{Fun}^L_{\mathcal{B}}(\mathcal{M}, \mathcal{B}) \otimes_{\mathcal{A}} \mathcal{M} \arrow[r] \arrow[d] \arrow[ur, "\simeq"] & \mathrm{Fun}^L_{\mathcal{A}}(\mathcal{M}, \mathcal{B}) \otimes_{\mathcal{A}} \mathcal{M} \arrow[r] \arrow[d] \arrow[u, "\varepsilon\otimes\mathrm{id}_{\mathcal{M}}"'] & \mathcal{B} \arrow[u, "\varepsilon"'] \arrow[d, "="] \\
\mathrm{Fun}^L_{\mathcal{B}}(\mathcal{M}, \mathcal{B}) \otimes_{\mathcal{B}} \mathcal{M} \arrow[r] & \mathrm{Fun}^L_{\mathcal{A}}(\mathcal{M}, \mathcal{B}) \otimes_{\mathcal{B}} \mathcal{M} \arrow[r] & \mathcal{B}
\end{tikzcd}$$

commutes and $\mathrm{Fun}^L_{\mathcal{B}}(\mathcal{M}, \mathcal{B}) \to \mathrm{Fun}^L_{\mathcal{A}}(\mathcal{M}, \mathcal{B}) \xrightarrow{\varepsilon} \mathrm{Fun}^L_{\mathcal{A}}(\mathcal{M}, \mathcal{A})$ is the equivalence deduced above. As in Lemma 4.1.7, these factorisations produce a $\mathcal{B}$-linear duality. □

Remark 4.3.4. Actually, none of these statements used property 3 of the definition of locally rigid algebras, just that $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{B} \to \mathcal{B}$ has a right adjoint whose lax $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{B}$-module structure is strong.

Corollary 4.3.5. *Let $\mathcal{B}$ be an algebra in $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Pr}^L)$ which satisfies*

1. *$\mathcal{B} \otimes_{\mathcal{A}} \mathcal{B} \to \mathcal{B}$ admits a right adjoint Δ.*

2. *The lax $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{B}$-module structure on Δ is strong.*

3' *$\mathcal{B}$ is dualizable in $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Pr}^L)$.*

Then $\mathcal{B}$ is locally rigid in $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Pr}^L)$.

Proof. From the previous proposition and remark, we learn that $\mathcal{B}$ is $\mathcal{B}$-linearly dualizable, and its dual canonically identifies with $\mathrm{Fun}^L_{\mathcal{B}}(\mathcal{B}, \mathcal{B}) = \mathcal{B}$. Under $\mathcal{B}^\vee \otimes_{\mathcal{A}} \mathcal{B} \simeq \mathrm{Fun}_{\mathcal{A}}(\mathcal{B}, \mathcal{B})$, the coevaluation of the canonical duality is the identity functor, i.e. the composite $\mathcal{A} \to \mathcal{B} \xrightarrow{\mathrm{id}}$

$\mathrm{Fun}^L_{\mathcal{B}}(\mathcal{B},\mathcal{B}) \to \mathrm{Fun}^L_{\mathcal{A}}(\mathcal{B},\mathcal{B})$, where the last functor is the forgetful map. Identifying the last two terms with $\mathcal{B}^\vee \otimes_{\mathcal{B}} \mathcal{B} \to \mathcal{B}^\vee \otimes_{\mathcal{A}} \mathcal{B}$, the map becomes the right adjoint to the canonical map $\mathcal{B}^\vee \otimes_{\mathcal{A}} \mathcal{B} \to \mathcal{B}^\vee \otimes_{\mathcal{B}} \mathcal{B}$. Under the identification $\mathcal{B}^\vee \simeq \mathcal{B}$, this becomes the multiplication map. So in total, we have identified the coevaluation with $\mathcal{A} \to \mathcal{B} \xrightarrow{\Delta} \mathcal{B} \otimes_{\mathcal{A}} \mathcal{B}$, proving that $\mathcal{B}$ is locally rigid. □

Example 4.3.6. Let R be a commutative ring spectrum. Then a stable, presentable R-linear category $\mathcal{C}$ is by definition a module over $\mathrm{Mod}(R)$ in Pr^L. The last assertion implies that $\mathcal{C}$ is dualizable relative to $\mathrm{Mod}(R)$ iff it is absolutely dualizable (that is relative to spectra). In particular for R an ordinary ring one could model the R-linear theory by dg-categories and thus conclude that being dualizable in this world is the same as being dualizable relative to the sphere.

Lemma 4.3.7. *Let $\mathcal{B}$ be a locally rigid algebra in $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Pr}^L)$, and assume $\mathcal{M} \to \mathcal{N}$ is a map of $\mathcal{B}$-modules which admits a right adjoint. Assume furthermore that the lax $\mathcal{A}$-module structure on the right adjoint is strong. Then also the lax $\mathcal{B}$-module structure is strong.*

Proof. Base-changing the comultiplication $\mathcal{B} \to \mathcal{B} \otimes_{\mathcal{A}} \mathcal{B}$ along $\mathcal{M} \otimes_{\mathcal{B}} (-)$, we obtain a coaction map $\mathcal{M} \to \mathcal{M} \otimes_{\mathcal{A}} \mathcal{B}$. Base-changing the adjunction

$$\begin{tikzcd} \mathcal{B} \otimes_{\mathcal{A}} \mathcal{B} \arrow[r, shift left] & \mathcal{B} \arrow[l, shift left] \end{tikzcd}$$

we obtain an adjunction

$$\begin{tikzcd} \mathcal{M} \otimes_{\mathcal{A}} \mathcal{B} \arrow[r, shift left] & \mathcal{M} \arrow[l, shift left] \end{tikzcd}$$

between the action and coaction maps. Base-changing the diagram

$$\begin{tikzcd} \mathcal{B} \arrow[rr, "\Delta"] \arrow[dr, "\mathrm{id} \otimes \mathrm{coev}"'] & & \mathcal{B} \otimes_{\mathcal{A}} \mathcal{B} \\ & \mathcal{B} \otimes_{\mathcal{A}} \mathcal{B} \otimes_{\mathcal{A}} \mathcal{B}, \arrow[ur, "\mu \otimes \mathrm{id}"'] & \end{tikzcd}$$

we witness that the coaction map $\mathcal{M} \to \mathcal{M} \otimes_{\mathcal{A}} \mathcal{B}$ and the action map $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{B} \to \mathcal{M}$ are related through "dualizing the $\mathcal{B}$ over". Starting with the commutative diagram of left adjoints

$$\begin{tikzcd} \mathcal{M} \otimes_{\mathcal{A}} \mathcal{B} \arrow[r] \arrow[d, "f \otimes \mathrm{id}"] & \mathcal{M} \arrow[d, "f"] \\ \mathcal{N} \otimes_{\mathcal{A}} \mathcal{B} \arrow[r] & \mathcal{N}, \end{tikzcd}$$

we obtain a diagram of right adjoints

$$\begin{tikzcd} \mathcal{M} \otimes_{\mathcal{A}} \mathcal{B} & \mathcal{M} \arrow[l] \\ \mathcal{N} \otimes_{\mathcal{A}} \mathcal{B} \arrow[u, "R_f \otimes \mathrm{id}"] & \mathcal{N}, \arrow[l] \arrow[u, "f"'] \end{tikzcd}$$

where the strong $\mathcal{A}$-linearity of R_f enters in the claim that the right adjoint of $f \otimes \mathrm{id}$ is $R_f \otimes \mathrm{id}$. Now, we may dualize over the $\mathcal{B}$ factors in the horizontal direction to turn this into a diagram witnessing that R_f also commutes with the action map, proving the claim. $\square$

Given a 2-category $\mathcal{X}$, denote by $\mathcal{X}^{iL}$ the full subcategory of the underlying 1-category on the internal left adjoints.

Lemma 4.3.8. *For $\mathcal{A} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ the forgetful functor $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Pr}^L)^{iL} \to \mathrm{Pr}^L$ creates colimits.*

Proof. We already know that $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Pr}^L) \to \mathrm{Pr}^L$ creates colimits, so it remains to show that $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Pr}^L)^{iL} \to \mathrm{Mod}_{\mathcal{A}}(\mathrm{Pr}^L)$ does too. Let $\mathcal{C}_\bullet : I \to \mathrm{Mod}_{\mathcal{A}}(\mathrm{Pr}^L)^{iL}$ be a diagram and $\mathcal{C}$ its colimit in $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Pr}^L)$. Moreover, let $\mathcal{C}_\bullet \Rightarrow \mathrm{const}\,\mathcal{D}$ be a cone in $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Pr}^L)^{iL}$, i.e. where each $\mathcal{C}_i \to \mathcal{D}$ is also an internal left adjoint. We need to show that the colimit cone and also the map $\mathcal{C} \to \mathcal{D}$ in $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Pr}^L)$ induced by the universal property of $\mathcal{C}$ again lie in $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Pr}^L)^{iL}$.

Taking right adjoints we obtain a cone $\mathcal{D} \Rightarrow \mathcal{C}_\bullet$ which by assumptions also lies in $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Pr}^L)$. Hence the limit cone and induced map to the limit $\mathcal{D} \to \mathcal{C}$ also lie in $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Pr}^L)$, as desired. $\square$

Remark 4.3.9. Let $\mathcal{B} \to \mathcal{C}$ be a map in $\mathrm{CAlg}(\mathrm{Pr}^L)$ and $F : \mathcal{M} \to \mathcal{N}$ a lax $\mathcal{C} \otimes_{\mathcal{B}} \mathcal{C}$-linear colimit preserving functor (of $\mathcal{C} \otimes_{\mathcal{B}} \mathcal{C}$-modules), with structure maps

$$\alpha_{x,m} : x \otimes F(m) \to F(x \otimes m)$$

for $x \in \mathcal{C} \otimes_{\mathcal{B}} \mathcal{C}$ and $m \in \mathcal{M}$. By restricting along the algebra maps

$$\ell : \mathcal{C} \xrightarrow{(-,1)} \mathcal{C} \otimes_{\mathcal{B}} \mathcal{C} \qquad \text{and} \qquad r : \mathcal{C} \xrightarrow{(1,-)} \mathcal{C} \otimes_{\mathcal{B}} \mathcal{C}$$

we obtain lax $\mathcal{C}$-linear functors $\ell^* F : \ell^*\mathcal{M} \to \ell^*\mathcal{N}$ and $r^*F : r^*\mathcal{M} \to r^*\mathcal{N}$.

We claim that to show that F is strong $\mathcal{C} \otimes_{\mathcal{B}} \mathcal{C}$-linear, it suffices to show that ℓ^*F and r^*F are strong $\mathcal{C}$-linear. Indeed, note first that since F preserves colimits and $\mathcal{C} \otimes_{\mathcal{B}} \mathcal{C}$ is generated under colimits by the image of $\mathcal{C} \times \mathcal{C} \to \mathcal{C} \otimes \mathcal{C} \to \mathcal{C} \otimes_{\mathcal{B}} \mathcal{C}$, i.e. by elementary tensors, it suffices to check that each $\alpha_{c \otimes c', m}$ is an equivalence. Next, note that the lax structure maps for ℓ^*F respectively r^*F at (c, m) are given by $\alpha_{c \otimes 1, m}$ respectively $\alpha_{1 \otimes c, m}$. Moreover, since $c \otimes c' \simeq (c \otimes 1) \otimes (1 \otimes c')$ inside $\mathcal{C} \otimes_{\mathcal{B}} \mathcal{C}$, we have $\alpha_{c \otimes c', m} \simeq \alpha_{1 \otimes c', c \otimes m} \circ \alpha_{c \otimes 1, m}$. Since equivalences compose, this shows the claim.

Proposition 4.3.10 (Arinkin–Gaitsgory–Kazhdan–Raskin–Rozenblyum–Varshavsky)**.** *Let $\mathcal{A} \to \mathcal{B}$ be a map of commutative algebras in Pr^L, and assume that $\mathcal{B}$ is locally rigid as algebra in $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Pr}^L)$. Then a $\mathcal{B}$-algebra $\mathcal{C}$ is locally rigid in $\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^L)$ if and only if it is locally rigid in $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Pr}^L)$. In this case we have commutative diagrams*

$$\begin{tikzcd}
\mathcal{C} \otimes_{\mathcal{A}} \mathcal{C} \arrow[r] \arrow[rr, bend left, "\mu_{\mathcal{C}/\mathcal{A}}"] & \mathcal{C} \otimes_{\mathcal{B}} \mathcal{C} \arrow[r, "\mu_{\mathcal{C}/\mathcal{B}}"'] & \mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C} \arrow[rr, "\varepsilon_{\mathcal{C}/\mathcal{B}}"] \arrow[dr, "\varepsilon_{\mathcal{C}/\mathcal{A}}"'] & & \mathcal{B} \arrow[dl, "\varepsilon_{\mathcal{B}/\mathcal{A}}"] \\
& \mathcal{A} &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{C} \otimes_{\mathcal{A}} \mathcal{C} & \mathcal{C} \otimes_{\mathcal{B}} \mathcal{C} \arrow[l, "\mathcal{C} \otimes_{\mathcal{B}} \Delta_{\mathcal{B}/\mathcal{A}} \otimes_{\mathcal{B}} \mathcal{C}"'] & \mathcal{C} \arrow[l, "\Delta_{\mathcal{C}/\mathcal{B}}"'] \arrow[ll, bend left, "\Delta_{\mathcal{C}/\mathcal{A}}"]
\end{tikzcd}$$

and $\varepsilon_{\mathcal{C}/\mathcal{B}}$ is the $\mathcal{A}$-linear dual of the map $\mathcal{B} \to \mathcal{C}$.

Proof. First assume that $\mathcal{C}$ is locally rigid over $\mathcal{B}$. In particular, $\mathcal{C}$ is dualizable in $\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^L)$ and hence also in $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Pr}^L)$ by Proposition 4.3.3. Moreover, the multiplication $\mu_{\mathcal{C}/\mathcal{B}} : \mathcal{C} \otimes_{\mathcal{B}} \mathcal{C} \to \mathcal{C}$ respectively $\mu_{\mathcal{B}/\mathcal{A}} : \mathcal{B} \otimes_{\mathcal{A}} \mathcal{B} \to \mathcal{B}$ admits a $\mathcal{C} \otimes_{\mathcal{B}} \mathcal{C}$-linear right adjoint $\Delta_{\mathcal{C}/\mathcal{B}}$ respectively $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{B}$-linear right adjoint $\Delta_{\mathcal{B}/\mathcal{A}}$. Applying the 2-functors $\mathcal{C} \otimes_{\mathcal{B}} -$ and $- \otimes_{\mathcal{B}} \mathcal{C}$ to the latter adjunction shows also that $\mathcal{C} \otimes_{\mathcal{A}} \mathcal{C} \to \mathcal{C} \otimes_{\mathcal{B}} \mathcal{C}$ admits a $\mathcal{C} \otimes_{\mathcal{A}} \mathcal{C}$-linear right adjoint $\mathcal{C} \otimes_{\mathcal{B}} \Delta_{\mathcal{B}/\mathcal{A}} \otimes_{\mathcal{B}} \mathcal{C}$. Composing, we see that the $\mathcal{C} \otimes_{\mathcal{A}} \mathcal{C}$-linear multiplication map

$$\mathcal{C} \otimes_{\mathcal{A}} \mathcal{C} \to \mathcal{C} \otimes_{\mathcal{B}} \mathcal{C} \xrightarrow{\mu_{\mathcal{C}/\mathcal{B}}} \mathcal{C}$$

also admits a $\mathcal{C} \otimes_{\mathcal{A}} \mathcal{C}$-linear right adjoint as claimed. In view of Corollary 4.3.5, this shows that $\mathcal{C}$ is also locally rigid over $\mathcal{A}$.

For the converse, we assume that $\mathcal{C}$ is locally rigid over $\mathcal{A}$. Again it follows from Proposition 4.3.3 that $\mathcal{C}$ is dualizable in $\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^L)$. Next, note that Lemma 4.3.8 applies to show that $\mu_{\mathcal{C}/\mathcal{B}}$ is a left adjoint internal to $\mathrm{Mod}_{\mathcal{A}\otimes\mathcal{A}}(\mathrm{Pr}^L)$. Indeed, each of the maps $\mathcal{C} \otimes_{\mathcal{A}} \mathcal{C} \to \mathcal{C}$, $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{C} \to \mathcal{C}$ and $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{B} \to \mathcal{B}$ is such an internal left adjoint, hence $\mathcal{C} \otimes_{\mathcal{A}} \mathcal{B}^{\otimes_{\mathcal{A}} n} \otimes_{\mathcal{A}} \mathcal{C} \to \mathcal{C}$ and all maps in the $\Delta^{\mathrm{op}}_{\mathrm{inj}}$-indexed diagram are too.

If we combine Remark 4.3.9 and Lemma 4.3.7 applied to $\mathcal{C}/\mathcal{A}$ with the fact that $\Delta_{\mathcal{C}/\mathcal{B}}$ is now colimit-preserving and strong $\mathcal{A} \otimes \mathcal{A}$-linear, it follows that the lax $\mathcal{C} \otimes_{\mathcal{B}} \mathcal{C}$-linearity is in fact strong. Hence $\mathcal{C}$ is locally rigid over $\mathcal{B}$ by Corollary 4.3.5.

Finally, we check the statement that the counits compose as indicated. Note that by definition $\varepsilon_{\mathcal{C}/\mathcal{A}}$ and $\varepsilon_{\mathcal{B}/\mathcal{A}}$ are the $\mathcal{A}$-linear duals of $\mathcal{A} \to \mathcal{C}$ respectively $\mathcal{A} \to \mathcal{B}$. Similarly, $\varepsilon_{\mathcal{C}/\mathcal{B}}$ is by definition the $\mathcal{B}$-linear dual of $\mathcal{B} \to \mathcal{C}$, and hence in view of Corollary 4.1.8 also the $\mathcal{A}$-linear dual of $\mathcal{B} \to \mathcal{C}$. Hence the triangle commutes as it is the $\mathcal{A}$-linear dual of the given commutative triangle

$$\begin{tikzcd} \mathcal{C} & & \mathcal{B} \arrow[ll] \\ & \mathcal{A} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

$\square$

Corollary 4.3.11. *If $\mathcal{A} \to \mathcal{B}$ and $\mathcal{B} \to \mathcal{C}$ are rigid, so is the composite $\mathcal{A} \to \mathcal{C}$.*

Proof. Both internal left adjoints and locally rigid morphisms (by the above) are closed under composition. $\square$

Corollary 4.3.12. *Let $\mathcal{B} \to \mathcal{B}'$ be a map of $\mathcal{A}$-algebras in Pr^L which admits a fully faithful and $\mathcal{B}$-linear left adjoint. If $\mathcal{B}$ is locally rigid over $\mathcal{A}$, then so is $\mathcal{B}'$.*

Proof. Combine 4.2.10 and 4.3.10. $\square$

Remark 4.3.13. We want to think of those maps $\mathcal{B} \to \mathcal{B}'$ as 'open' maps. Therefore the last proposition is the assertion that being locally rigid is a property that holds for open subcategories of categories as well.

Definition 4.3.14. Let $\mathcal{A} \to \mathcal{B}$ be a map in $\mathrm{CAlg}(\mathrm{Pr}^L)$. Then a rigidification of $\mathcal{B}$ over $\mathcal{A}$ is a factorization

$$\mathcal{A} \to \overline{\mathcal{B}} \to \mathcal{B}$$

in $\mathrm{CAlg}(\mathrm{Pr}^L)$ where $\mathcal{B}$ is rigid over $\mathcal{A}$ and $\overline{\mathcal{B}} \to \mathcal{B}$ admits a $\overline{\mathcal{B}}$-linear left adjoint which is fully faithful.

Using this terminology we see that Corollary 4.3.12 states that if $\mathcal{B}$ admits a rigidification over $\mathcal{A}$ then $\mathcal{B}$ has to be locally rigid over $\mathcal{A}$. We will later see the converse. Namely that every locally rigid category $\mathcal{B}$ over some base $\mathcal{A}$ in $\mathrm{CAlg}(\mathrm{Pr}^L)$ admits a rigidification, in fact a universal one. Thus the locally rigid categories can also be characterized as those that admit a rigidification.

Next, let us investigate under which colimits in $\mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ locally rigid categories are closed. The case of rigid categories will be considered in the next section.

Proposition 4.3.15. *Let* $\mathrm{LocRig}^L \subset \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ *be the subcategory on locally rigid categories and symmetric monoidal strong left adjoint functors. Then the forgetful functor*

$$\mathrm{LocRig}^L \to \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$$

creates sifted colimits. Moreover, a finite coproduct (in $\mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$*) of locally rigid categories is again locally rigid. In particular, colimits of locally rigid categories along symmetric monoidal strong left adjoints in* $\mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ *are again locally rigid.*

Proof. We first check finite coproducts, i.e. finite tensor products. So let $\mathcal{C}$ and $\mathcal{D}$ be locally rigid. By Proposition 4.2.11 $\mathcal{C} \to \mathcal{C} \otimes \mathcal{D}$ is locally rigid, and hence by Proposition 4.3.10 also the composite $\mathrm{Sp} \to \mathcal{C} \otimes \mathcal{D}$ is, as desired.

Next, we check sifted colimits, so let $\mathcal{C}_\bullet : J \to \mathrm{LocRig}^L$ be a sifted diagram. Denote $\mathcal{C} := \mathrm{colim}_j^{\mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})} \mathcal{C}_j$ and write $\mu_j : \mathcal{C}_j \otimes \mathcal{C}_j \to \mathcal{C}_j$ and $\mu : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ for the multiplications. By siftedness we have $\mu \simeq \mathrm{colim}_j \mu_j$ in $\mathrm{Ar}(\mathrm{Pr}^L_{\mathrm{dual}})$ (where we use that also $\mathcal{C}_i \otimes \mathcal{C}_i \to \mathcal{C}_j \otimes \mathcal{C}_j$ are strong left adjoints). In particular, $\mathcal{C}$ is dualizable and μ is a strong left adjoint, with right adjoint Δ that is then Sp-linear and lax $\mathcal{C}$-linear. By restriction Δ is also lax $\mathcal{C}_j$-linear for each j, and so by local rigidity of the $\mathcal{C}_j$ and Lemma 4.3.7 Δ is also strong $\mathcal{C}_j$-linear for each j. Since Δ preserves colimits and $\mathcal{C}$ is generated under colimits by the images of the $\mathcal{C}_j$, it follows that Δ is also $\mathcal{C}$-linear. Thus $\mathcal{C}$ is locally rigid by Corollary 4.3.5.

Moreover, it follows from Lemma 4.3.8 for $\mathcal{A} = \mathrm{Sp}$ (equivalently, from Theorem 2.6.5) that the entire colimit cone already lies in LocRig^L, i.e. that the induced maps $\mathcal{C}_j \to \mathcal{C}$ are again strong left adjoints. Analogously, it follows that given a cone $\mathcal{C}_\bullet \Rightarrow \mathcal{D}$ in LocRig^L, the map $\mathcal{C} \to \mathcal{D}$ induced from the universal property of the colimit in $\mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ is automatically a strong left adjoint. This proves that the forgetful $\mathrm{LocRig}^L \to \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ creates sifted colimits. □

Remark 4.3.16. If we take the coproduct of locally rigid categories in $\mathrm{CAlg}(\mathrm{Pr}^L)$, then this will generally not be a coproduct in LocRig^L anymore, i.e. the structure maps $\mathcal{C} \to \mathcal{C} \otimes \mathcal{D} \leftarrow \mathcal{D}$ need not be strong left adjoints. Indeed, taking $\mathcal{C} = \mathrm{Sp}$ this would precisely require the unit map $\mathrm{Sp} \to \mathcal{D}$ to be a strong left adjoint, i.e. $\mathcal{D}$ to be rigid and not just locally rigid.

4.4 Trace class morphisms and rigidification

Definition 4.4.1. A map $f : X \to Y$ in a symmetric monoidal ∞-category $\mathcal{C}$ is called trace-class if there exists an object $Z \in \mathcal{C}$, morphisms $\alpha : \mathbb{1} \to Y \otimes Z$ and $\beta : Z \otimes X \to \mathbb{1}$ and a homotopy witnessing commutativity of the following diagram

$$\begin{tikzcd} & Y \otimes Z \otimes X \arrow[dr, "Y\otimes\beta"] & \\ X \arrow[ur, "\alpha\otimes X"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

Example 4.4.2. Let $\mathcal{C}$ be the 1-category Vect_k of vector spaces over a field k. We claim that $f : X \to Y$ is trace class precisely if it is finite rank, that is it factors over a finite dimensional subspace $U \subseteq Y$. First if we have such a factorization, then consider $X \xrightarrow{p} U \xrightarrow{i} W$. We set $Z = U^\vee$ and consider $\alpha : k \to Y \otimes U^\vee = [U, Y]$ given by $1 \mapsto i$ which is also the composite $k \to U \otimes U^\vee \to Y \otimes U^\vee$ and $\beta : U^\vee \otimes X \to k$ given by $\mathrm{ev} \circ (f^* \otimes \mathrm{id})$. Then we consider the diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{coev}\otimes\mathrm{id}"] \arrow[d] & U \otimes U^\vee \otimes X \arrow[r, "i\otimes\mathrm{id}"] \arrow[d] & Y \otimes U^\vee \otimes X \arrow[d, "\mathrm{id}\otimes f"] \\ U \arrow[r, "\mathrm{coev}\otimes\mathrm{id}"] \arrow[dr, "\mathrm{id}"] & U \otimes U^\vee \otimes U \arrow[r] \arrow[d] & Y \otimes U^\vee \otimes U \arrow[d] \\ & U \arrow[r] & Y \end{tikzcd}$$

to verify the claim. Before we prove the converse we want to state some general properties of trace class maps.

Let us note some easy consequences of the above definition.

Lemma 4.4.3. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category.*

1. *If f factors through a dualizable object D, then f is trace-class.*

2. *If $\mathcal{C}$ is closed symmetric monoidal with internal hom $[-,-] : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$, then it is equivalent to demand $Z = [X, \mathbb{1}]$ and $\beta : [X, \mathbb{1}] \otimes X \to \mathbb{1}$ to be the evaluation. By adjoining X over, we see that f is trace-class iff there exists a lift in the following diagram*

$$\begin{tikzcd} & {[X, \mathbb{1}] \otimes Y} \arrow[d] \\ \mathbb{1} \arrow[ur, dashed] \arrow[r, "\hat{f}"'] & {[X, Y]} \end{tikzcd}$$

3. *If $\mathcal{C}$ is closed symmetric monoidal then id_X is trace-class iff X is dualizable.*

4. *The trace-class morphisms form a 2-sided ideal.*

5. *Symmetric monoidal functors preserve trace-class morphisms.*

Proof. Claim (1) is the exact argument as given in the previous example. For (2) we need to show that if $f : X \to Y$ is trace class with witness Z then it is trace class with witness $[X, \mathbb{1}]$. We consider the adjunct of the map $\beta : Z \otimes X \to \mathbb{1}$, which is a map

$$\tilde{\beta} : Z \to [X, \mathbb{1}]\ .$$

Then we apply this to α to obtain a map

$$\alpha' : \mathbb{1} \to Y \otimes Z \xrightarrow{Y \otimes \tilde{\beta}} Y \otimes [X, \mathbb{1}]\ .$$

Now we assert that the composite $X \to Y \otimes [X, \mathbb{1}] \otimes X \to Y$ is given by f. This follows from the commutativity of the diagram

$$\begin{tikzcd}
 & Y \otimes Z \otimes X \arrow[d, dashed, "Y\otimes\tilde{\beta}\otimes X"] \arrow[dr, "Y\otimes\beta"] & \\
X \arrow[ur, "\alpha\otimes X"] \arrow[r, "\alpha'\otimes X"'] & Y \otimes [X, \mathbb{1}] \otimes X \arrow[r, "Y\otimes \mathrm{ev}"'] & Y
\end{tikzcd}$$

Since we are in a closed symmetric monoidal setting, (3) follows from Lemma 2.9.1. Next, point (4) is easily seen from (2) and the fact that $[X, \mathbb{1}] \otimes Y \to [X, Y]$ is natural in both X and Y. Finally (5) is clear from the original definition of trace-class maps. □

Example 4.4.4. We continue the example of vector spaces. Let $f : X \to Y$ be a trace class map of vector spaces and write $Y = \operatorname{colim} U_i$. Then we have a lift of $f \in [X, Y]$ to $X^\vee \otimes Y$. This lift will then lie in some $X^\vee \otimes U_i$ and thus we have a factorization $X \to U_i \to Y$. In fact, we see that the map $X \to U_i$ is even trace class.

More generally the argument of the last example shows:

Lemma 4.4.5. *Let $\mathcal{C}$ be a closed symmetric monoidal presentable category that is dualizable with compact tensor unit. Then every trace class map is compact.*

Proof. Let $X \to Y$ be trace class witness $\mathbb{1} \to X^\vee \otimes Y$ and a map $Y \to \operatorname{colim} D_i$. The composite $X \to \operatorname{colim} D_i$ then is trace class, as witnessed by $\mathbb{1} \to X^\vee \otimes \operatorname{colim} D_i$. Note that since $\mathbb{1}$ is compact and tensor product commutes with filtered colimits we find a factorization $\mathbb{1} \to X^\vee \otimes D_i$ that shows the claim that $X \to \operatorname{colim} D_i$ factors over a finite stage. □

We note that the converse also holds: if every trace class map is compact, then the tensor unit is compact. This easily follows from the fact that the identity of $\mathbb{1}$ is trace class. Together this shows the first part in the following result:

Theorem 4.4.6. *Assume that $\mathcal{C}$ is a dualizable, stable ∞-category which carries a closed symmetric monoidal structure. Then*

1. *The unit is compact precisely if every trace class morphism is compact.*

2. $\mathcal{C}$ is locally rigid precisely if every compact morphism is trace class.

3. $\mathcal{C}$ is rigid precisely if the compact morphisms and trace class morphisms agree.

The third assertion of the theorem is a combination of the first two. Thus it remains to show (2), which we will do in the next section. We would like to first present some examples and important consequences.

Example 4.4.7. Assume $\mathcal{C}$ is stable, presentably symmetric-monoidal and compactly generated. Then

1. The unit is compact if and only if every dualizable object is compact.

2. $\mathcal{C}$ is locally rigid if and only if every compact object is dualizable.

3. $\mathcal{C}$ is rigid if the compact and dualizable objects agree.

Proof. Since an object is dualizable if and only if its identity is trace class, the first statement follows from the first statement of Theorem 4.4.6. Similarly, the "only if" direction of the second statement follows directly from the theorem, whereas for the "if" direction we assume that every compact object is dualizable and let $f : X \to Y$ be a compact morphism. Then it factors through a compact object, which is dualizable, and so f is trace-class.

Finally, the third statement is again a combination of the first two. $\square$

Remark 4.4.8. While in a compactly generated category, the compact morphisms are exactly the ones which factor through a compact object, the trace-class morphisms are not necessarily the ones which factor through dualizable objects (unless the category is also rigid). Consider $\mathcal{D}(\mathbb{Z})_p^\wedge$, which is compactly generated by $\mathbb{Z}/p$, and whose dualizable objects are generated as a thick subcategory by $\mathbb{Z}_p^\wedge$. The map

$$\widehat{\bigoplus}\mathbb{Z}_p \xrightarrow{(p^n)} \widehat{\bigoplus}\mathbb{Z}_p$$

which we can think of as a diagonal matrix with entries p^n, is trace class, but does not factor through a dualizable object since for example its rank after tensoring with $\mathbb{Q}_p$ is infinite.

A useful consequence of the above internal characterisation of rigid categories is that every symmetric monoidal functor between rigid categories automatically preserves compact morphisms. In particular, if we write $\mathrm{Rig} \subseteq \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ for the full subcategory on rigid categories, then we also have a fully faithful inclusion $\mathrm{Rig} \subseteq \mathrm{LocRig}^L$ into the category of locally rigid categories and symmetric monoidal strong left adjoints we considered in Proposition 4.3.15.

Corollary 4.4.9. $\mathrm{Rig} \subseteq \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ *is closed under colimits.*

Proof. Again we show this separately for finite coproducts and sifted diagrams. The case of finite coproducts follows immediately from Proposition 4.2.11 and Corollary 4.3.11.

Next, let $\mathcal{C}_\bullet : I \to \mathrm{Rig}$ be a sifted diagram, and $\mathcal{C}$ its colimit in $\mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$. As mentioned above, this diagram actually factors through $\mathrm{LocRig}^L \subset \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$, and hence the colimit cone lies again in LocRig^L. In particular, $\mathcal{C}$ is locally rigid, so it remains to see that the unit is compact. Pick any $i \in I$. Then the composite $\mathrm{Sp} \to \mathcal{C}_i \to \mathcal{C}$ is one of symmetric monoidal strong left adjoints, so the unit in $\mathcal{C}$ is compact. $\square$

Example 4.4.10. The inclusion $\mathrm{Rig} \subseteq \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ is certainly not closed under limits, as is easily seen using Example 4.4.7. For example, $\prod_{\mathbb{N}} \mathrm{Sp}$ is again compactly generated, where the compact objects are precisely the finitely supported pointwise compact objects as we have seen in Example 2.7.15. On the other hand, the dualizable objects in $\prod_{\mathbb{N}} \mathrm{Sp}$ are just the pointwise dualizable ones, so this is clearly locally rigid but not rigid.

Another nice example is that of Borel G-spectra $\mathrm{Sp}^{BG} = \lim_{BG} \mathrm{Sp}$ for a finite group G. It is compactly generated by the "induced" sphere spectrum $\mathbb{S}[G] = i_!\mathbb{S}$ where $i_! : \mathrm{Sp} \to \mathrm{Sp}^{BG}$ is left Kan extension along $i : * \to BG$. Moreover, one checks that

$$(\mathrm{Sp}^{BG})^\omega \subseteq (\mathrm{Sp}^\omega)^{BG} = (\mathrm{Sp}^{\mathrm{dbl}})^{BG} = (\mathrm{Sp}^{BG})^{\mathrm{dbl}},$$

so it is locally rigid. However, the unit $p^*\mathbb{S}$ for $p : BG \to *$ is not compact, as otherwise $p_!p^*\mathbb{S} = \mathbb{S}[BG] \in \mathrm{Sp}$ would have to be.

The above suggests that there should exist a right adjoint $(-)^{\mathrm{rig}} : \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}}) \to \mathrm{Rig}$ to the inclusion $\mathrm{Rig} \subseteq \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$. Indeed we will be able to construct this using the compactly assembled core we introduced in Theorem 2.7.5.

Lemma 4.4.11. *Let*

$$\begin{tikzcd} A_0 \arrow[r] & A_1 \arrow[r] & A_2 \\ B_0 \arrow[r] \arrow[u] \arrow[d] & B_1 \arrow[r] \arrow[u] \arrow[d] & B_2 \arrow[u] \arrow[d] \\ C_0 \arrow[r] & C_1 \arrow[r] & C_2 \end{tikzcd}$$

be a diagram in a stable presentably symmetric-monoidal category $\mathcal{C}$, where the horizontal maps are trace-class. Denote the vertical pushouts by $P_0 \to P_1 \to P_2$. Then $P_0 \to P_2$ is trace-class.

Proof. Since the maps $A_1 \to P_2$ and $C_1 \to P_2$ are trace-class, we obtain lifts

$$\mathbb{1} \to A_1^\vee \otimes P_2, \quad \mathbb{1} \to C_1^\vee \otimes P_2.$$

Composing with the maps to $B_1^\vee \otimes P_2$, we get two maps $\mathbb{1} \rightrightarrows B_1^\vee \otimes P_2$ which by commutativity of the above diagram become homotopic in $[B_1, P_2]$. Now the fact that $B_0 \to B_1$ is trace-class gives us a diagram

$$\begin{tikzcd} B_1^\vee \otimes P_2 \arrow[r] \arrow[d] & {[B_1, P_2]} \arrow[d] \arrow[dl, dashed] \\ B_0^\vee \otimes P_2 \arrow[r] & {[B_0, P_2]} \end{tikzcd}$$

where the dashed map is multiplication with a lift $\mathbb{1} \to B_0^\vee \otimes B_1$, so the maps become homotopic in $B_0^\vee \otimes P_2$ and we obtain a commutative diagram

$$\begin{tikzcd} \mathbb{1} \arrow[r] \arrow[d] & A_0^\vee \otimes P_2 \arrow[d] \\ C_0^\vee \otimes P_2 \arrow[r] & B_0^\vee \otimes P_2. \end{tikzcd}$$

Since $P_0^\vee \otimes P_2$ is the pullback, we obtain a map $\mathbb{1} \to P_0^\vee \otimes P_2$ lifting the map $P_0 \to P_2$. $\square$

Proposition 4.4.12. *Let $\mathcal{C}$ be a stable, presentably symmetric-monoidal ∞-category. Then the class S of trace-class morphisms forms a precompact ideal as in Definition 2.7.1.*

Proof. The fact that it is an ideal was verified before. To see that it is accessible assume that $\mathcal{C}$ is κ-accessible and such that the unit $\mathbb{1}$ is κ-compact. Consider a trace class map $X \to Y$. Write $Y = \operatorname{colim} Y_i$ for Y_i being κ-compact. Then we find a lift $\mathbb{1} \to X^\vee \otimes Y_i$ for some i, hence a lift of f through a κ-compact object. In particular every morphism in $S^{\mathbb{Q}}$ factors through a κ-compact object. To check that the ideal S is precompact we have to further verify that given a diagram $F_0 \leftarrow F_1 \to F_2$ of functors $[0,1] \cap \mathbb{Q} \to \mathcal{C}$ with all positive-length morphisms in S, the pushout $F_0 \amalg_{F_1} F_2$ takes $0 \to 1$ to a morphism in S. This follows directly from Lemma 4.4.11. $\square$

Corollary 4.4.13. *Let X be a locally compact Hausdorff space. Then* $\mathrm{Shv}(X; \mathrm{Sp})$ *is locally rigid.*

Proof. By Theorem 4.4.6, it suffices to check that all compact morphisms are trace-class. Note that an entirely analogous argument as for 2.7.18(1.) shows that the precompact ideal of compact maps in $\mathrm{Shv}(X; \mathrm{Sp})$ is generated by the morphisms $\Sigma^\infty_+ \underline{U} \to \Sigma^\infty_+ \underline{V}$ for $U \subseteq V$ with a compact inbetween. So by Proposition 4.4.12 it suffices to check that these morphisms. We claim that already all inclusions with $\overline{U} \subseteq V$ are trace class in $\mathrm{Shv}(X; \mathrm{Sp})$ (without the compactness of $\overline{U}$).

Here $\Sigma^\infty_+ \underline{U}$ is the spectral sheaf which is the sheafification of the presheaf taking $W \subseteq X$ to $\mathbb{S}$ if $W \subseteq U$, and 0 otherwise. This is zero on the open $X \setminus \overline{U}$, and the constant sheaf with value $\mathbb{S}$ on U, which is the unit. We thus have that $(\Sigma^\infty_+ \underline{U})^\vee \otimes \Sigma^\infty_+ \underline{V} \to [\Sigma^\infty_+ \underline{U}, \Sigma^\infty_+ \underline{V}]$ is an equivalence after restriction to $X \setminus \overline{U}$, since there both are zero. It is also an equivalence after restriction to V, since there $\Sigma^\infty_+ \underline{V}$ is the unit. So in particular the map $\Sigma^\infty_+ \underline{U} \to \Sigma^\infty_+ \underline{V}$ is trace class if $\overline{U} \subseteq V$, so $X \setminus \overline{U} \cup V = X$. $\square$

This proves directly (without investing the $f_!$ or $f^!$ functors as before!) that $\mathrm{Shv}(X; \mathrm{Sp})$ is locally rigid. In particular, it is self-dual.

Definition 4.4.14. For a continuous map of locally compact Hausdorff spaces $f : X \to Y$, we define an adjunction

$$\mathrm{Shv}(X; \mathrm{Sp}) \underset{f^!}{\overset{f_!}{\rightleftarrows}} \mathrm{Shv}(Y; \mathrm{Sp}),$$

by letting the left adjoint $f_!$ be the $\mathrm{Pr}^L_{\mathrm{st}}$ dual of the left adjoint f^*. Equivalently it is the counit of the relatively locally rigid algebra defined by f^* (see Proposition 4.3.10), since the duality is f^*-linear.

Lemma 4.4.15.

1. *If $f : X \to Y$ is proper, then $f_* \simeq f_!$.*

2. *If $f : X \to Y$ is an open immersion, then $f^! \simeq f^*$.*

Proof. Note that f^* exhibits $\mathrm{Shv}(X; \mathrm{Sp})$ as algebra over $\mathrm{Shv}(Y; \mathrm{Sp})$, and the right adjoint f_* is $\mathrm{Shv}(Y; \mathrm{Sp})$-linear by the projection formula. By Proposition 4.3.10, $\mathrm{Shv}(X; \mathrm{Sp})$ is also locally rigid over $\mathrm{Shv}(Y; \mathrm{Sp})$, and $f_!$ is also the $\mathrm{Shv}(Y; \mathrm{Sp})$-linear dual of f^*. Since f^* is the unit of $\mathrm{Shv}(X; \mathrm{Sp})$ as locally rigid $\mathrm{Shv}(Y; \mathrm{Sp})$-algebra, $f_!$ is the counit.

If f is proper, f_* preserves colimits. So it is a morphism in Pr^L, and thus agrees with the counit $f_!$ (Compare Definition 4.2.5). If f is an open immersion, then f^* admits a fully faithful left adjoint, namely extension by zero. In this case, Proposition 4.2.10 applies to show that this left adjoint agrees with the counit $f_!$, and so on right adjoints $f^* \simeq f^!$. □

This shows that the $f_!, f^!$ functors obtained from dualizing coincide with the usual characterisation as further adjoints. It also gives a recipe to compute it, as we may factor any $f : X \to Y$ as open immersion $X \to \overline{X}$ followed by a proper map $\overline{X} \to Y$, for example by letting $\overline{X}$ be the preimage $Y \times_{\beta Y} \beta X$ in the Stone-Cech compactification. This recipe does of course not clearly lead to well-defined functors independent of the choice of factorisation, which we get for free from the description as duals.

We now come to the promised right adjoint $(-)^{\mathrm{rig}}$ to the inclusion $\mathrm{Rig} \subseteq \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$.

Theorem 4.4.16. *Let $\mathcal{C}$ be a stable, presentably symmetric monoidal ∞-category. Then there is a universal rigid symmetric monoidal ∞-category $\mathcal{C}^{\mathrm{rig}}$ with a symmetric monoidal functor $\mathcal{C}^{\mathrm{rig}} \to \mathcal{C}$. Universality means that for any rigid presentable category $\mathcal{D}$ the induced map*

$$\mathrm{Fun}^{L,\otimes}(\mathcal{D}, \mathcal{C}^{\mathrm{rig}}) \to \mathrm{Fun}^{L,\otimes}(\mathcal{D}, \mathcal{C})$$

is an equivalence. If $\mathcal{C}$ is locally rigid, then $\mathcal{C}^{\mathrm{rig}} \to \mathcal{C}$ admits a fully faithful left adjoint, i.e. $\mathcal{C}^{\mathrm{rig}} \to \mathcal{C}$ is a rigidifcation of $\mathrm{Sp} \to \mathcal{C}$ as in Definition 4.3.14. If $\mathbb{1} \in \mathcal{C}$ is compact then $\mathcal{C}^{\mathrm{rig}} \to \mathcal{C}$ is fully faithful.

We call the rigid category $\mathcal{C}^{\mathrm{rig}}$ the rigid core of $\mathcal{C}$.

Proof. By Proposition 4.4.12, the trace class morphisms in $\mathcal{C}$ form a precompact ideal S, and we may form $(\mathcal{C}, S)^{\mathrm{ca}}$. Now compactly assembled functors $\mathcal{D} \to (\mathcal{C}, S)^{\mathrm{ca}}$ are the same as colimit-preserving functors $\mathcal{D} \to \mathcal{C}$ taking compact morphisms to trace-class morphisms.

Since trace class morphisms are closed under the monoidal structure, and $j\mathbb{1} \in (\mathcal{C}, S)^{\mathrm{ca}}$, we also see that $(\mathcal{C}, S)^{\mathrm{ca}} \subseteq \mathrm{Ind}(\mathcal{C})$ is a symmetric-monoidal full subcategory, and the functor $k : (\mathcal{C}, S)^{\mathrm{ca}} \to \mathcal{C}$ is symmetric-monoidal.

Next, we prove that $(\mathcal{C}, S)^{\mathrm{ca}}$ is itself rigid. It suffices to check that the unit is compact and that compact morphisms are trace-class. As the unit is $j\mathbb{1}$, which is even compact in

$\mathrm{Ind}(\mathcal{C})$, the first part is clear. For the second part, we check as in the proof of Theorem 2.7.5 that a compact morphism $X \to Y$ in $(\mathcal{C}, S)^{\mathrm{ca}}$ factors through $\mathrm{colim}_{\alpha<\beta}\, jZ_\alpha$ for some S-exhaustible diagram Z_α and some $\beta \in \mathbb{Q}$. But then $X \to Y$ factors through the morphism $\mathrm{colim}_{\alpha<\beta}\, jZ_\alpha \to \mathrm{colim}_{\beta+\frac{2}{3}<\alpha<\beta+1}\, jZ_\alpha$. We now directly witness this as trace-class by taking $W = \mathrm{colim}_{\beta+\frac{1}{3}>\alpha>\beta}\, j(Z_\alpha^\vee)$, and using the "coevaluation"

$$j\mathbb{1} \to j(Z_{\beta+\frac{1}{3}}^\vee) \otimes jZ_{\beta+\frac{2}{3}} \to \mathrm{colim}_{\beta+\frac{1}{3}>\alpha>\beta}\, j(Z_\alpha^\vee) \otimes \mathrm{colim}_{\beta+\frac{2}{3}<\alpha<\beta+1}\, jZ_\alpha$$

with the "evaluation"

$$\mathrm{colim}_{\alpha<\beta}\, jZ_\alpha \otimes \mathrm{colim}_{\beta+\frac{1}{3}>\alpha>\beta}\, j(Z_\alpha^\vee) \to jZ_\beta \otimes j(Z_\beta^\vee) \to j\mathbb{1}.$$

Setting $\mathcal{C}^{\mathrm{rig}} = (\mathcal{C}, S)^{\mathrm{ca}}$, the equivalence

$$\mathrm{Fun}^{L,\otimes}(\mathcal{D}, \mathcal{C}^{\mathrm{rig}}) \to \mathrm{Fun}^{L,\otimes}(\mathcal{D}, \mathcal{C})$$

is now given in one direction by postcomposition with k, and in the other by applying $(-)^{\mathrm{rig}}$ and using that if $\mathcal{D}$ is rigid, $\mathcal{D}^{\mathrm{rig}} \to \mathcal{D}$ is an equivalence since the trace class morphisms in $\mathcal{D}$ then coincide with the compact ones by Theorem 4.4.6.

If $\mathcal{C}$ is locally rigid, $C \subseteq S$ where C denotes the compact morphisms, so $\mathcal{C}^{\mathrm{rig}} \to \mathcal{C}$ admits a fully faithful left adjoint by Addendum 2.7.7. If $\mathbb{1}$ is compact then all trace class morphisms are compact,then $\mathcal{C}^{\mathrm{rig}} \to \mathcal{C}$ is fully faithful by Addendum 2.7.6. □

Example 4.4.17. For $\mathcal{C} = \mathcal{D}(\mathbb{Z})_p^\wedge$, we have $\mathcal{C}^{\mathrm{rig}} = \widetilde{\mathrm{Nuc}}(\mathbb{Z}_p)$. For example, this follows since by the universal property, $(-)^{\mathrm{rig}}$ turns the limit $\mathcal{D}(\mathbb{Z})_p^\wedge \simeq \lim \mathcal{D}(\mathbb{Z}/p^n)$ into a limit in rigid categories, which is formed in $\mathrm{Pr}^L_{\mathrm{dual}}$.

Let us mention one more simple statement which is useful to know:

Lemma 4.4.18. *Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$. Viewing $\mathcal{C}$ and $\mathcal{C}^{\mathrm{rig}}$ as full subcategories of $\mathrm{Ind}(\mathcal{C})$, we have*

$$\mathcal{C}^{\mathrm{dbl}} = \mathcal{C} \cap (\mathcal{C}^{\mathrm{rig}})^{\mathrm{dbl}} = \mathcal{C} \cap (\mathcal{C}^{\mathrm{rig}})^{\omega} = \mathcal{C} \cap \mathcal{C}^{\mathrm{rig}}.$$

Proof. As before, let S denote the precompact ideal of trace-class morphisms in $\mathcal{C}$. Since $\mathcal{C}^{\mathrm{rig}} \subseteq \mathrm{Ind}(\mathcal{C})$ is fully faithful and preserves colimits, it detects compact objects. Since $\mathcal{C} = \mathrm{Ind}(\mathcal{C})^\omega$, it follows immediately that $\mathcal{C} \cap \mathcal{C}^{\mathrm{rig}} = \mathcal{C} \cap (\mathcal{C}^{\mathrm{rig}})^\omega = \mathcal{C} \cap (\mathcal{C}^{\mathrm{rig}})^{\mathrm{dbl}}$. Now if $X \in \mathcal{C}^{\mathrm{dbl}}$, then jX is S-exhaustible by the identity maps, hence $\mathcal{C}^{\mathrm{dbl}} \subseteq \mathcal{C} \cap \mathcal{C}^{\mathrm{rig}}$. It remains to see that if $jX \in \mathcal{C}^{\mathrm{rig}}$, then X must be dualizable. So write $jX = \mathrm{colim}_i\, Z_i$ is a filtered colimit of S-exhaustible objects $Z_i = \mathrm{colim}_{\alpha\in\mathbb{Q}}\, jZ_{i,\alpha}$. Using compactness of jX, we can then factor id_{jX} as follows:

$$\begin{tikzcd}
jZ_{i,\alpha} \arrow[d] \arrow[dr] & jX \arrow[l, dashed] \arrow[d, dashed] \arrow[r, equal] \arrow[dr] & jX \\
jZ_{i,\alpha+1} \arrow[r] & Z_i = \mathrm{colim}_{\alpha\in\mathbb{Q}}\, jZ_{i,\alpha} \arrow[r] & \mathrm{colim}_i\, Z_i \arrow[u]
\end{tikzcd}$$

Since j is fully faithful and $Z_{i,\alpha} \to Z_{i,\alpha+1}$ is trace-class, it follows that id_X is trace-class, hence X is dualizable, as desired. □

4.5 Proof of Theorem 4.4.6

We adapt the proof of Maxime Ramzi from [Ram, Proposition 3.16]. In op.cit. the author derives a good part of the theory of compactly assembled categories in a $\mathcal{V}$-enriched setting, where compact maps are replaced with the notion of $\mathcal{V}$-atomic maps. For the proof of Theorem 4.4.6 we will introduce a slightly weaker but more easily defined notion, which for lack of a better name we call weakly $\mathcal{V}$-atomic maps (and indeed one may check that Ramzi's atomic maps are always weakly atomic in our sense). Let us begin with some short recollections on enriched categories we will need in the following.

Recall that if $\mathcal{V} \in \mathrm{CAlg}(\mathrm{Pr}^L)$ and $\mathcal{M}$ is an $\mathcal{V}$-module, then $\mathcal{M}$ admits an $\mathcal{V}$-enrichment, and in particular a functor

$$\hom_{\mathcal{M}} = \hom_{\mathcal{M}}^{\mathcal{V}} : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{V}$$

so that we have a natural equivalence

$$\mathrm{Map}_{\mathcal{V}}(a, \hom_{\mathcal{M}}(x, y)) \simeq \mathrm{Map}_{\mathcal{M}}(a \otimes x, y).$$

The $\mathcal{V}$-linear functors $f : \mathcal{M} \to \mathcal{N}$ will be compatible with this enrichment in that they induce natural transformations $\hom_{\mathcal{M}}(-,-) \to \hom_{\mathcal{N}}(f(-), f(-))$. Moreover, we will also need that if $L : \mathcal{M} \rightleftarrows \mathcal{N} : R$ is an internal adjunction in $\mathrm{Mod}_{\mathcal{V}}(\mathrm{Pr}^L)$, then we obtain an adjunction equivalence on the mapping objects by the Yoneda Lemma:

$$\begin{aligned}
\mathrm{Map}_{\mathcal{V}}(a, \hom_{\mathcal{N}}(Lx, y)) &\simeq \mathrm{Map}_{\mathcal{N}}(a \otimes Lx, y) \\
&\simeq \mathrm{Map}_{\mathcal{N}}(L(a \otimes x), y) \\
&\simeq \mathrm{Map}_{\mathcal{M}}(a \otimes x, Ry) \\
&\simeq \mathrm{Map}_{\mathcal{V}}(a, \hom_{\mathcal{M}}(x, Ry)).
\end{aligned}$$

One shows analogously that a fully faithful $\mathcal{A}$-linear functor also induces equivalences on the hom-objects.

Definition 4.5.1. Let $\mathcal{V} \in \mathrm{CAlg}(\mathrm{Pr}^L)$ and $\mathcal{M}$ be an $\mathcal{V}$-module.

1. An object $x \in \mathcal{M}$ is $\mathcal{V}$-atomic if $\mathcal{V} \xrightarrow{-\otimes x} \mathcal{M}$ is an internal left adjoint in $\mathrm{Mod}_{\mathcal{V}}(\mathrm{Pr}^L)$.
2. A morphism $f : x \to y$ is weakly $\mathcal{V}$-atomic if for all $a \in \mathcal{V}, z \in \mathcal{M}$ and all small diagrams $(z_i)_i : I \to \mathcal{M}$ the indicated lifts exist in the following diagrams

$$\begin{tikzcd}
v \otimes \hom_M(y,z) \arrow[r] \arrow[d] & v \otimes \hom_M(x,z) \arrow[d] \\
\hom_M(y, v \otimes z) \arrow[r] \arrow[ur, dashed] & \hom_M(x, v \otimes z)
\end{tikzcd}
\qquad
\begin{tikzcd}
\operatorname{colim}_i \hom_M(y,z_i) \arrow[r] \arrow[d] & \operatorname{colim}_i \hom_M(x,z_i) \arrow[d] \\
\hom_M(y, \operatorname{colim}_i z_i) \arrow[r] \arrow[ur, dashed] & \hom_M(x, \operatorname{colim}_i z_i)
\end{tikzcd}$$

Again we remark some easy consequences.

Lemma 4.5.2. *Let $\mathcal{V} \in \mathrm{CAlg}(\mathrm{Pr}^L)$ and $f : \mathcal{M} \to \mathcal{N}$ a functor in $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Pr}^l)$.*

1. *An object $x \in \mathcal{M}$ is $\mathcal{V}$-atomic if and only if id_x is weakly $\mathcal{V}$-atomic.*
2. *The weakly $\mathcal{V}$-atomic maps in $\mathcal{M}$ form a 2-sided ideal in $\mathcal{M}$. In particular, a morphism factoring over a $\mathcal{V}$-atomic object is weakly $\mathcal{V}$-atomic.*
3. *If f is an internal left adjoint in $\mathrm{Mod}_{\mathcal{V}}(\mathrm{Pr}^L)$, then it preserves weakly $\mathcal{V}$-atomic maps.*
4. *If f is fully faithful then it reflects weakly $\mathcal{V}$-atomic maps.*

Proof. For the first claim, note that in the diagrams defining what it means for id_x to be weakly $\mathcal{V}$-atomic, the vertical maps are identities now. Hence by 2-out-of-6 it follows that id_x is weakly $\mathcal{V}$-atomic if and only if all involved maps are equivalences, which precisely states that $\hom_{\mathcal{M}}(x, -)$ is $\mathcal{B}$-linear and colimit-preserving. Since this functor is right-adjoint to $\mathcal{B} \xrightarrow{-\otimes x} \mathcal{M}$, this is equivalent to asking that x is $\mathcal{V}$-atomic.

Claim (2) is clear. Part (3) follows from the commutative diagram

$$\begin{tikzcd}
v \otimes \hom_N(fy, z) \arrow[rrr] \arrow[dr, equal] \arrow[ddd] & & & v \otimes \hom_N(x, z) \arrow[dl, equal] \arrow[ddd] \\
& v \otimes \hom_M(y, f^R z) \arrow[r] \arrow[d] & v \otimes \hom_M(x, f^R z) \arrow[d] & \\
& \hom_M(y, v \otimes f^R z) \arrow[r] \arrow[ur, dashed] \arrow[dl, equal] & \hom_M(x, v \otimes f^R z) \arrow[dr, equal] & \\
\hom_N(fy, v \otimes z) \arrow[rrr] & & & \hom_N(fx, v \otimes z)
\end{tikzcd}$$

where we crucially use $\mathcal{A}$-linearity of the right adjoint f^R of f. The case for colimits is entirely analogous and uses that f^R also preserves them. Part (4) is very similar to part (3); we consider the diagram

$$\begin{tikzcd}
v \otimes \hom_M(y, z) \arrow[rrr] \arrow[dr, equal] \arrow[ddd] & & & v \otimes \hom_M(x, z) \arrow[dl, equal] \arrow[ddd] \\
& v \otimes \hom_N(fy, fz) \arrow[r] \arrow[d] & v \otimes \hom_N(fx, fz) \arrow[d] & \\
& \hom_N(fy, v \otimes fz) \arrow[r] \arrow[ur, dashed] \arrow[dl, equal] & \hom_N(fx, v \otimes fz) \arrow[dr, equal] & \\
\hom_M(y, v \otimes z) \arrow[rrr] & & & \hom_M(x, v \otimes z)
\end{tikzcd}$$

and similarly for the colimit case. □

Proposition 4.5.3. *Let $\mathcal{V} \in \mathrm{CAlg}(\mathrm{Pr}^L)$.*

1. *The class of weakly $\mathcal{V}$-atomic maps in $\mathcal{V}$ agrees with the trace-class maps in $\mathcal{V}$.*
2. *If $F : \mathcal{M} \to \mathcal{N}$ is a lax $\mathcal{V}$-linear functor and $x \to y$ is trace-class, then for any $m \in \mathcal{M}$ we can find a lift*

$$\begin{tikzcd} x \otimes Fm \arrow[r] \arrow[d] & y \otimes Fm \arrow[d] \\ F(x \otimes m) \arrow[r] \arrow[ur, dashed] & F(y \otimes m) \end{tikzcd}$$

Proof. We begin with (1.). Note that now $\hom_{\mathcal{V}}(-,-) = [-,-] : \mathcal{V}^{\mathrm{op}} \times \mathcal{V} \to \mathcal{V}$ is the internal hom of $\mathcal{V}$. Let $f : x \to y$ be a morphism in $\mathcal{V}$. We first show that if f is weakly $\mathcal{V}$-atomic, then it is trace-class. In fact, we do not even need the full power of being weakly $\mathcal{V}$-atomic, just the part requiring $\mathcal{V}$-linearity. Namely, we can build a lift as in condition (2.) of Lemma 4.4.3 as follows:

$$\begin{tikzcd} & y \otimes [y, \mathbb{1}] \arrow[r] \arrow[d] & y \otimes [x, \mathbb{1}] \arrow[d] \\ \mathbb{1} \arrow[r, "\widehat{\mathrm{id}}"] \arrow[rr, bend right, "\widehat{f}"'] & {[y, y]} \arrow[r] \arrow[ur, dashed] & {[x, y]} \end{tikzcd}$$

For the converse, suppose that f is trace-class and choose a lift $\alpha : \mathbb{1} \to [x, \mathbb{1}] \otimes y$ of $\widehat{f} : \mathbb{1} \to [x, y]$. Given $v \in \mathcal{V}$ and a small diagram $(z_i)_i : I \to \mathcal{V}$, one checks that the composites

$$[y, v \otimes z] \xrightarrow{\alpha} [x, \mathbb{1}] \otimes y \otimes [y, v \otimes z] \xrightarrow{\mathrm{ev}_y} [x, \mathbb{1}] \otimes v \otimes z \to v \otimes [x, z]$$

and

$$[y, \mathrm{colim}_i\, z_i] \xrightarrow{\alpha} [x, \mathbb{1}] \otimes y \otimes [y, \mathrm{colim}_i\, z_i] \xrightarrow{\mathrm{ev}_y} [x, \mathbb{1}] \otimes \mathrm{colim}_i\, z_i \to \mathrm{colim}_i [x, z_i]$$

define the desired lifts in the diagrams

$$\begin{tikzcd} v \otimes [y, z] \arrow[r] \arrow[d] & v \otimes [x, z] \arrow[d] \\ {[y, v \otimes z]} \arrow[r] \arrow[ur, dashed] & {[x, v \otimes z]} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{colim}_i [y, z_i] \arrow[r] \arrow[d] & \mathrm{colim}_i [x, z_i] \arrow[d] \\ {[y, \mathrm{colim}_i\, z_i]} \arrow[r] \arrow[ur, dashed] & {[x, \mathrm{colim}_i\, z_i]} \end{tikzcd}$$

For point (2.), one again checks that the composite

$$F(x \otimes m) \to [x, \mathbb{1}] \otimes y \otimes F(x \otimes m) \to y \otimes F([x, \mathbb{1}] \otimes x \otimes m) \to y \otimes Fm$$

is the desired lift. □

Corollary 4.5.4. *Let $\mathcal{V} \in \mathrm{CAlg}(\mathrm{Pr}^L)$ and $\alpha : x \to y$ a trace-class map and $\beta : v \to w$ a compact map. Then $\beta \otimes \alpha : v \otimes x \to w \otimes y$ is compact.*

Proof. Note that α is weakly $\mathcal{V}$-atomic by Proposition 4.5.3(1). Now let $(z_i)_i : I \to \mathcal{V}$ be a filtered diagram. We can then paste lifts obtained from α and β to obtain one for $\beta \otimes \alpha$ as follows

$$\begin{tikzcd}
\operatorname{colim}_i \mathrm{Map}_{\mathcal{V}}(w \otimes y, z_i) \arrow[rr] \arrow[d, equal] & & \operatorname{colim}_i \mathrm{Map}_{\mathcal{V}}(v \otimes x, z_i) \arrow[d, equal] \\
\operatorname{colim}_i \mathrm{Map}_{\mathcal{V}}(w, [y, z_i]) \arrow[r] \arrow[d] & \operatorname{colim}_i \mathrm{Map}_{\mathcal{V}}(w, [x, z_i]) \arrow[r] \arrow[d] & \operatorname{colim}_i \mathrm{Map}_{\mathcal{V}}(v, [x, z_i]) \arrow[d] \\
\mathrm{Map}_{\mathcal{V}}(w, \operatorname{colim}_i [y, z_i]) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{V}}(w, \operatorname{colim}_i [x, z_i]) \arrow[r] \arrow[d] \arrow[ur, dashed] & \mathrm{Map}_{\mathcal{V}}(v, \operatorname{colim}_i [x, z_i]) \arrow[d] \\
\mathrm{Map}_{\mathcal{V}}(w, [y, \operatorname{colim}_i z_i]) \arrow[r] \arrow[d, equal] \arrow[ur, dashed] & \mathrm{Map}_{\mathcal{V}}(w, [x, \operatorname{colim}_i z_i]) \arrow[r] & \mathrm{Map}_{\mathcal{V}}(v, [x, \operatorname{colim}_i z_i]) \arrow[d, equal] \\
\mathrm{Map}_{\mathcal{V}}(w \otimes y, \operatorname{colim}_i z_i) \arrow[rr] & & \mathrm{Map}_{\mathcal{V}}(v \otimes x, \operatorname{colim}_i z_i)
\end{tikzcd}$$

□

Proof of Theorem 4.4.6. As mentioned there, it remains to show that $\mathcal{C} \in \mathrm{Pr}^L_{\mathrm{dual}}$ is locally rigid if and only if every trace-class map in $\mathcal{C}$ is compact.

Suppose first that $\mathcal{C}$ is locally rigid, and let $f : x \to y$ in $\mathcal{C}$ be a compact morphism. We obtain a factorization $\widehat{\jmath}x \to jy \to \widehat{\jmath}y$ in $\mathrm{Ind}(\mathcal{C}^{\omega_1})$. Since jy is compact, we see that $\mathrm{Sp} \xrightarrow{-\otimes jy} \mathrm{Ind}(\mathcal{C}^{\omega_1})$ is an internal left adjoint in $\mathrm{Pr}^L_{\mathrm{st}}$. Basechanging, we get that $\mathcal{C} \xrightarrow{-\boxtimes jy} \mathcal{C} \otimes \mathrm{Ind}(\mathcal{C}^{\omega_1})$ is an internal left-adjoint in $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^L)$, i.e. $\mathbb{1}_{\mathcal{C}} \boxtimes jy \in \mathcal{C} \otimes \mathrm{Ind}(\mathcal{C}^{\omega_1})$ is $\mathcal{C}$-atomic. Since $\mathbb{1}_{\mathcal{C}} \boxtimes \widehat{\jmath}f$ factors through this $\mathcal{C}$-atomic object, it is a weakly $\mathcal{C}$-atomic map. Tensoring the internal adjunction $\widehat{\jmath} \dashv k$ with $\mathcal{C}$, we obtain the following adjunction internal to $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^L)$:

$$\begin{tikzcd}
\mathcal{C} \otimes \mathcal{C} \arrow[r, shift left=1.5, "-\boxtimes \widehat{\jmath}"] \arrow[r, phantom, "\perp"] & \mathcal{C} \otimes \mathrm{Ind}(\mathcal{C}^{\omega_1}) \arrow[l, shift left=1.5, "-\boxtimes k"]
\end{tikzcd}$$

In particular $- \boxtimes \widehat{\jmath}$ is a fully faithful $\mathcal{C}$-linear functor, and hence reflects weakly $\mathcal{C}$-atomic maps. It follows that $\mathbb{1} \boxtimes f$ is weakly $\mathcal{C}$-atomic. Since $\mathcal{C}$ is locally rigid, $\mu : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ is an internal left adjoint in $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^L)$, and thus preserves weakly $\mathcal{C}$-atomic maps. It follows that f is weakly $\mathcal{C}$-atomic in $\mathcal{C}$, and thus trace-class by Proposition 4.5.3(1).

For the converse, assume that $\mathcal{C}$ is dualizable and compact maps are trace-class. We begin by showing that $\mu : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ is strongly left adjoint. By Propositions 2.6.1 and 2.12.2 we have to show that if $f : X \to Y$ and $f' : X' \to Y'$ are compact maps in $\mathcal{C}$, then $f \otimes f' : X \otimes X' \to Y \otimes Y'$ is again compact. But by assumption compact maps are trace-class, so it follows from the previous Corollary that $f \otimes f'$ is again compact.

In view of Remark 4.3.9 it remains to see that Δ is left $\mathcal{C}$-linear, as right $\mathcal{C}$-linearity is entirely analogous. Since we already know that Δ preservrs colimits, we can reduce to checking this on compactly exhausted objects. Since compact maps are trace-class, this follows from Proposition 4.5.3(2) by a cofinality argument. $\square$

4.6 Verdier Duality

In this section, we make the self-duality of $\mathrm{Shv}(X;\mathrm{Sp})$ more explicit.

Remark 4.6.1. All results of this section do admit analogues for $\mathrm{Shv}(X;\mathcal{C})$ with coefficients in a rigid category $\mathcal{C}$ (and $\mathcal{C}$-linear duals etc.), but we focus on Sp for simplicity.

We will require some properties of the functors $f_!$ and $f^!$. These can be stated very coherently (in the language of 6 functor formalisms), but we won't need to do that yet.

Lemma 4.6.2. *For $i : U \to X$ and $j : Z \to X$ complementary open and closed inclusions ($U = X \setminus Z$), we have cofiber sequences*

$$i_! i^! \to \mathrm{id}_{\mathrm{Shv}(X;\mathrm{Sp})} \to j_* j^*.$$

Proof. Since $i_!$ is left adjoint to $i^! = i^*$, is is given by the extension-by-zero functor. Using this description, one directly checks the claim. $\square$

Lemma 4.6.3. $\mathrm{Fun}^L(\mathrm{Shv}(X;\mathrm{An}),\mathcal{C}) = \mathrm{coShv}(X;\mathcal{C})$*, in particular* $\mathrm{Fun}^L(\mathrm{Shv}(X;\mathrm{Sp}),\mathrm{Sp}) \simeq \mathrm{coShv}(X;\mathrm{Sp})$.

Proof. The first statement has been established in Example 2.1.25. For the second, we observe that

$$\mathrm{coShv}(X;\mathrm{Sp}) \simeq \mathrm{Fun}^L(\mathrm{Shv}(X;\mathrm{An}),\mathrm{Sp}) \simeq \mathrm{Fun}^L(\mathrm{Shv}(X;\mathrm{An})\otimes\mathrm{Sp},\mathrm{Sp}) \simeq \mathrm{Fun}^L(\mathrm{Shv}(X;\mathrm{Sp}),\mathrm{Sp}),$$

using that Sp is an idempotent algebra in Pr^L. $\square$

This means that the dual of $\mathrm{Shv}(X;\mathrm{Sp})$ is canonically $\mathrm{coShv}(X;\mathrm{Sp})$. For X locally compact Hausdorff, we now also have an identification of $\mathrm{Shv}(X;\mathrm{Sp})^\vee \simeq \mathrm{Shv}(X;\mathrm{Sp})$. Composing these, we obtain Lurie's version of Verdier duality.

Theorem 4.6.4 ([Lur17a, Theorem 5.5.5.1])**.** *There is an equivalence*

$$\mathbb{D} : \mathrm{Shv}(X;\mathrm{Sp}) \simeq \mathrm{coShv}(X;\mathrm{Sp})$$

taking $\mathcal{F}$ to the cosheaf $U \mapsto t_{U,!} i_U^* \mathcal{F}$, where $i_U : U \to X$ is the inclusion and $t_U : U \to \mathrm{pt}$ the constant map.

Proof. Given a functor $\mathrm{Shv}(X;\mathrm{Sp}) \to \mathrm{Sp}$, the associated cosheaf has value on U given by evaluating on $\Sigma^\infty_+ \underline{U}$.

The composite $\mathrm{Shv}(X;\mathrm{Sp}) \to \mathrm{Shv}(X;\mathrm{Sp})^\wedge \to \mathrm{coShv}(X;\mathrm{Sp})$ thus takes $\mathcal{F}$ to the cosheaf taking U to $b(\mathcal{F}, \Sigma^\infty_+ \underline{U})$, where $b : \mathrm{Shv}(X;\mathrm{Sp}) \times \mathrm{Shv}(X;\mathrm{Sp}) \to \mathrm{Sp}$ is the pairing of the locally rigid algebra $\mathrm{Shv}(X;\mathrm{Sp})$. Writing this as $\varepsilon \circ \mu$ in terms of the multiplication and counit, and recalling that the counit is by definition $t_!$, we get

$$t_!(\mathcal{F} \otimes \Sigma^\infty_+ \underline{U}).$$

Now recall that for the open inclusion $i = i_U : U \to X$, $i_!$ agrees with the left adjoint of i^*, and so by Yoneda, $\Sigma^\infty_+ \underline{U} = i_! \mathbb{1}$. Also, $i_!$ is i^*-linear since it is the dual of i^*, and so $\mathcal{F} \otimes i_! \mathbb{1} \simeq i_! i^* \mathcal{F}$. Finally, $t_! i_! = t_{U,!}$, so we learn that $\mathcal{DF}$ takes $U \mapsto t_{U,!} i_U^* \mathcal{F}$. □

Lemma 4.6.5. *For a continuous map $f : X \to Y$, we have an adjunction $f_+ \dashv f^+$ with $f_+ : \mathrm{coShv}(X;\mathrm{Sp}) \to \mathrm{coShv}(Y;\mathrm{Sp})$ the pushforward $f_+(\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$. We have*

$$\mathbb{D} \circ f_! \simeq f_+ \circ \mathbb{D}$$
$$\mathbb{D} \circ f^! \simeq f^+ \circ \mathbb{D}$$

Proof. As $\mathbb{D}$ is an equivalence, it suffices to check that the first relation holds. By definition, $f_!$ is the dual to f^* under the self-duality of $\mathrm{Shv}(-;\mathrm{Sp})$. So the first statement is equivalent to check that f_+ is the dual to f^* under the canonical duality $\mathrm{Shv}(-;\mathrm{Sp})^\vee \simeq \mathrm{coShv}(-;\mathrm{Sp})$. Since the diagram

$$\begin{tikzcd}
\mathrm{Open}(Y) \arrow[r] \arrow[d, "f^{-1}"] & \mathrm{Shv}(Y) \arrow[r, "\Sigma^\infty_+"] \arrow[d, "f^*"] & \mathrm{Shv}(Y;\mathrm{Sp}) \arrow[d, "f^*"] \\
\mathrm{Open}(X) \arrow[r] & \mathrm{Shv}(X) \arrow[r, "\Sigma^\infty_+"] & \mathrm{Shv}(X;\mathrm{Sp})
\end{tikzcd}$$

commutes, the dual to f^* on cosheaves is given by precomposing with f^{-1}, hence is given by f_+ as claimed. □

Now recall that for $t : X \to \mathrm{pt}$, $t_* t^* : \mathrm{Sp} \to \mathrm{Sp}$ computes (sheaf) cohomology of X with coefficients in a constant coefficient system. For example, we may apply this to the spectrum $\mathbb{Z}$ to obtain a spectrum whose homotopy groups are the sheaf cohomology of X with coefficients in $\mathbb{Z}$. Analogously, we may think of $t_+ t^+$ as computing *homology* with constant coefficients. Indeed, due to the reversed adjunction, this is covariantly functorial in maps $f : X \to Y$, still vanishes on contractible spaces, and has Mayer-Vietoris sequences. So it agrees with singular homology on nice enough spaces.

Since $\mathbb{D}$ is an equivalence, we also see that this "cosheaf homology" agrees with $t_! t^!$.

Proposition 4.6.6. *If X is a CW complex,*

1. *$t_* t^* A$ agrees with $\mathrm{map}_{\mathrm{Sp}}(\Sigma^\infty_+ X, A)$, in particular the singular cohomology of X with coefficients in A if we take A to be an abelian group.*

2. *$t_+t^+A = t_!t^!A$ agrees with $\Sigma^\infty_+ X \otimes A$, in particular the singular homology of X with coefficients in A if we take A to be an abelian group.*

The appearance of X in two different roles here (once as actual topological space indexing $\mathrm{Shv}(X)$, once as anima in $\Sigma^\infty_+ X$) is best explained by the notion of shape (of a topos): Sometimes, the functor t_*t^* is corepresented by an anima, which is then called the shape. (More generally, it can be pro-corepresented, leading to a pro-anima, which appears for example in the notion of étale homotopy type.) The above statements hold for X which is more general than CW complexes, namely X of "locally contractible shape".

There are two mixed versions of the above that we can write down, namely $t_!t^*$ and $t_*t^!$. The first is *cohomology with compact support*, while the second one is a version of *locally finite homology* (also known as Borel-Moore homology).

Definition 4.6.7. We define a functor

$$D : \mathrm{Shv}(X; \mathrm{Sp}) \to \mathrm{Shv}(X; \mathrm{Sp})^{\mathrm{op}}$$

by composing $\mathbb{D}$ with the functor

$$\mathrm{coShv}(X; \mathrm{Sp}) \to \mathrm{Shv}(X; \mathrm{Sp})^{\mathrm{op}}$$

which pointwise applies the dual $\mathrm{map}(-, \mathbb{S})$.

This is of course no equivalence anymore, but can be expressed completely in terms of sheaves. Writing $\underline{Hom}_{\mathrm{coShv}(X;\mathrm{Sp})}(\mathcal{F}, \mathcal{G})$ for the *sheaf* taking U to $\mathrm{map}(\mathcal{F}|_U, \mathcal{G}|_U)$, we may write

$$D\mathcal{F} \simeq \underline{Hom}_{\mathrm{coShv}(X;\mathrm{Sp})}(\mathbb{D}\mathcal{F}, t^+\mathbb{S}) \simeq \underline{Hom}_{\mathrm{Shv}(X;\mathrm{Sp})}(\mathcal{F}, t^!\mathbb{S}).$$

Definition 4.6.8. We call $D\mathcal{F}$ the Verdier dual sheaf of $\mathcal{F}$, and $t^!\mathbb{S} =: \omega_X$ the *dualizing complex* of X.

Proposition 4.6.9. *For $f : X \to Y$, we have*

1. *$f_*D\mathcal{F} = Df_!\mathcal{F}$ for $\mathcal{F} \in \mathrm{Shv}(X; \mathrm{Sp})$*

2. *$\underline{\mathrm{Hom}}(\mathcal{F}, D\mathcal{G}) = \underline{Hom}(\mathcal{G}, D\mathcal{F})$, in particular D is left adjoint to D^{op}.*

3. *$f^!D\mathcal{F} = Df^*\mathcal{F}$ for $\mathcal{F} \in \mathrm{Shv}(Y; \mathrm{Sp})$*

Proof. We have $f_+\mathbb{D}\mathcal{F} = \mathbb{D}f_!\mathcal{F}$ in $\mathrm{coShv}(Y; \mathrm{Sp})$. Now applying $\mathrm{map}(-; \mathbb{S})$ pointwise commutes with pushforward, i.e. takes f_+ to f_*, and we obtain the first identity. For the second one, we have

$$\underline{\mathrm{Hom}}(\mathcal{F}, D\mathcal{G}) \simeq \underline{\mathrm{Hom}}(\mathcal{F} \otimes \mathcal{G}, \omega_X) \simeq \underline{\mathrm{Hom}}(\mathcal{G}, D\mathcal{F}).$$

Now $Df_!$ is left adjoint to $(f^!D)^{\mathrm{op}}$, and $(f_*)^{\mathrm{op}}D$ is left adjoint to $(Df^*)^{\mathrm{op}}$, so the last claim follows from the first by passing to right adjoints. □

In particular, $t_*t^!DA = Dt_!t^*A$, so we always have that locally finite homology is the dual of compactly supported cohomology.

When X is compact, $t_! = t_*$ and this therefore expresses that homology is the dual of cohomology. This is opposite to the situation in singular (co)homology, where cohomology is the dual of homology.

Example 4.6.10. Let $X = \prod_{\mathbb{N}}\{0,1\}$ be the Cantor set. Then $t^*\mathbb{Z}$ is the sheafification of the constant presheaf $\mathbb{Z}$, which coincides with sheaf of continuous maps $C(-,\mathbb{Z})$: This is a sheaf, receives a map from the constant presheaf, and this map is an isomorphism on stalks. By a result of Nöbeling [Nöb68] (see also [Sch19, Theorem 5.4]), $C(X;\mathbb{Z})$ is free, indeed on countably many generators since X is a sequential limit of finite sets. So $t_*t^*\mathbb{Z}$ is a countably generated free abelian group. As X is compact, the sheaf homology $t_!t^!\mathbb{Z}$ is its dual, which is a countable product of copies of $\mathbb{Z}$.

So for general X, even compact, it is not true that cohomology is the dual of homology. In general, we have

$$t_*D^2t^*A \simeq Dt_!t^!DA,$$

which exhibits cohomology as dual of homology if $D^2t^*A \simeq t^*A$. For dualizable A, this reduces to $\underline{\mathrm{Hom}}(\omega_X,\omega_X) \simeq t^*\mathbb{S}$. Under $\mathbb{D}$, this corresponds to $\underline{\mathrm{Hom}}_{\mathrm{coShv}}(t^+\mathbb{S}, t^+\mathbb{S})$, which on U has the value $\mathrm{map}(t_{U,+}t_U^+\mathbb{S},\mathbb{S})$, i.e. the dual of homology of U. For good X (like a CW complex), this is the constant sheaf $t^*\mathbb{S}$, such that cohomology is indeed the dual of homology. Combining both statements, (co)homology is dualizable on compact good X.

In general, ω_X is mysterious. However, it admits a very nice description in the case of manifolds. In that case, we will see that ω_X is actually invertible. Moreover, we will see that $f^!$ and f^* agree up to a twist. Since $f_*f^!$ is locally finite homology, and f_*f^* is cohomology, this statement gives a version of Poincaré duality. We abstract this situation by the following definition.

Definition 4.6.11. Let X be a locally compact Hausdorff space, and $f : X \to \mathrm{pt}$ the constant map. We call X *cohomologically smooth* if the following conditions are satisfied:

1. The natural transformation

$$f^!(\mathbb{S}) \otimes f^*(-) \to f^!(-)$$

   obtained from the f^*-linearity of $f_!$, is an equivalence.

2. The object $f^!(\mathbb{S}) \in \mathrm{Shv}(X;\mathrm{Sp})$ is invertible.

If X is cohomologically smooth and compact, we get $f_!f^!(A) = f_*(f^!(\mathbb{S}) \otimes f^*(A)) = f_*(\omega_X \otimes f^*(A))$, i.e. a statement of the form "$H_*(X;A) \cong H^*(X;A\otimes\omega_X)$. In the case of manifolds and $\mathbb{Z}$-linear coefficients, this is the well-known form of Poincaré duality, with ω_X contributing a shift of n and potentially an orientation character. If X is not compact, we still get $f_!f^!(A) = f_!(\omega_X \otimes f^*(A))$, which identifies homology with (twisted) compactly supported cohomology, and $f_*f^!(A) = f_*(\omega_X \otimes f^*(A))$, which identifies locally finite homology with (twisted) cohomology.

We now want to see that manifolds are indeed cohomologically smooth. To see this, we analyze the two conditions separately.

Lemma 4.6.12. *Let $\mathcal{F} \in \mathrm{Shv}(X; \mathrm{Sp})$ be an object. Then the following are equivalent:*

1. $f_!(\mathcal{F} \otimes -) : \mathrm{Shv}(X; \mathrm{Sp}) \to \mathrm{Sp}$ *is strongly left adjoint.*
2. $\underline{\mathrm{Hom}}(\mathcal{F}, f^!(-)) : \mathrm{Sp} \to \mathrm{Shv}(X; \mathrm{Sp})$ *preserves colimits.*
3. $f^*(-) \otimes \mathcal{F} : \mathrm{Sp} \to \mathrm{Shv}(X; \mathrm{Sp})$ *has a left adjoint.*
4. *There exists $\mathcal{G} \in \mathrm{Shv}(X; \mathrm{Sp})$ with maps*
$$\Delta_!(\mathbb{1}) \to p_1^* \mathcal{F} \otimes p_2^* \mathcal{G}$$
in $\mathrm{Shv}(X \times X; \mathrm{Sp})$, and
$$f_!(\mathcal{G} \otimes \mathcal{F}) \to \mathbb{S},$$
with the snake identities.
5. *There exists a map*
$$\Delta_!(\mathbb{1}) \to p_1^* \mathcal{F} \otimes p_2^* D\mathcal{F}$$
satisfying the snake identity with the canonical map
$$f_!(D\mathcal{F} \otimes \mathcal{F}) \to \mathbb{S}.$$

Furthermore, if $\mathcal{F}$ satisfies these conditions, the left adjoint of $f^*(-) \otimes \mathcal{F}$ is $f_!(- \otimes D\mathcal{F})$, we have $D^2\mathcal{F} \simeq \mathcal{F}$, and
$$\underline{\mathrm{Hom}}(\mathcal{F}, f^!(-)) \simeq f^*(-) \otimes D\mathcal{F}.$$

Proof. $1 \Leftrightarrow 2$ is simply the observation that $\underline{\mathrm{Hom}}(\mathcal{F}, f^!(-))$ is the right adjoint of $f_!(\mathcal{F} \otimes -)$.

Since the pairing on $\mathrm{Shv}(X; \mathrm{Sp})$ is given by $f_!(- \otimes -)$, the dual of the functor $f^*(-) \otimes \mathcal{F}$ is given by $f_!(f^*(\mathbb{S}) \otimes \mathcal{F} \otimes -) = f_!(\mathcal{F} \otimes -)$. So $f_!(\mathcal{F} \otimes -)$ is an internal left adjoint if and only if $f^*(-) \otimes \mathcal{F}$ is a right adjoint.

Before we discuss parts 4 and 5, we prove the additional statements. Since $\underline{\mathrm{Hom}}(\mathcal{F}, f^!(-))$ dualizes to $f_!(\underline{\mathrm{Hom}}(\mathcal{F}, f^!(\mathbb{S})) \otimes -) = f_!(D\mathcal{F} \otimes -)$, this is the left adjoint of $f^*(-) \otimes \mathcal{F}$ if it exists. Now the first three parts together imply that $f_!(- \otimes \mathcal{F})$ has an internal right adjoint iff $f_!(- \otimes D\mathcal{F})$ has, and the right adjoints are then dual to each other. Inserting $D\mathcal{F}$ for $\mathcal{F}$ must therefore interchange the right adjoints, and so

$$\underline{\mathrm{Hom}}(\mathcal{F}, f^!(-)) \simeq f^*(-) \otimes D\mathcal{F}$$
$$\underline{\mathrm{Hom}}(D\mathcal{F}, f^!(-)) \simeq f^*(-) \otimes \mathcal{F}.$$

Inserting the unit into the second equivalence, we in particular learn $DD\mathcal{F} \simeq \mathcal{F}$.

For the equivalence to 4, we observe more generally that $\mathrm{Fun}^L(\mathrm{Shv}(X; \mathrm{Sp}), \mathrm{Shv}(Y; \mathrm{Sp})) \simeq \mathrm{Shv}(X \times Y; \mathrm{Sp})$, with an object $\mathcal{K} \in \mathrm{Shv}(X \times Y; \mathrm{Sp})$ corresponding to $p_{2,!}(p_1^*(-) \otimes \mathcal{K})$. Under this correspondence, an adjoint for $f_!(\mathcal{F} \otimes -)$ corresponds to $\mathcal{G}$ as claimed. Since the adjoint is necessarily of the form $f^* \otimes D\mathcal{F}$, we $\mathcal{G}$ is necessarily $D\mathcal{F}$, showing the equivalence to 5. $\square$

In [Sch23], the analogue of this condition on an object $\mathcal{F}$ (of $\mathrm{Shv}(X \times Y)$ in the relative case $f : X \to Y$) is called f-smooth. We see that X is cohomologically smooth if and only if the unit $f^*(\mathbb{S})$ is f-smooth, and $f^!(\mathbb{S})$ is invertible. The first condition has nice closure properties.

Lemma 4.6.13. *If $\mathcal{F} \in \mathrm{Shv}(X; \mathrm{Sp})$ and $\mathcal{G} \in \mathrm{Shv}(Y; \mathrm{Sp})$ are f-smooth, $p_1^*\mathcal{F} \otimes p_2^*\mathcal{G} \in \mathrm{Shv}(X \times Y; \mathrm{Sp})$ is f-smooth.*

Proof. As observed above, the functor $f_!(\mathcal{F} \otimes -)$ is dual to $f^*(-) \otimes \mathcal{F}$. This means that its base-change $\mathrm{Shv}(X \otimes Y; \mathrm{Sp}) \to \mathrm{Shv}(Y)$ is dual to $p_2^*(-) \otimes \mathcal{F}$. The composite is then dual to $f^*(-) \otimes p_1^*\mathcal{F} \otimes p_2^*\mathcal{G}$. Since strong left adjoints compose, this shows that $p_1^*\mathcal{F} \otimes p_2^*\mathcal{G}$ is f-smooth. □

Lemma 4.6.14. *1. Retracts of f-smooth objects are f-smooth.*

2. If $i : Y \to X$ is a retract with retraction $r : X \to Y$, and $\mathcal{F} \in \mathrm{Shv}(X; \mathrm{Sp})$ is f-smooth, $i^*\mathcal{F} \in \mathrm{Shv}(X; \mathrm{Sp})$ is f-smooth.

Proof. If $\mathcal{F}$ is a retract of $\mathcal{G}$, then $f^*(-) \otimes \mathcal{F}$ is a retract of $f^*(-) \otimes \mathcal{G}$ (in $\mathrm{Fun}^L(\mathrm{Sp}, \mathrm{Shv}(X))$), which proves that it has a left adjoint as well.

For the other statement, observe that $f^*(-) \otimes i^*(\mathcal{F})$ is a retract of $f^*(-) \otimes \mathcal{F}$ in $\mathrm{Ar}(\mathrm{Pr}^L)$. As in the proof of 2.3.22, this also proves existence of a left adjoint. □

In particular, spaces where $f^*(\mathbb{S})$ is f-smooth (the first half of cohomological smoothness) are closed under products and retracts.

Proposition 4.6.15. *For $X = \mathbb{R}$, the unit $f^*(\mathbb{S})$ is f-smooth.*

Proof. We claim one can check condition (4) of Lemma 4.6.12, with $\mathcal{G} = f^*(\mathbb{S}[1])$. Indeed, write f as the composite $\mathbb{R} \xrightarrow{i} S^1 \xrightarrow{t} \mathrm{pt}$ and observe the fiber sequence

$$i_!\mathbb{S} \to \mathbb{S} \to j_*\mathbb{S},$$

of sheaves on S^1, where $j : \{\infty\} \to S^1$ is the inclusion of the complement of $\mathbb{R}$. Applying $t_! = t_*$, it follows from Proposition 4.6.6 that also

$$f_!\mathbb{S} \to \mathrm{map}(S^1, \mathbb{S}) \to \mathbb{S},$$

is a fiber sequence, and hence $f_!\mathbb{S} = \mathbb{S}[-1]$. So

$$\mathrm{map}(f_!(\mathbb{S}[1]), \mathbb{S}) \simeq \mathbb{S}.$$

Next, $\Delta : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ is a proper map, so $\Delta_! = \Delta_*$ and we have a sequence

$$i_!\mathbb{S} \to \mathbb{S} \to \Delta_*\mathbb{S},$$

where $i : U = \mathbb{R} \times \mathbb{R} \setminus \Delta(\mathbb{R}) \to \mathbb{R} \times \mathbb{R}$ is the inclusion of the complement of the diagonal. We have $\mathrm{map}_{\mathbb{R}\times\mathbb{R}}(i_!\mathbb{S}, \mathbb{S}) = \mathrm{map}_U(\mathbb{S}, \mathbb{S}) = \mathbb{S} \oplus \mathbb{S}$, since U consists of two contractible components. We also have $\mathrm{map}_{\mathbb{R}\times\mathbb{R}}(\mathbb{S}, \mathbb{S}) = \mathbb{S}$. So we learn that

$$\mathrm{map}(\Delta_*\mathbb{S}, \mathbb{S}) \simeq \mathrm{fib}(\mathbb{S} \to \mathbb{S} \oplus \mathbb{S}) \simeq \mathbb{S}[-1],$$

or

$$\mathrm{map}(\Delta_!(\mathbb{S}), \mathbb{S}[1]) \simeq \mathbb{S}.$$

We now take the structure maps for the "adjunction" between $\mathcal{F} = \mathbb{S}$ and $\mathcal{G} = \mathbb{S}[1]$ to be the equivalences to $\mathbb{S}$ above. One can check that these indeed satisfy the snake identities, but we refrain from doing so here. $\square$

Proposition 4.6.16. *For X a manifold (or more generally an euclidean neighbourhood retract), the unit $f^*(\mathbb{S})$ is f-smooth.*

Proof. Knowledge of f-smoothness on $\mathbb{R}$ and compatibility with products immediately shows it for $\mathbb{R}^n$. Next, condition (2) in Lemma 4.6.12 shows that f-smoothness is a local property, and thus follows for general manifolds. Finally, compatibility with retracts shows it for euclidean neighbourhood retracts. $\square$

The first half of cohomological smoothness is therefore satisfied for a quite large class of spaces. Invertibility of ω_X is more restrictive. We will see that it can be checked on stalks for euclidean neighbourhood retracts. For this, we will need a portion of the above theory also with $\mathcal{D}(\mathbb{Z})$ and $\mathcal{D}(\mathbb{F}_p)$-coefficients.

Lemma 4.6.17. *Let X be such that the unit $f^*(\mathbb{S}) \in \mathrm{Shv}(X; \mathrm{Sp})$ is f-smooth.*

1. *For any presentable $\mathcal{C}$, we have an adjunction*

$$\begin{tikzcd} \mathrm{Shv}(X;\mathcal{C}) \arrow[r, shift left, "f_{\mathcal{C},!}"] & \mathcal{C} \arrow[l, shift left, "f_{\mathcal{C}}^!"] \end{tikzcd}$$

base-changed from the one for Sp *coefficients. We have*

$$f_{\mathcal{C}}^!(X) = f^!(\mathbb{S}) \otimes f_{\mathcal{C}}^*(X).$$

2. *If $\mathcal{C}$ is locally rigid, the analogous $\mathcal{C}$-linear properties from Lemma 4.6.12 hold, with*

$$D_{\mathcal{C}}\mathcal{F} \simeq \underline{\mathrm{Hom}}(\mathcal{F}, f_{\mathcal{C}}^!(\mathbb{1}_{\mathcal{C}})) = D\mathcal{F} \otimes \mathbb{1}_{\mathcal{C}}.$$

Proof. Under the assumptions, $f^! : \mathrm{Sp} \to \mathrm{Shv}(X; \mathrm{Sp})$ preserves colimits and coincides with $f^!(\mathbb{S}) \otimes f^*(-)$. So $f_! \dashv f^!$ is an internal adjunction, and $f^!(-)$ base-changes to a functor agreeing with $f^!(\mathbb{S}) \otimes f_{\mathcal{C}}^*(-)$.

For the second part, we observe that the internal adjunctions base-change. $\square$

Proposition 4.6.18. *Let X be a euclidean neighbourhood retract such that there exists n with the property that for every $x \in X$, the stalk of $f^!(\mathbb{S})$ at x is equivalent to $\mathbb{S}[n]$. Then X is cohomologically smooth. Then $f^!(\mathbb{S})$ is locally isomorphic to the constant sheaf $\mathbb{S}[n]$.*

Proof. We use that euclidean neighbourhood retracts are hypercomplete, so we may check equivalences of sheaves on stalks. Let $x \in X$ be a point, and

$$\mathbb{S}[n] \to f^!(\mathbb{S})|_U$$

a map in $\mathrm{Shv}(U; \mathrm{Sp})$ inducing an equivalence on stalks at x. We may assume U to be connected, since X is locally connected. We prove that the above map is already an equivalence on all of U.

Since all stalks are connective spectra and we can detect equivalences on stalks, it suffices to check this after base-change to $\mathbb{Z}$, i.e. that the map

$$\mathbb{Z}[n] \to f^!_{\mathbb{Z}}(\mathbb{Z})|_U$$

is an equivalence (using the fact established above that if the unit is f-smooth, $f^!$ base-changes). Finally, since all stalks are finite-type $\mathbb{Z}$-complexes, we may check that the above map is an equivalence by checking that for every p,

$$\mathbb{F}_p[n] \to f^!_{\mathbb{F}_p}(\mathbb{F}_p)|_U$$

is an equivalence. Denote the target by $\mathcal{F}$. All its stalks are coconnective, so it is coconnective (this uses hypercompleteness again!). Let $Z \subseteq U$ be the closed subset where this map is nonzero on stalks, and $i : U \setminus Z \to U$ and $j : Z \to U$ the open and closed inclusions. Since $i_!\mathbb{F}_p \to \mathbb{F}_p \to \mathcal{F}$ is zero on all stalks, its image sheaf is trivial, so this map is zero. The sequence

$$i_!\mathbb{F}_p \to \mathbb{F}_p \to j_*\mathbb{F}_p$$

then tells us that we get a factorisation

$$\mathbb{F}_p \to j_*\mathbb{F}_p \to \mathcal{F}.$$

The map $i_!i^!\mathcal{F} \to \mathcal{F} \to \mathrm{cofib}(j_*\mathbb{F}_p \to \mathcal{F})$ now induces equivalences on all stalks: On Z, both are zero, while on $U \setminus Z$, they agree with the stalks of $\mathcal{F}$. So this map is an equivalence and we obtain a split cofiber sequence

$$j_*\mathbb{F}_p \to \mathcal{F} \to i_!i^!\mathcal{F}.$$

But since we know $\underline{\mathrm{Hom}}_{\mathrm{Shv}(U;\mathcal{D}(\mathbb{F}_p))}(\mathcal{F}, \mathcal{F}) = D_{\mathbb{F}_p}D_{\mathbb{F}_p}\mathbb{F}_p = \mathbb{F}_p$ and assumed U connected, $\pi_0 \operatorname{map}(\mathcal{F}, \mathcal{F}) = \mathbb{F}_p$. In particular, $\mathcal{F}$ is indecomposable. Since $j_*\mathbb{F}_p$ is nontrivial, $j_*\mathbb{F}_p \to \mathcal{F}$ has to be an equivalence and $Z = U$, so we are done. $\square$

Remark 4.6.19. Assume $Z \subseteq X$ is closed. For any sheaf $\mathcal{F}$, we have a sequence

$$i_!i^!\mathcal{F} \to \mathcal{F} \to j_*j^*\mathcal{F},$$

where $i : U = X \setminus Z \to X$ is the complementary open inclusion. Applying $t_!$ where $t : X \to \mathrm{pt}$ is the unique map, and using that $j_* = j_!$, we get $t_{U,!} i^! \mathcal{F} \to t_{X,!} \mathcal{F} \to t_{Z,!} j^* \mathcal{F}$.

This is a sequence relating compactly supported cohomology of $\mathcal{F}|_U$, $\mathcal{F}$ and $\mathcal{F}|_Z$. In the special case where $\mathcal{F} = t_X^! A$ and $Z = \{x\}$, it expresses the stalk $(t_X^! A)_x$ in terms of a cofiber sequence

$$t_{U,!} t_U^! A \to t_{X,!} t_X^! A \to (t_X^! A)_x,$$

i.e. as "local homology $H_*(X, X \setminus \{x\}; A)$".

If U and X satisfy some shape condition, this is really singular homology, or rather the spectrum valued version $\mathrm{cofib}(\Sigma_+^\infty (X \setminus \{x\}) \to \Sigma_+^\infty X) \otimes A$. Since this is connective, to see that it agrees with $\mathbb{S}[n]$ it suffices to know that the base-change to $\mathbb{Z}$ agrees with $\mathbb{Z}[n]$ (by Hurewicz). So to apply the above proposition to an ENR X (which satisfies the required shape conditions automatically), it suffices to know that $H_*(X, X \setminus \{x\}; \mathbb{Z}) = \mathbb{Z}[n]$ for some n and every point x. This is the usual definition of homology manifolds.

Corollary 4.6.20. *Homology manifolds (i.e. ENRs with $H_*(X, X \setminus \{x\}) = \mathbb{Z}[n]$ for some n and every $x \in X$) are cohomologically smooth.*

Proof. We have seen in Proposition 4.6.16 that the unit is f-smooth, and in Proposition 4.6.18 together with the above Remark that $f^!(\mathbb{S})$ is invertible, so the result follows. □

4.7 The rigidity of NcMot

Recall that a localizing invariant is a functor $\mathrm{Pr}^L_{\mathrm{dual}} \to \mathcal{D}$ into a stable ∞-category which takes Verdier sequences to cofiber sequences, and is called finitary if it additionally preserves filtered colimits.

We have seen in Theorem 3.6.14 that if $\mathcal{D}$ is dualizable, any localizing invariant can be computed on $\mathrm{Shv}(X; \mathcal{C})$ for compactly assembled $\mathcal{C}$ by the formula

$$F(\mathrm{Shv}(X; \mathcal{C})) \simeq \Gamma_c(X, f^* F(\mathcal{C})).$$

(We may now also write this as $f_! f^* F(\mathcal{C})$ for $f : X \to \mathrm{pt}$, and $f_!$ the base-change of $f_! : \mathrm{Shv}(X; \mathrm{Sp}) \to \mathrm{Sp}$.)

We also introduced the target NcMot of the *universal* finitary localizing invariant, as full subcategory of $\mathrm{Fun}(\mathrm{Pr}^L_{\mathrm{dual}}, \mathrm{Sp}^{\mathrm{op}})$ on those functors which are finitary localizing invariants with values in $\mathrm{Sp}^{\mathrm{op}}$. This is a Bousfield localisation generated by the equivalences $\mathrm{cofib}(j\mathcal{A} \to j\mathcal{B}) \to j\mathcal{C}$ and $\mathrm{colim}_i\, j\mathcal{A}_i \to j(\mathrm{colim}_i\, \mathcal{A}_i)$, i.e. enforcing that the spectral Yoneda embedding becomes a finitary localizing invariant. We call the resulting functor $\mathrm{Pr}^L_{\mathrm{dual}} \to \mathrm{NcMot}$ simply M and think of $M\mathcal{C}$ as the "motive" of $\mathcal{C}$.

In order to apply Theorem 3.6.14 to the universal finitary localizing invariant, we would need to know that NcMot is dualizable. This would give $M(\mathrm{Shv}(X; \mathcal{C})) = \Gamma_c(X; f^* M(\mathcal{C}))$. If X has nice topology (shape conditions), this simplifies further, for example for $X = \mathbb{R}^n$, we have $\Gamma_c(X; f^* M(\mathcal{C})) = \Sigma^{-n} M(\mathcal{C})$. This implies $F(\mathrm{Shv}(\mathbb{R}^n; \mathcal{C})) = \Sigma^{-n} F(\mathcal{C})$ for *all* finitary localizing invariants!

In this section, we will prove more:

Theorem 4.7.1 (Efimov)**.** NcMot *is rigid, in particular dualizable.*

We first remember that NcMot is actually stably generated by the image of the restriction of M to $\mathrm{Pr}^L_{\mathrm{st},\omega} \to \mathrm{NcMot}$, since every dualizable $\mathcal{C}$ fits into a Verdier sequence

$$\mathcal{C} \to \mathrm{Ind}(\mathcal{C}^{\omega_1}) \to \mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}}(\mathcal{C}))$$

where the rightmost two terms are compactly generated. (And in fact, the original definition of NcMot due to Blumberg-Gepner-Tabuada works with $\mathrm{Cat}^{\mathrm{perf}}_\infty \simeq \mathrm{Pr}^L_{\mathrm{st},\omega}$).

The main ingredient in the proof of 4.7.1 will be that $\mathrm{Pr}^L_{\mathrm{st},\omega}$ is compactly generated:

Lemma 4.7.2. $\mathrm{Pr}^L_{\mathrm{st},\omega}$ *is generated by* Sp*, which is compact.*

Proof. Equivalently, it suffices to check that Sp^ω is a compact generator of $\mathrm{Cat}^{\mathrm{perf}}_\infty$. Indeed, $\mathrm{Map}(\mathrm{Sp}^\omega, \mathcal{C}) = \mathcal{C}^\simeq$, and since filtered colimits in $\mathrm{Cat}^{\mathrm{perf}}_\infty$ are formed underlying, it is compact. To see that it is a generator, we need to check that maps out of it preserve equivalences, so that a functor $\mathcal{C} \to \mathcal{D}$ between small stable ∞-categories is an equivalence if $\mathcal{C}^\simeq \to \mathcal{D}^\simeq$ is an equivalence. Obviously this implies essential surjectivity. To see that $\mathcal{C} \to \mathcal{D}$ is also fully faithful, take objects $x, y \in \mathcal{C}$ and observe that $\mathrm{Map}(x, y)$ is a retract of $\mathrm{Aut}(x \oplus y)$, by inclusion $\mathrm{Map}(x, y) \to \mathrm{Aut}(x \oplus y, x \oplus y)$ as "upper triangular matrices" (with identities on the diagonal). So if

$$\mathrm{Aut}(x \oplus y) \simeq \mathrm{Aut}(Fx \oplus Fy),$$

we learn $\mathrm{Map}(x, y) \simeq \mathrm{Map}(Fx, Fy)$. □

This on its own does not help much though, since it is not a priori clear how the functor $M : \mathrm{Pr}^L_{\mathrm{st},\omega} \to \mathrm{NcMot}$ interacts with compact objects. We will utilize below that it does interact well with trace-class morphisms (since M is symmetric-monoidal).

Definition 4.7.3 (Kontsevich)**.**

1. $\mathcal{D} \in \mathrm{Pr}^L_{\mathrm{st},\omega}$ is called *proper* if the evaluation $\mathcal{D}^\vee \otimes \mathcal{D} \to \mathrm{Sp}$ is strongly left adjoint.

2. $\mathcal{D} \in \mathrm{Pr}^L_{\mathrm{st},\omega}$ is called *smooth* if the coevaluation $\mathrm{Sp} \to \mathcal{D} \otimes \mathcal{D}^\vee$ is strongly left adjoint.

Lemma 4.7.4 (Toen-Vezzosi)**.**

1. *If $\mathcal{A}$ is a compact object in $\mathrm{Pr}^L_{\mathrm{st},\omega}$, it is smooth.*

2. *If $\mathcal{A}$ is smooth and proper (i.e. dualizable in $\mathrm{Pr}^L_{\mathrm{st},\omega}$), it is compact.*

Proof. If $\mathcal{A}$ is compact, then it is Mod_A for some $\mathbb{E}_1$-ring A. Indeed, $\mathcal{A}$ is compactly generated, by some set of objects X_i. As in the proof of Proposition 2.10.14 we can then write $\mathcal{A}$ is a filtered colimit of $\langle X_i \rangle_{i \in I}$ where I ranges over finite subsets of indices, and so $\mathcal{A}$ is a retract of a category with finitely many generators, and thus itself generated by finitely many generators. Passing to the sum, we may assume $\mathcal{A}$ to be generated by a single X, and thus $\mathcal{A} \simeq \mathrm{Mod}_A$ with $A = \mathrm{end}(X)$.

Next, compactness of $\mathcal{A}$ implies that $\mathrm{Fun}^L(\mathcal{A},-) : \mathrm{Pr}^L_{\mathrm{st},\omega} \to \widehat{\mathrm{Cat}}_\infty$ commutes with filtered colimits. Since we may write it as pullback, this also implies that *pointed* functors $\mathrm{Fun}^L_*(\mathcal{A},-)$ commute with filtered colimits. In particular, since

$$\mathrm{Fun}^L_*(\mathrm{Mod}_A, \mathrm{Mod}_R) = \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp})}(A, R),$$

and $\mathrm{colim}_i \mathrm{Mod}_{R_i} = \mathrm{Mod}_{\mathrm{colim}\, R_i}$ by Lemma 2.10.12, compactness of $\mathcal{A}$ implies compactness of $A \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp})$.

The $\mathbb{E}_1$ cotangent complex $L^{\mathbb{E}_1}_{A/\mathbb{S}}$ is characterized by

$$\mathrm{Map}_{\mathrm{BiMod}(A,A)}(L^{\mathbb{E}_1}_{A/\mathbb{S}}, M) = \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_1}}(A, A \oplus M)_{\mathrm{id}},$$

where the subscript indicates that we take the fiber over $\mathrm{id} \in \mathrm{Map}_{\mathrm{Alg}^{E_1}}(A,A)$. One has a fiber sequence

$$L^{E_1}_{A/\mathbb{S}} \to A \otimes A \to A,$$

and so the first two terms being compact implies that A is also compact in $\mathrm{BiMod}(A,A)$. But this is exactly what is required to know that $\mathrm{Sp} \to \mathcal{A} \otimes \mathcal{A}^\vee$ is strongly left adjoint.

Compactness of the unit $\mathrm{Sp} \in \mathrm{Pr}^L_\omega$ was done in Lemma 4.7.2, and so every dualizable object is compact. □

Definition 4.7.5. We call $\mathcal{B} \in \mathrm{Pr}^L_{\mathrm{st},\omega}$ *nuclear* if for all compact objects $\mathcal{A}$ the map

$$\underline{\mathrm{Hom}}(\mathcal{A}, \mathrm{Sp}) \otimes \mathcal{B} \to \underline{\mathrm{Hom}}(\mathcal{A}, \mathcal{B})$$

is an equivalence.

Lemma 4.7.6. *If $\mathcal{A}$ is smooth, then the above comparison functor is fully faithful.*

Proof. Recall from Definition 2.12.5 that $\underline{\mathrm{Hom}}(\mathcal{A},\mathcal{B}) = (\mathrm{Fun}^L(\mathcal{A},\mathcal{B}), S)^{\mathrm{ca}}$ for a certain precompact ideal S. In particular, we have the counit

$$\varepsilon : \underline{\mathrm{Hom}}(\mathcal{A},\mathcal{B}) \subseteq \mathrm{Ind}(\mathrm{Fun}^L(\mathcal{A},\mathcal{B})) \xrightarrow{k} \mathrm{Fun}^L(\mathcal{A},\mathcal{B})$$

that sends compact maps into S. Since Fun^L is the internal hom in Pr^L, we have an evaluation $\mathrm{ev} : \mathcal{C} \otimes \mathrm{Fun}^L(\mathcal{C},\mathcal{D}) \to \mathcal{D}$, and analogously we have a compactly assembled evaluation $\mathrm{ev}_{\mathrm{ca}} : \mathcal{C} \otimes \underline{\mathrm{Hom}}(\mathcal{C},\mathcal{D}) \to \mathcal{D}$ which is equivalently given as $\mathrm{ev}_{\mathrm{ca}} = \mathrm{ev} \circ (\mathcal{C} \otimes \varepsilon)$. In general, the comparison functor ε agrees with the following composite

$$\underline{\mathrm{Hom}}(\mathcal{A},\mathcal{B}) \to \mathcal{A}^\vee \otimes \mathcal{A} \otimes \underline{\mathrm{Hom}}(\mathcal{A},\mathcal{B}) \xrightarrow{\mathrm{ev}_{\mathrm{ca}}} \mathcal{A}^\vee \otimes \mathcal{B} \simeq \mathrm{Fun}^L(\mathcal{A},\mathcal{B}).$$

Under the hypothesis that $\mathcal{A}$ is smooth, the first functor and hence the whole composite ε is actually a strong left adjoint. It follows by the Addendum 2.7.6 that ε is fully faithful.

Finally, one checks that the assembly-type functor in the above definition of nuclear categories fits into a commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(\mathcal{A},\mathrm{Sp}) \otimes \mathcal{B} \arrow[r] \arrow[d, hook, "\varepsilon\otimes\mathcal{B}"'] & \underline{\mathrm{Hom}}(\mathcal{A},\mathcal{B}) \arrow[d, hook, "\varepsilon"] \\
\mathrm{Fun}^L(\mathcal{A},\mathrm{Sp}) \otimes \mathcal{B} \arrow[r, equal] & \mathrm{Fun}^L(\mathcal{A},\mathcal{B})
\end{tikzcd}$$

and is hence fully faithful itself, as desired. □

Lemma 4.7.7. *In* NcMot*, the unit* $M(\mathrm{Sp})$ *is compact. In particular, trace-class morphisms are compact.*

Proof. By Blumberg-Gepner-Tabuada, $M(\mathrm{Sp})$ corepresents K-theory, i.e. $\mathrm{map}(M(\mathrm{Sp}), M(\mathcal{C})) = K(\mathcal{C})$. Since K-theory commutes with filtered colimits, the unit $M(\mathrm{Sp})$ of NcMot is indeed compact. □

Now recall that our goal is to prove the rigidity of NcMot. For this, we will show the conditions in Theorem 4.4.6. We have seen above that the unit in NcMot is compact, so it remains to see that every compact morphism is trace class. Instead, we will show that NcMot is generated by basic nuclears. Since compact maps are trace-class, these are compactly exhaustible (hence closed under finite colimits) and it follows that NcMot is dualizable. Moreover, given any compact map $X \to Z$, we can then factor it into two compact maps $X \to Y \to Z$. Writing Z as filtered colimit of basic nuclears, $Y \to Z$ factors through some Z_i, and writing the latter as a sequential colimit along trace-class maps, we see that $X \to Z$ factors through a trace-class map and is thus itself trace-class. This proves that to show rigidity of NcMot, it remains to see that it is generated by basic nuclears. In fact, nuclear objects suffice:

Lemma 4.7.8. *Assume* $\mathcal{C}$ *is a symmetric-monoidal compactly generated category with compact unit* $\mathbb{1}$*. Then every nuclear object* B *is a filtered colimit of basic nuclear objects (i.e. sequential colimits* $\mathrm{colim}_{\mathbb{N}} A_n$ *where all nonidentity maps are trace-class.)*

Proof. Indeed, write $B = \mathrm{colim}_{i \in I} A_i$ where the A_i are compact and I is filtered. Without limiting generality we may assume I to be a poset, see Kerodon 02QA. Now let $J \subset I$ be the non-full subcategory consisting of all objects, identities and those $i \to j$ for which $A_i \to A_j$ is trace-class. We claim $J \subset I$ is cofinal. For this, using the fact that trace-class morphisms form a 2-sided ideal, it suffices to find for every i a j such that $A_i \to A_j$ is trace-class. The map $A_i \to B$ may be lifted to a map $\mathbb{1} \to [A_i, \mathbb{1}] \otimes B$ by nuclearity. Writing $B = \mathrm{colim}_{i \to j} A_j$ using that I is filtered, we obtain a factorisation $\mathbb{1} \to [A_i, \mathbb{1}] \otimes A_j$. If the composite $\mathbb{1} \to [A_i, A_j]$ agrees with the map $A_i \to A_j$, we have found a witness for the latter being trace-class. In general, we only know this to be true in the colimit over j (since by construction the composite $\mathbb{1} \to [A_i, B]$ agrees with the map $A_i \to B$), so by again using compactness of $\mathbb{1}$, we may replace j by a bigger j' for which $A_i \to A_{j'}$ is trace-class.

So we may write B as colimit $\mathrm{colim}_{j \in J} A_j$. Next, consider the ω_1-filtered poset $\mathcal{P}_\omega(J)$ of countable filtered subposets $J_0 \subseteq J$ under inclusion. Analogously to the fact that one can write any colimit as an ω_1-filtered colimit of countable colimits [Lur17b, 4.2.3.11], one can use [Lur17b, 4.2.3.4, 4.2.3.8] to show that there exists a diagram $\mathcal{B}_\bullet : \mathcal{P}_\omega(J) \to \mathrm{Pr}^L_{\mathrm{st},\omega}$ which sends a countable directed subposet $J_0 \subseteq J$ to $\mathcal{B}_{J_0} \coloneqq \mathrm{colim}_{i \in J_0} \mathcal{A}_i$, and such that $\mathrm{colim}_{J_0 \in \mathcal{P}_\omega(J)} \mathcal{B}_{J_0} \simeq \mathrm{colim}_j \mathcal{A}_j \simeq \mathcal{B}$. Finally, it is an easy exercise that any countable filtered poset J_0 admits a cofinal functor $\mathbb{N} \to J_0$, so by construction each $\mathcal{B}_{J_0}$ is basic nuclear, as desired. □

Since M is symmetric monoidal and preserves filtered colimits, it thus suffices to check that NcMot is generated by $M\mathcal{B}$ for $\mathcal{B}$ nuclear.

Lemma 4.7.9. *The nuclear objects of* $\mathrm{Pr}^L_{\mathrm{st},\omega}$ *are closed under:*

1. *Filtered colimits*
2. *Semi-orthogonal decompositions*
3. *Full subcategories*

Proof. The first point is clear from the definition. If we have a semi-orthogonal decomposition, i.e. a diagram of adjunctions

$$\mathcal{B}_1 \leftrightarrows \mathcal{B}_2 \leftrightarrows \mathcal{B}_3$$

where $\mathrm{map}_{\mathcal{B}_2}(x, y) = 0$ for $x \in \mathcal{B}_1$ and $y \in \mathcal{B}_3$, we obtain semi-orthogonal decompositions

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(\mathcal{A}, \mathrm{Sp}) \otimes \mathcal{B}_1 \arrow[r, shift left] \arrow[d] & \underline{\mathrm{Hom}}(\mathcal{A}, \mathrm{Sp}) \otimes \mathcal{B}_2 \arrow[l, shift left] \arrow[r, shift left] \arrow[d] & \underline{\mathrm{Hom}}(\mathcal{A}, \mathrm{Sp}) \otimes \mathcal{B}_3 \arrow[l, shift left] \arrow[d] \\
\underline{\mathrm{Hom}}(\mathcal{A}, \mathcal{B}_1) \arrow[r, shift left] & \underline{\mathrm{Hom}}(\mathcal{A}, \mathcal{B}_2) \arrow[l, shift left] \arrow[r, shift left] & \underline{\mathrm{Hom}}(\mathcal{A}, \mathcal{B}_3). \arrow[l, shift left]
\end{tikzcd}$$

If $\mathcal{B}_1$ and $\mathcal{B}_3$ are nuclear, the outer maps are equivalences. The middle vertical is always fully faithful by Lemma 4.7.6. Given $b \in \underline{\mathrm{Hom}}(\mathcal{A}, \mathcal{B}_2)$, we have a fiber sequence $a \to b \to c$ where a lies in the image of the left inclusion and c in the image of the right one. Rotating once, we have $b \to c \to a[1]$, and since the outer maps are equivalences and the diagram commutes, the objects $c, a[1]$ lie in the full subcategory $\underline{\mathrm{Hom}}(\mathcal{A}, \mathrm{Sp}) \otimes \mathcal{B}_2$. But then by fully faithfulness also the map $c \to a[1]$, and hence the fiber b lies in that subcategory, so the middle vertical is also essentially surjective.

For the last part, if $\mathcal{B}_1 \subseteq \mathcal{B}_2$ is fully faithful, we have a diagram

$$\begin{tikzcd}
 & \underline{\mathrm{Hom}}(\mathcal{A}, \mathrm{Sp}) \otimes \mathcal{B}_1 \arrow[r] \arrow[d, hook] & \underline{\mathrm{Hom}}(\mathcal{A}, \mathrm{Sp}) \otimes \mathcal{B}_2 \arrow[r] \arrow[d, "\simeq"] & \underline{\mathrm{Hom}}(\mathcal{A}, \mathrm{Sp}) \otimes \mathcal{B}_3 \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \underline{\mathrm{Hom}}(\mathcal{A}, \mathcal{B}_1) \arrow[r] & \underline{\mathrm{Hom}}(\mathcal{A}, \mathcal{B}_2) \arrow[r] & \underline{\mathrm{Hom}}(\mathcal{A}, \mathcal{B}_3), &
\end{tikzcd}$$

and the same diagram chase as in the 5-lemma shows that the left-hand vertical functor is essentially surjective. □

Lemma 4.7.10. *For an* $\mathbb{E}_1$ *ring* R*, let* $\widetilde{R}$ *be the graded ring (i.e. algebra in* $\mathrm{Sp}^{\mathbb{Z}}$*) with*

$$R_n = \begin{cases} R, & n > 0 \\ \mathbb{S}, & n = 0 \\ 0, & n < 0. \end{cases}$$

Then $\mathrm{Mod}^{\leq 0}_{\widetilde{R}}$*, the category of* $\widetilde{R}$*-modules concentrated in nonpositive degrees, is nuclear.*

Proof. We write $\mathcal{C} = \mathrm{Mod}_{\widetilde{R}}$ and define $\mathcal{C}^{\geq n}$ as modules concentrated in degrees $\geq n$, $\mathcal{C}^{\leq n}$ as modules concentrated in degrees $\leq n$, and $\mathcal{C}^{[n,m]}$ as modules concentrated in degrees between n and m. Now $\mathcal{C}^{\geq n}$ is generated by shifted free modules $\widetilde{R}(m)$ for $m \geq n$. The right orthogonal subcategory

$$(\mathcal{C}^{\geq n})^{\perp} := \{y \in \mathcal{C} \mid \mathrm{map}(x, y) = 0 \text{ for all } x \in \mathcal{C}^{\geq n}\}$$

is therefore given by $\mathcal{C}^{\leq n-1}$. In particular, we have

$$\mathcal{C}^{\geq n} \cap (\mathcal{C}^{\geq n+1})^{\perp} = \mathcal{C}^{[n,n]} \simeq \mathrm{Sp},$$

since it consists of $\widetilde{R}$-modules concentrated in a single degree, and $\widetilde{R} = \mathbb{S}$ in degree 0.

We have semiorthogonal decompositions

$$\mathcal{C}^{[0,0]} \to \mathcal{C}^{[-1,0]} \to \mathcal{C}^{[-1,-1]},$$
$$\mathcal{C}^{[-1,0]} \to \mathcal{C}^{[-2,0]} \to \mathcal{C}^{[-2,-2]}$$

etc., and so inductively all $\mathcal{C}^{[-n,0]}$ and the filtered colimit $\mathcal{C}^{\leq 0}$ are nuclear. □

Lemma 4.7.11. *There is a left Bousfield localisation* $\mathrm{Mod}_{\widetilde{R}}^{\leq 0} \to \mathrm{Mod}_R$*.*

Proof. Consider the functor $F : \mathrm{Mod}_R \to \mathrm{Mod}_{\widetilde{R}}^{\leq 0}$ which takes N to the graded object which is N in degrees ≤ 0, with the obvious $\widetilde{R}$-action. This functor has both adjoints. Its left adjoint L takes the truncation of $(\widetilde{R}(m) \otimes M)^{\leq 0}$ for $m \leq 0$ simply to M. Furthermore, since $FN = \mathrm{colim}_m(\widetilde{R}(m) \otimes N)^{\leq 0}$, we learn $LFN = N$, and thus F is fully faithful. So L is a left Bousfield localisation. □

Proof of Theorem 4.7.1. As discussed above, it suffices to check that NcMot is generated by $M\mathcal{B}$ for $\mathcal{B}$ nuclear. Since NcMot is generated by $M\mathcal{A}$ for $\mathcal{A}$ compact, and those are of the form Mod_R for some R (in fact, compact $\mathbb{E}_1$ rings as seen above), it suffices to check that those $M\mathrm{Mod}_R$ are in the stable subcategory generated by nuclear objects. We have a Verdier sequence

$$0 \to \mathrm{Ker} \to \mathrm{Mod}_{\widetilde{R}}^{\leq 0} \to \mathrm{Mod}_R \to 0,$$

and by the above, the middle and left terms are nuclear. □

This proves the rigidity of NcMot. In particular, it is dualizable and we learn $M\,\mathrm{Shv}(X; \mathcal{C}) = p_! p^* M\mathcal{C}$.

Definition 4.7.12. For compactly assembled categories $\mathcal{A}$ and $\mathcal{B}$, their *bivariant K-theory* is given by

$$\mathrm{KK}^{\mathrm{cont}}(\mathcal{A}, \mathcal{B}) = \mathrm{map}_{\mathrm{NcMot}}(M\mathcal{A}, M\mathcal{B}).$$

The relation to K-theory is through the aforementioned result

Theorem 4.7.13 (Blumberg-Gepner-Tabuada)**.** $M\,\mathrm{Sp}$ *corepresents K-theory, so*

$$\mathrm{KK}^{\mathrm{cont}}(\mathrm{Sp}, \mathcal{B}) \simeq K^{\mathrm{cont}}(\mathcal{B}).$$

This also suggests the following definition:

Definition 4.7.14. For a compactly assembled category $\mathcal{A}$, we call

$$\mathrm{KK}^{\mathrm{cont}}(\mathcal{A}, \mathrm{Sp})$$

the *K-homology* of $\mathcal{A}$.

The naming here is of course inspired by the fact that we think of ordinary K-theory as cohomology theory: It is covariant in categories, but contravariant for example in schemes. The K-homology is covariant (and has finitary Zariski codescent for the same reasons as K-theory having finitary Zariski descent).

Bivariant K-theory has a precursor in K-theory of C^* algebras, where it is called Kasparov KK-theory. The possibility of defining bi- and covariant K-theory of categories was clear after Blumberg-Gepner-Tabuadas work, but it was not really possible to compute this in nontrivial examples.

We now can compute K-homology of sheaf categories:

Proposition 4.7.15. $\mathrm{KK}^{\mathrm{cont}}(\mathrm{Shv}(X; \mathrm{Sp}), \mathcal{C}) = p_* p^! K(\mathcal{C})$*, which for good X agrees with locally finite homology on X with coefficients $K(\mathcal{C})$.*

Proof. By definition,

$$\mathrm{KK}^{\mathrm{cont}}(\mathrm{Shv}(X; \mathrm{Sp}), \mathcal{C}) = \mathrm{map}_{\mathrm{NcMot}}(M \,\mathrm{Shv}(X; \mathrm{Sp}), M\mathcal{C}).$$

We have $M \,\mathrm{Shv}(X; \mathrm{Sp}) = p_! p^* M \,\mathrm{Sp}$, by applying Theorem 3.6.14. Here we write $p_!$ and p^* for the functors between $\mathrm{Shv}(X; \mathrm{NcMot})$ and NcMot obtained by base-changing the ones for spectrum-valued functors. Now we claim

$$\underline{\mathrm{Hom}}_{\mathrm{NcMot}}(p_! p^* M \,\mathrm{Sp}, M\mathcal{C}) = p_* p^! M\mathcal{C}.$$

Indeed, for an arbitrary $Z \in \mathrm{NcMot}$ we have

$$\begin{aligned} \mathrm{map}_{\mathrm{NcMot}}(Z, \underline{\mathrm{Hom}}(p_! p^* M \,\mathrm{Sp}, M\mathcal{C})) &\simeq \mathrm{map}_{\mathrm{NcMot}}(Z \otimes p_! p^* M \,\mathrm{Sp}, M\mathcal{C}) \\ &\simeq \mathrm{map}_{\mathrm{NcMot}}(p_! p^* Z, M\mathcal{C}) \\ &\simeq \mathrm{map}_{\mathrm{NcMot}}(Z, p^! p_* M\mathcal{C}), \end{aligned}$$

where in the second step we use linearity of $p_!$ and p^* as well as that $M \,\mathrm{Sp}$ is the unit.

It follows that

$$\begin{aligned} \mathrm{KK}^{\mathrm{cont}}(\mathrm{Shv}(X; \mathrm{Sp}), \mathcal{C}) &\simeq \mathrm{map}_{\mathrm{NcMot}}(p_! p^* M \,\mathrm{Sp}, M\mathcal{C}) \\ &\simeq \mathrm{map}_{\mathrm{NcMot}}(M \,\mathrm{Sp}, \underline{\mathrm{Hom}}_{\mathrm{NcMot}}(p_! p^* M \,\mathrm{Sp}, M\mathcal{C})) \\ &\simeq \mathrm{map}_{\mathrm{NcMot}}(M \,\mathrm{Sp}, p_* p^! M\mathcal{C}) \\ &\simeq p_* p^! \,\mathrm{map}_{\mathrm{NcMot}}(M \,\mathrm{Sp}, M\mathcal{C}) \\ &\simeq p_* p^! K(\mathcal{C}). \end{aligned}$$

Here, for the second-to-last step, we use that the functors $p^*, p_!$ are base-changed from their Sp-coefficient counterparts, so commute with the unit functor $\mathrm{Sp} \to \mathrm{NcMot}$, and hence their right adjoints commute with the right adjoint $\mathrm{map}_{\mathrm{NcMot}}(M \,\mathrm{Sp}, -)$ of the unit functor. $\square$

This is one of the cases in which we can understand bivariant K-theory. In another direction, one can ask the following question: Is $\underline{\mathrm{Hom}}_{\mathrm{NcMot}}(M\mathcal{C}, M\mathcal{D})$ itself M of some functor category? In general, this is wrong. However, there is the following fact, which we will not prove here.

Proposition 4.7.16 (Efimov)**.** *If $\mathcal{C} \in \mathrm{Pr}^L_{\mathrm{dual}}$ is proper and ω_1-compact, we have*

$$M\underline{\mathrm{Hom}}^{\mathrm{dual}}(\mathcal{C}, \mathcal{D}) = \underline{\mathrm{Hom}}_{\mathrm{NcMot}}(M\mathcal{C}, M\mathcal{D})$$

for any $\mathcal{D} \in \mathrm{Pr}^L_{\mathrm{dual}}$.

This is some kind of projectivity statement. If this holds, we have

$$\mathrm{KK}^{\mathrm{cont}}(\mathcal{C}, \mathcal{D}) = K^{\mathrm{cont}}(\underline{\mathrm{Hom}}^{\mathrm{dual}}(\mathcal{C}, \mathcal{D})).$$

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